REMARKS ON HOPF BIFURCATION FORMULÆ(*)

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To the Memory of Ugo Barbuti

SOMMARIO.- *Mostriamo come la complessità delle formule di Hassard, Kazarinoff e Wan* (1981) *riguardanti la biforcazione di Hopf in sistemi di equazioni differenziali ordinarie possa essere ridotta usando una opportuna tecnica formale nel calcolo delle derivate parziali di ordine superiore richiesto dal detto metodo.*

SUMMARY.- *We outline how the complexity of the formulæ by Hassard, Kazarinoff and Wan* (1981) *concerning the Hopf bifurcation in systems of ordinary differential equations can be reduced computing the pertinent higher- order partial derivatives in a suitable formal way.*

1. Introduction

The quantitative analysis of a Hopf bifurcation of periodic solutions from an equilibrium point, e.g. for an autonomous system of ordinary differential equations, is usually much more difficult that just proving the existence of such a bifurcation. We contribute to this problem showing that the cumbersome computation of higher order partial derivative of some *composite* mappings arising for instance in the method proposed by Hassard, Kazarinoff and Wan [6] can be avoided and substituted by a procedure which is simpler in many cases: see Sect. 2. After a first reference example concerning the Hopf bifurcation in the Lorenz system (Sect. 3), we apply this idea in Sect. 4 to a single loop feedback control system of large dimension (= 50), which has been proposed and studied by Sparrow [12] in a biological framework and has recently received further attention by Medio and the author as a continuous model in economics [7]. Studying the Hopf bifurcation, the particular form of this system allows the application fo the harmonic balance techniques developed by Allwright [1, 2]. We show that the general bifurcation formulæ can equally be applied without heavy computations. Applications to the n-dimensional metabolic cellular control Goodwin system will be considered in a forthcoming paper

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[8]. Applications to fully nonlinear or multiple feed-back systems (where no special technique is available) can be considered as well.

2. Remarks on Bifurcation Formulæ

Consider an autonomous system of ordinary differential equations

$$dx/dt = f(x; \nu) \tag{2.1}$$

where $x \in R^n$ and ν is a real parameter on an open interval J, and assume that a Hopf bifurcation occurs at $\nu = \nu_c$. Namely, assume that, for f sufficiently smooth, there is a point $x^* = x^*(\nu)$ such that $f(x^*(\nu), \nu) = 0$ for all ν near ν_c, and that a pair of simple complex conjugate eigenvalues of the jacobian matrix $A(\nu)_{ij} = (\partial_j f_i)(x^*(\nu); \nu)$ crosses the imaginary axis at $\nu = \nu_c$, while the remaining eigenvalues have negative real part. Then it is well known that the sub-, super- or critical character of the bifurcation and the period and the Floquet exponents of the bifurcating orbits can be calculated by means of the so-called *bifurcation formulæ*. In particular Hassard, Kazarinoff and Wan have fully displayed these formulæ in their book [6] to which we refer also for the formal statement of the Hopf Bifurcation Theorem for ordinary differential equations (see also [11]). For short, we shall call "HKW procedure" the Recipe-Summary presented in the Ch. 2 of [6]. We recall that the first step in this HKW procedure is the reduction of the jacobian matrix $A(\nu_c)$ to its real Jordan form. Therefore we need to compute the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(\nu_c)$, where λ_1 and $\lambda_2 = \bar{\lambda}_1$ is the pair crossing the imaginary axis at $\nu = \nu_c$; let u_1 be the eigenvector corresponding to λ_1. Then we have to form an $n \times n$ matrix P whose first and second columns are respectively $Re\ u_1$ and $-Im\ u_1$ while the other columns span the union of the (generalized) eigenspaces of $A(\nu_c)$ corresponding to $\lambda_j, j \geq 3$. This leads to the composite mapping

$$F(y) = Q f(x^* + Py), \quad Q = P^{-1},$$

and the next step is the computation of at least the 3-rd partial derivatives of F with respect to the new variable y, to be evaluated at $\nu = \nu_c, y = 0$. In some degenerate cases these derivatives have to be computed up to the 5-th order, or more. In several practical situation, this requires a tremendous effort of symbolic manipulation, so that the analytic evaluation of the bifurcation formulæ becomes sometimes practically impossible. In these cases, for example, Hassard et al. ([6], Ch. 3] suggest the possible use of numerical differencing to approximate the partial derivatives.

