NON-ORDERED LOWER AND UPPER SOLUTIONS AND SOLVABILITY OF THE PERIODIC PROBLEM FOR THE LIÉNARD AND THE RAYLEIGH EQUATIONS(*)

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To the Memory of Ugo Barbuti

SUMMARY.- *In this paper we study the existence of periodic solutions for the Liénard and the Rayleigh equations in presence of lower and upper solutions which are not necessarily ordered. In this way several previous results in the literature are completed and extended.*

SOMMARIO.- *In questo lavoro si studia l'esistenza di soluzioni periodiche per le equazioni di Liénard e di Rayleigh in presenza di sotto e sopra soluzioni non necessariamente ordinate. In questo modo si completano ed estendono vari risultati precedentemente apparsi in letteratura.*

A.M.S. SUBJECT CLASSIFICATION: 34 C 25.

1. Introduction

In this paper we study the existence of 2π-periodic solutions to the second order ordinary differential equations of Liénard type

$$-x'' = f(x)\,x' + g(x) + h(t, x, x') \tag{1.1}$$

and of Rayleigh type

$$-x'' = F(x') + g(x) + h(t, x, x'), \tag{1.2}$$

where $f, g, F : \boldsymbol{R} \to \boldsymbol{R}$ are continuous functions and $h : \boldsymbol{R}^3 \to \boldsymbol{R}$ is continuous, bounded and 2π-periodic in the first variable. It is a classical result (see [K_1], [K_2], [R-M], [G-M], [L-L-V] that equation (1.1), or (1.2), has a 2π-periodic solution if there exists a pair of lower and upper solutions α, β (of class C^2 and 2π-periodic) such that

$$\alpha(t) \leq \beta(t), \text{ for every } t. \tag{1.3}$$

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Here, we will investigate the case where α, β do not satisfy this ordering condition. We stress that in this frame one cannot generally conclude existence. Indeed, the linear equation

$$-x'' = x + \sin(t)$$

admits constant lower and upper solutions, but does not possess any 2π-periodic solution. This phenomenon becomes clear looking at the eigenvalues of the linear differential operator $\mathcal{L}_a : x \to -x'' - ax$ ($a \in \boldsymbol{R}$ fixed), acting on 2π-periodic functions x. Namely, if $a = 0$, the spectrum of $\mathcal{L}_0$ is the set $\{n^2 : n = 0, 1, ...\}$, while, if $a \neq 0$, the only (real) eigenvalue of $\mathcal{L}_a$ is 0. From this point of view, the assumption of the existence of a pair of lower and upper solutions can be viewed as a control on the behaviour of the nonlinearities at the right hand side of (1.1), or (1.2), with respect to the eigenvalue 0. More precisely, the ordering condition (1.3) expresses the fact that the right hand sides lie, in some sense, at the left of the spectrum. Hence in order to achieve solvability, when (1.3) fails, it is natural to put some additional hypotheses on the nonlinear terms to prevent interference with the rest of the spectrum. The conditions we shall introduce will correspond, for the linear equation

$$-x'' = ax' + bx + h(t),$$

with $h : \boldsymbol{R} \to \boldsymbol{R}$ continuous and 2π-periodic, either to $b < 1$ and $a \in \boldsymbol{R}$ arbitrary (cf. (i_1), (j_1) below), or to $a \neq 0$ and $b \in \boldsymbol{R}$ arbitrary (cf. (i_2), (j_2) below). Actually, the existence results for (1.1) and (1.2) we are going to present will be derived from a more general theorem for the equation

$$-x'' = \varphi(t, x, x'),$$

where $\varphi : \boldsymbol{R}^3 \to \boldsymbol{R}$ is a (possibly) unbounded continuous function, 2π-periodic in the first variable. We point out that the case of a bounded nonlinearity φ was already considered in [A-A-M], while the case of an unbounded φ was discussed in [G-O], but in presence of constant lower and upper solutions and using a different approach.

2. Existence results

Let us consider the second order ordinary differential equation

$$-x'' = \varphi(t, x, x'), \tag{2.1}$$

where $\varphi : \boldsymbol{R}^3 \to \boldsymbol{R}$ is continuous and 2π-periodic in the first variable. We recall that a 2π-periodic function $\alpha : \boldsymbol{R} \to \boldsymbol{R}$ (resp. $\beta : \boldsymbol{R} \to \boldsymbol{R}$) of class C^2 is called a (strict) lower (resp. upper) solution of (2.1) if

$$-\alpha''(t) < \varphi(t, \alpha(t), \alpha'(t)), \text{ for every } t. \tag{2.2}$$

$$(\text{resp. } -\beta''(t) > \varphi(t, \beta(t), \beta'(t)), \text{ for every } t. \tag{2.2'})$$

As far as one is concerned with the existence of 2π-periodic solutions to (2.1), the following result is classical (see e.g. [R-M, vol. II], [G-M, ch. V]).