We propose here an alternative technique. Namely, it is possible to compute the p-th order partial derivatives of F without computing its lower order derivatives, including F itself: It is enough to know the p-th order derivatives of the original map f at the equilibrium x^*, and solve some algebraic linear systems, each having P as coefficient matrix but different right hand sides. If n is large, this last step may possibly require numerical techniques. Anyway, a numerical linear systems solver is conceptually simpler and less complex than a numerical differencing routine.

In fact we have the following result.

PROPOSITION 2.1. *Suppose that* $f : \mathbf{R}^n \supseteq U \to \mathbf{R}^n$ *is a* C^p *map, with* U *open. Let* P *be a real constant* $n \times n$ *nonsingular matrix,* $Q = P^{-1}$, *and let* $x^* \in \mathbf{R}^n$. *Then the map* F *defined by*

$$F(y) = Qf(x^* + Py)$$

is C^p *on a neighborhood of* 0, *and for any multi-index* $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ *with* $\alpha_1 + \alpha_2 + ... + \alpha_n = p$, *we have*

$$(\partial^\alpha F_1(0), \partial^\alpha F_2(0), ..., \partial^\alpha F_m(0)) = (\xi_1, \xi_2, ..., \xi_m)$$

where ξ *is the solution of the linear system* $P\xi = b$ *with right hand side*

$$b = D^p f(x^*)[P_1]^{\alpha_1}[P_2]^{\alpha_2} ... [P_n]^{\alpha_n}$$

(P_r is the r-th column of P, $1 \le r \le n$).

Proof. Consider first $G(y) = f(x^* + Py)$. Then G is C^p and its p-th differential at y is defined by

$$D^p G(y)[w_1, w_2, ..., w_p] = D^p f(x^* + Py)[Pw_1, Pw_2, ..., Pw_p]$$

(see [3], p. 74). The theorem relating the p-th differential to linear maps (see [5], p. 86) gives that $D^p(QG)(y) = Q\{D^p G(y)\}$. Therefore we have

$$D^p F(y)[w_1, w_2, ..., w_p] = QD^p f(x^* + Py)[Pw_1, Pw_2, ..., Pw_p],$$

and, evaluating at $y = 0$,

$$D^p F(0)[w_1, w_2, ..., w_p] = QD^p f(x^*)[Pw_1, Pw_2, ..., Pw_p],$$

In particular, the components of $D^p F(0)$ in terms of the standard basis are the p-th order partial derivatives, so that

$$\partial^\alpha F(0) = D^p F(0)\,[e_1]^{\alpha_1}[e_2]^{\alpha_2} \dots [e_n]^{\alpha_n}$$

$$= Q\,D^p f(x^*)\,[Pe_1]^{\alpha_1}[Pe_2]^{\alpha_2} \dots [Pe_n]^{\alpha_n} = Q\,D^p f(x^*)\,[P_1]^{\alpha_1}[P_2]^{\alpha_2} \dots [P_n]^{\alpha_n},$$

for any multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_1 + \alpha_2 + \dots + \alpha_n = p$, where P_r is the r- th column of P.

As special cases, the 2-nd and 3-rd order derivatives of the components of F are

$$(\partial_i\partial_j F_1(0), \partial_i\partial_j F_2(0), \dots, \partial_i\partial_j F_n(0)) = D^2F(0)\,[e_i]\,[e_j]$$

$$= Q\,D^2 f(x^*)\,[P_i]\,[P_j],$$

$$(\partial_i\partial_j\partial_k F_1(0), \partial_i\partial_j\partial_k F_1(0), \dots, \partial_i\partial_j\partial_k F_n(0)) = D^3F(0)\,[e_i]\,[e_j]\,[e_k]$$

$$= Q\,D^3 f(x^*)\,[P_i]\,[P_j]\,[P_k].$$

Therefore, once we have computed the partial derivatives of f, the n-vector $\partial_i\partial_j F(0)$ is the solution ζ of the linear system

$$P\zeta = D^2 f(x^*)\,[P_i]\,[P_j], \tag{2.2}$$

and the n-vector $\partial_i\partial_j\partial_k F(0)$ is the solution of the linear system

$$P\zeta = D^3 f(x^*)\,[P_i]\,[P_j]\,[P_k]. \tag{2.3}$$

Remark that all these linear systems are independent one to each other, hence, they can be solved by parallel algorithms. At this point we can go back to the standard HKW procedure without supplementary heavy non-linear computations.

3. A first example: the Lorenz system

We briefly consider, as a reference example, the classical Lorenz system:

$$\dot{x} = f(x; b, \sigma, r), \quad x = (x_1, x_2, x_3),$$

$$f(x; b, \sigma, r) = (\sigma(x_2 - x_1), -x_1x_3 + rx_1 - x_2, x_1x_2 - bx_3),$$

where b, σ are fixed positive numbers, $\sigma > b + 1$. The Hopf bifurcation in this system has been already analyzed, e.g. in [6]. The bifurcation parameter is r. It is well known ([6], [11], ...) that, if $r = r_c = \sigma(\sigma + b + 3)/(\sigma - b - 1)$, then a Hopf bifurcation from the equilibrium

$$x^*(r) = ((b(r-1))^{1/2}, (b(r-1))^{1/2}, r - 1)$$

comes out. The eigenvalues of the jacobian matrix A^0 of $f(x^*(r_c); r_c)$ at bifurcation are

$$\lambda_1(r_c) = \bar{\lambda}_2(r_c) = i\omega, \text{ with } \omega = (b(r_c + \sigma))^{1/2},$$

$$\lambda_3(r_c) = -(\sigma + b + 1).$$

The columns of the real 3×3 matrix $P = (p_{ij})$ transforming the jacobian A^0 into its real canonical form are:

$$P_1 = (p_{11}, p_{21}, p_{31}) = (1, 1, \omega^2/(\sigma x_1^*))$$
$$P_2 = (p_{12}, p_{22}, p_{32}) = (0, -\omega/\sigma, \omega(1 + 1/\sigma)/x_1^*),$$
$$P_3 = (p_{13}, p_{23}, p_{33}) = (1, -(b+1)/\sigma, (1 + (\sigma + b)(b + 1))/x_1^*),$$

where $x_1^* = (b(r-1))^{1/2}$. The vectors $P_1 - iP_2$ and P_3 are the eigenvectors of A^0 corresponding respectively to the eigenvalues $\lambda_1(r_c)$ and $\lambda_3(r_c)$. It is also possible to compute formally

$$\alpha_1 = Re\ \lambda'_1(r_c) = 0.5 \cdot b[\sigma - b - 1]/[\omega^2 + (\sigma + b + 1)^2],$$

$$\omega_1 = Im\ \lambda'_1(r_c) = 0.5 \cdot [b\omega^2 + 2b\sigma(\sigma + b + 1)]/[\omega^2 + (\sigma + b + 1)^2],$$

$$(' = d/dr).$$

Since f is quadratic in x, the 2nd differential of f at the considered equilibrium and $r = r_c$, is constant, while the 3rd one vanishes. Moreover, only the following partials of f at $x = x^*(r_c), r = r_c$

$$\partial_{13}f_2 = \partial_{31}f_2 = 1, \quad \partial_{12}f_3 = \partial_{21}f_3 = 1$$

are non-vanishing. Using Proposition 2.1, we compute for each i, j from 1 to 3 the vector

$$b^{ij} = D^2 f\,(x^*(r_c);\,(r_c)\,[P_i]\,[P_j].$$

Obviously $b_1^{ij} = 0$, while

$$b_2^{ij} = \Sigma_{h,m=1,3}\,\partial_{hm} f_2\, p_{hi} p_{mj} = -\,p_{1i} p_{3j} - p_{3i} p_{1j},$$

$$b_3^{ij} = \Sigma_{h,m=1,3}\,\partial_{hm} f_3\, p_{hi} p_{mj} = p_{1i} p_{2j} + p_{2i} p_{1j}.$$