THEOREM 1. *Suppose that*
(h_1) *for every* $M > 0$, *there exists* $N = N(M) > 0$ *such that, if* x *is a* 2π*-periodic solution to*

$$-x'' = \lambda\varphi(t, x, x'),$$

for some $\lambda \in [0, 1]$, *with* $\max_t |x(t)| \le M$, *then* $\max_t |x'(t)| \le N$.

Moreover, assume that there exists a pair α, β *of lower and upper solutions such that*
(h_2) $\alpha(t) < \beta(t)$, *for every* t.

Then equation (2.1) *has at least one* 2π*- periodic solution.*

The next theorem shows that assumption (h_2) can be dropped, provided that (h_1) is replaced by certain technical conditions (cf. (k_1) and (k_2) below), which permit to avoid the resonance phenomena mentioned in the introduction. Of course, theorem 2 is significant only when (h_2) is violated. In the sequel, for any continuous 2π-periodic function x, we denote by Px the mean value of x on a period, i.e. $Px = (2\pi)^{-1} \int_{[0,2\pi]} x(t)\, dt$.

THEOREM 2. *Suppose that*
(k_1) *for every* $M > 0$, *there exists* $N = N(M) > 0$ *such that, if* x *is a* 2π*-periodic solution to*

$$-x'' = \lambda[\varphi(t, x, x') - P\varphi(., x, x')],$$

for some $\lambda \in [0, 1]$, *with* $|Px| \le M$, *then* $\max_t |x'(t)| \le N$,
and
(k_2) *there exist* $0 \le A < 1$, $B \ge 0$ *such that, if* x *is a* 2π*-periodic solution to*

$$-x'' = \varphi(t, x, x') - P\varphi(., x, x'),$$

with $Px \cdot P\varphi(., x, x') \leq 0$, *then* $\max_t |x(t) - Px| \leq A\,|Px| + B$.

Moreover, assume that there exists a pair α, β *of lower and upper solutions. Then equation* (2.1) *has at least one* 2π*-periodic solution.*

REMARK 1. Condition (k_1) implies condition (h_1). Indeed, if x is a 2π-periodic solution to

$$-x'' = \lambda\varphi(t, x, x'),$$

for some $\lambda \in [0, 1]$, with $\max_t |x(t)| \leq M$, then it is a solution to

$$-x'' = \lambda\,[\varphi(t, x, x') - P\varphi(., x, x')],$$

for the same λ, with $|Px| \leq M$. Hence, $\max_t |x'(t)| \leq N$.

REMARK 2. Weak inequalities in (2.2), (2.2′) and (h_2) are allowed (see [G-M, ch. V]; cf. also [M-Wi]).

Now we produce some applications of theorem 2 to the existence of 2π-periodic solutions to the Liénard and to the Rayleigh equations.

PROPOSITION 1. *Suppose that*

(i_1) $$\limsup_{|s| \to +\infty} g(s)/s < 1.$$

Moreover, assume that there exists a pair of lower and upper solutions. Then equation (1.1) *has at least one* 2π*-periodic solution.*

PROPOSITION 2. *Suppose that*

(i_2) $$\inf_s |f(s)| > 0.$$

Moreover, assume that there exists a pair of lower and upper solutions. Then equation (1.1) *has at least one* 2π*-periodic solution.*

PROPOSITION 3. *Suppose that*

(j_1) $$g \text{ is of class } C^1 \text{ and } \sup_s g'(s) < 1.$$

Moreover, assume that there exists a pair of lower and upper solutions. Then equation (1.2) *has at least one* 2π*-periodic solution.*

PROPOSITION 4. *Suppose that*

(j_2) *there exist constants a, b, c, with* $a > \sup_{t,s,r} |h(t,s,r)|$, *such that*

$$F(s) \cdot s \geq a|s| + bs + c, \text{ for every } s.$$

Moreover, assume that there exists a pair of lower and upper solutions. Then equation (1.2) *has at least one* 2π*-periodic solution.*

REMARK 3. Proposition 1 holds true even for a more general equation of the type

$$-x'' = f(x)x' + g(t,x) + h(t,x,x'),$$

where $g : \boldsymbol{R}^2 \to \boldsymbol{R}$ is continuous and 2π-periodic in the first variable, provided that condition (i_1) is replaced (for instance) by: *there are constants* $c, d > 0$ *and* $k \in [0,1[$ *such that* $-c \leq g(t,s)\,sgn(s) \leq k|s|$, *for every t and every* $|s| \geq d$. In this case the proof proceeds like that of proposition 1, but starting from step (3.4).

REMARK 4. It is easy to see that condition (j_2) is satisfied if, for instance,

$\lim_{s\to+\infty} F(s) = +\infty$ *and there is* $d > 0$ *such that* $F(s) \leq d$, *for* $s \leq 0$,

(resp. $\lim_{s\to-\infty} F(s) = +\infty$ *and there is* $d > 0$ *such that* $F(s) \leq d$, *for* $s \geq 0$)

or

$\lim_{s\to+\infty} F(s) = -\infty$ *and there is* $d > 0$ *such that* $F(s) \geq -d$, *for* $s \leq 0$

(resp. $\lim_{s\to-\infty} F(s) = -\infty$ *and there is* $d > 0$ *such that* $F(s) \geq -d$, *for* $s \geq 0$).

REMARK 5. Propositions 1-4 complete and extend in various directions previous results contained e.g. in [R], [D], [G-M], [M-Wa], [R-M], [B-M], [I- Z], [Z]. In particular, we stress that substantial improvements of the main existence theorems in [N] are obtained.

REMARK 6. When the lower and upper solutions are constants, propositions 1-4 can be (essentially) derived from [G-O], where more general conditions than (i_1) are also considered. Yet, the extension of the results

in [G-O] to the case of non-constant lower and upper solutions does not seem trivial, using the method of proof there introduced.

REMARK 7. Extensions of these results to boundary value problems for second order semilinear elliptic equations will appear in a forthcoming joint paper with Jean-Pierre Gossez.

3. Proofs

Proof of Theorem 1. We borrow some ideas from [A-A-M]. We also use a result in [F-P], which we recall below as a lemma.

LEMMA. *Let E, F be real Banach spaces, let $L : E \to F$ be a linear isomorphism and let $N : \boldsymbol{R} \times E \to F$ be a completely continuous operator. Assume that*

(l_1) *the set* $\{w \in E : Lw = \lambda N(0, w), \lambda \in [0, 1]\}$ *is bounded*

and

(l_2) *for every* $M > 0$, *the set* $\{w \in E : Lw = N(v, w), |v| \leq M\}$ *is bounded.*

Then equation

$$Lw = N(v, w) \tag{3.1}$$

admits a connected set Σ $(\subset \boldsymbol{R} \times E)$ *of solutions whose projection on* $\boldsymbol{R}$ *is onto (i.e.* $\mathrm{proj}_{\boldsymbol{R}} \Sigma = \{v \in \boldsymbol{R} : (v, w) \in \Sigma\} = \boldsymbol{R}$*).*

In order to apply this lemma, we set $E = \{w : \boldsymbol{R} \to \boldsymbol{R}$ of class C^2, 2π-periodic and such that $Pw = 0\}$ and $F = \{w : \boldsymbol{R} \to \boldsymbol{R}$ continuous, 2π-periodic and such that $Pw = 0\}$. Of course, E and F, endowed respectively with the C^2 and C^0 norms, are Banach spaces. We also set $L : E \to F$, $Lw = -w''$, and $N : \boldsymbol{R} \times E \to F$, $N(v, w) = \varphi(., v + w, w') - P\varphi(., v + w, w')$. Clearly, L is a linear isomorphism and N is completely continuous. Decomposing any 2π-periodic continuous function $x : \boldsymbol{R} \to \boldsymbol{R}$ in the form $x = v + w$, with $v = Px$ and $w = x - Px$, it is easy to see that (l_1) and (l_2) follow from (k_1). Accordingly, there exists a connected set Σ $(\subset \boldsymbol{R} \times E)$ of solutions to the equation (equivalent to (3.1))