Only six 3×3 linear systems $P\zeta^{ij} = b^{ij}$ $(i \le j)$ have to be solved. Then we get the 2nd and 3rd partials of $F\,(y) = P^{-1} f\,(x^* + Py)$ in $y = 0, r = r_c$, thus:

$$\partial_{ij} F_m = \zeta_m^{ij},\ \partial_{ijh} F_m = 0.$$

The linear systems have been solved by a library routine on a simple *pcHP*9816*S*, and the solutions have been substituted in the formulæ of

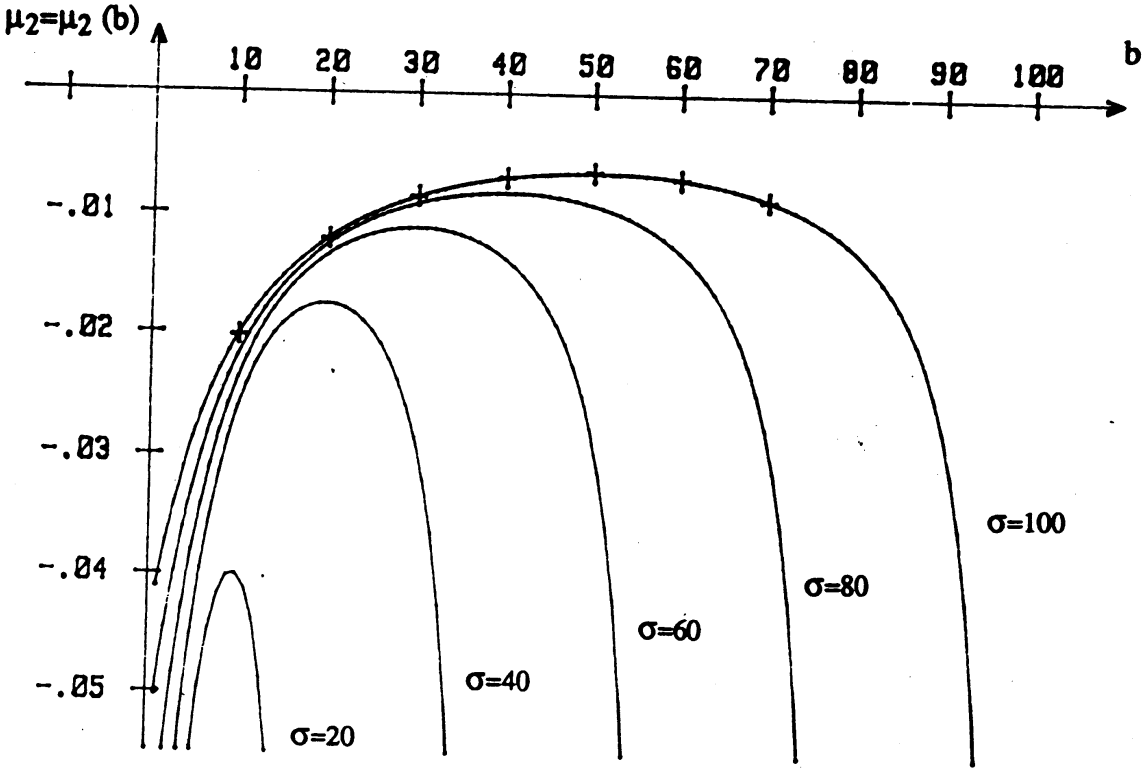

Figure 1

Hassard et al., getting the computed value of the coefficient μ_2 as function of b for $\sigma = 20, 40, 60, 80, 100$.

The results are plotted in Fig. 1, where, for comparison, the signs " + " indicate the values obtained by Hassard et al. by the code BIFOR for $\sigma = 100$ and $b = 10, 20, ..., 70$, as presented in [6; pag. 159, Table 3.4].