$$-w'' = \varphi(t, v + w, w') - P\varphi(., v + w, w'),$$

such that $\mathrm{proj}_{\boldsymbol{R}}\Sigma = \boldsymbol{R}$. Hence, equation (2.1) has a 2π-periodic solution $x = v + w$, if there exists $(v, w) \in \Sigma$ such that $P\varphi(., v + w, w') = 0$. Define $\Psi : \Sigma \to \boldsymbol{R}$, by $\Psi(v, w) = P\varphi(., v + w, w')$; of course, Ψ is continuous and therefore $\Psi(\Sigma)$ is an interval. Thus, three cases may occur: $0 \in \Psi(\Sigma)$, $\Psi(\Sigma) \subset \boldsymbol{R}^+$, $\Psi(\Sigma) \subset \boldsymbol{R}^-$. Since in the first case a solution exists, we assume e.g. $\Psi(\Sigma) \subset \boldsymbol{R}^+$. This implies that, for each $(v, w) \in \Sigma$, setting $x = v + w$,

$$-x''(t) = \varphi(t, x(t), x'(t)) - P\varphi(., x, x') < \varphi(t, x(t), x'(t)),$$

for every t, i.e. x is a lower solution. Then let us take $v^* \leq 0$ so large that

$$(1 - A)\, v^* + B < \min_t \beta(t).$$

Since $\mathrm{proj}_{\boldsymbol{R}}\Sigma = \boldsymbol{R}$, there exists $w^* \in E$ such that $(v^*, w^*) \in \Sigma$, with $v^* \cdot P\varphi(., v^* + w^*, w^{*\prime}) \leq 0$. By (h_2), we have, setting $\alpha^* = v^* + w^*$,

$$\alpha^*(t) \leq v^* + \max_t |w^*(t)| \leq v^* + A\,|v^*| + B < \min_t \beta(t) \leq \beta(t),$$

for every t. Therefore there exists a pair α^*, β of lower and upper solutions satisfying (h_2). Hence, using remark 1, it follows that all the assumptions of theorem 1 are satisfied and then equation (2.1) has at least one 2π-periodic solution. Similarly, one argues if $\Psi(\Sigma) \subset \boldsymbol{R}^-$.

Q.E.D.

Proof of Proposition 1. We start observing, like in [G-O], that, if $g(s) \cdot sgn(s)$ is unbounded from below on $\boldsymbol{R}$, then there exist either arbitrarily large negative constant lower solutions or arbitrarily large positive upper solutions, and hence there is a pair of lower and upper solutions satisfying (h_2). Moreover, condition (h_1) is also fulfilled in this case, because if x is a 2π-periodic solution to

$$-x'' = \lambda\,[f(x)\,x' + g(x) + h(t, x, x')],$$

for some $\lambda \in [0, 1]$, with $\max_t |x(t)| \leq M$, multiplying the equation by x and integrating, we immediately get

$$\int_{[0,2\pi]} |x'(t)|^2 dt \leq 2\pi M\,(\max_{|s| \leq M} |g(s)| + \sup_{t,s,r} |h(t, s, r)|).$$

Hence, using again the equation, we obtain $\max_t |x'(t)| \leq N$, for some $N = N(M) > 0$. Then theorem 1 applies and yields the conclusion.

Therefore, assume that there exists a constant $c > 0$ such that

$$g(s) \cdot sgn(s) \geq -c, \tag{3.3}$$

for every s. From (i_1) and (3.3), it follows (cf. [M-Wa]) that there exist a constant $0 \leq k < 1$ and continuous functions $\gamma, \delta : \boldsymbol{R} \to \boldsymbol{R}$, with $0 \leq \gamma(s) \leq k$, for every s, and $\sup_s |\delta(s)| < +\infty$, such that

$$g(s) = \gamma(s)\, s + \delta(s), \tag{3.4}$$

for every s. Now we are going to apply theorem 2. Let x be a 2π-periodic solution to

$$-x'' = \lambda [f(x)\, x' + g(x) + h(t, x, x') - P(g(x) + h(., x, x'))], \tag{3.5}$$

for some $\lambda \in [0, 1]$, with $|Px| \leq M$. Multiplying the equation by $x - Px$ and using (3.4), we get

$$\int_{[0,2\pi]} |x'(t)|^2 dt = \lambda \int_{[0,2\pi]} \gamma(x(t))\, |x(t) - Px|^2\, dt +$$

$$+\lambda \int_{[0,2\pi]} (\gamma(x(t)) \cdot Px + \delta(x(t)) + h(t, x(t), x'(t)))\, (x(t) - Px)\, dt$$

$$\leq k \int_{[0,2\pi]} |x(t) - Px|^2 dt + (2\pi)^{1/2} (k \cdot M + \sup_{t,s,r} |\delta(s) +$$

$$+ h(t, s, r)|) \left(\int_{[0,2\pi]} |x(t) - Px|^2 dt\right)^{1/2}.$$

This implies that there exist constants $c_1, c_2 > 0$ (depending on M) such that

$$\int_{[0,2\pi]} |x'(t)|^2 dt \leq c_1$$

and then

$$\max_t |x(t) - Px| \leq c_2.$$

Using again the equation, after multiplication by x'', we obtain

$$\max_t |x'(t)| \leq N,$$

for some $N = N(M) > 0$. Hence condition (k_1) holds. Next, let us prove that condition (k_2) is also fulfilled. Let x be a 2π-periodic solution to (3.5), with $Px \cdot P\varphi(.,x,x') = Px \cdot P(g(x) + h(.,x,x')) \leq 0$. Multiplying equation (3.5) by $x - 2Px$ (cf. [M-Wa]) and using the decomposition (3.4) we get

$$\int_{[0,2\pi]} |x'(t)|^2 dt =$$

$$= \lambda \int_{[0,2\pi]} \gamma(x(t)) \, |x(t) - Px|^2 dt - \lambda \int_{[0,2\pi]} \gamma(x(t)) \, |Px|^2 dt +$$

$$+ \lambda \int_{[0,2\pi]} (\delta(x(t)) +$$

$$+ h(t, x(t), x'(t))) (x(t) - Px) \, dt - \lambda \int_{[0,2\pi]} (\delta(x(t)) +$$

$$+ h(t, x(t), x'(t))) \, Px \, dt +$$

$$+ 2\pi \, Px \, P(g(x) + h(., x, x'))$$

$$\leq k \int_{[0,2\pi]} |x(t) - Px|^2 dt + c_3 \left(\int_{[0,2\pi]} |x(t) - Px|^2 dt\right)^{1/2} + c_4 \, |Px|,$$

for some constants $c_3, c_4 > 0$. Hence, we derive

$$\int_{[0,2\pi]} |x'(t)|^2 dt \leq c_5 \, |Px| + c_6,$$

for some constants $c_5, c_6 > 0$. Accordingly, $\max_t |x(t) - Px| = O(|Px|^{1/2})$ and then there exist constants $0 \leq A < 1$, $B \geq 0$ such that

$$\max_t |x(t) - Px| \leq A \, |Px| + B,$$

that is (k_2) holds. Therefore theorem 2 applies and yields the conclusion.

Q.E.D.

Proof of Proposition 2. We apply theorem 2 again. It is easy to see that conditions (k_1) and (k_2) (with $A = 0$) are satisfied, for any 2π-periodic solution x to equation (3.5), just multiplying the equation by $(sgn\, f) \cdot x'$, integrating and using (i_2).

Q.E.D.

Proof of Proposition 3. Let x be any 2π-periodic solution to the equation

$$-x'' = \lambda [F(x') + g(x) + h(t, x, x') - P(F(x') + g(x) + h(., x, x'))] \tag{3.6}$$

Multiply (3.6) by $-x''$ and integrate. Using (j_1) we immediately realize that conditions (k_1) and (k_2) (with $A = 0$) are fulfilled, so that theorem 2 applies.

Q.E.D.

Proof of Proposition 4. Multiplying equation (3.6) by x', integrating and using (j_2), we obtain, for some constant $c_1 > 0$,

$$\int_{[0,2\pi]} |x'(t)|\, dt \leq c_1. \tag{3.7}$$

Hence, condition (k_2) (with $A = 0$) is fulfilled. From (3.7) and $|Px| \leq M$, it follows

$$\max_t |x(t)| \leq c_2,$$

for some constant $c_2 = c_2(M) > 0$. Then multiplying equation (3.6) by $-x''$ and integrating, we find

$$\left(\int_{[0,2\pi]} |x''(t)|^2 dt\right)^{1/2} \leq (2\pi)^{1/2} \left(\max_{|s| \leq c_2} |g(s)| + \sup_{t,s,r} |h(t, s, r)|\right)$$

and then

$$\max_t |x'(t)| \leq N,$$

for some $N = N(M) > 0$. Thus condition (k_1) is also satisfied. Hence, theorem 2 applies and yields the conclusion.

Q.E.D.

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