4. Application to a Single Loop Feedback Control System

We study in this section the quantitative properties of the Hopf bifurcation of periodic solution from a positive equilibrium state for the following one-loop n-dimensional feedback control system

$$dx_1/dt = n\,(f\,(x_n) - x_1), \tag{4.1}$$
$$dx_j/dt = n\,(x_{j-1} - x_j), \quad j = 2, ..., n, \tag{4.2}$$

where $f\,(s) = \nu s e^{-s}$, $\nu > 0$, and $n \geq 3$. The bifurcation parameter is ν. We shall see that the bifurcation occurs at $\nu = \exp\{1 + [\cos(\pi/n)]^{-n}\}$. Equations (4.1)-(4.2) are a simple model for many natural phenomena where a distributed time lag is present. The "essential" variables are the "input" x_1 and the "output" x_n, which controls the evolution of x_1. The remaining "hidden" variables represent the lag. A prototype of this kind of equations are the Goodwin equations [5] in dimension $n = 3$. This 3-dimensional model describes (with a different type of "hump" nonlinearity, namely $f\,(s) = 1/(1 + s^{\rho})$) the dynamics of end-product inhibition of gene activity: messenger RNA codes for an enzyme, one of whose metabolic products inhibits further synthesis of mRNA. See [10] for the derivation and the interpretation of (4.1)-(4.2) in a biological framework. The same systems occurs in some economic models: see [7], where a detailed study of multiple distributed lags is presented. In the same paper [7] it is proved that, for any n, (4.1)-(4.2) possesses a maximal bounded, compact and connected, attracting set which attracts the bounded subsets of the open positive orthant, and its Lyapunov dimension is determined for $n = 50$ by means of numerical estimates of the Lyapunov characteristic exponents. Previously, in 1980, Sparrow [12] studied (4.1)-(4.2), and he found a complex or chaotic behavior of orbits in this high dimension for $\nu \approx 22$, together with a first Hopf bifurcation for $\nu = 8.197$ and further bifurcations and other interesting facts which are still not completely understood (see the pertinent remarks in [9]).

The technique used in [12] for the analysis of the first Hopf bifurcation was developed by Allwright in 1977 [1] (see also [2]). It is based on the method of harmonic balance, and its results are correct to the first order in the bifurcation parameter.

In alternative, we use here the HKW procedure as modified in Sect. 2. This modification allows us to avoid any numerical help, except for a single algebraic linear system and some elementary complex arithmetic.

The system (4.1)-(4.2) has a non-zero equilibrium

$$x^* = x^*(\nu) = (\log(\nu), \log(\nu), ..., \log(\nu))$$

as soon as $\nu > 1$, which is asymptotically stable if $1 < \nu < \nu_c$, where

$$\nu_c = \exp\{1 + [\cos(\pi/n)]^{-n}\}. \tag{4.3}$$

In fact, given a (monic) polynomial $a(s) = a_0 + a_1 s + ... + a_{n-1}s^{n-1} + s^n$, and its companion matrix C_a, we have $\det(C_a - \mu I) = (-1)^n a(\mu)$. The jacobian matrix (with respect to the variable x) of the vector field

$$f(x;\nu) = n(f(x_n) - x_1, x_1 - x_2, x_2 - x_3, ..., x_{n-1} - x_n)$$

at $x^* = x^*(\nu)$ is $A(\nu) = n({}^tC_a - I)$, with $a_0 = -(df/ds)(x_n)$, $a_j = 0$ for $j \geq 1$.

Therefore the characteristic polynomial of ${}^tC_a - I$ is

$$\det({}^tC_a - (\lambda + 1)I) = (-1)^n a(\lambda + 1) = (-1)^n\{(\lambda + 1)^n - (df/ds)(x_n)\}$$

$$= (-1)^n\{(\lambda + 1)^n - (1 - \log(\nu))\},$$

and the eigenvalues of $A(\nu)$ for $\nu > e$ (remark that $\nu_c > e$) are

$$\lambda_k(\nu) = n\{(\log(\nu) - 1)^{1/n} e^{i(2k-1)\pi/n} - 1\}, \tag{4.4}$$

$(k = 1, ..., n)$. Since $(\log(\nu_c) - 1)^{1/n} = 1/\cos(\pi/n)$, we have

$$\alpha_1(\nu) = Re\lambda_1(\nu) = n\{(\log(\nu) - 1)^{1/n}\cos(\pi/n) - 1\},$$

$$\alpha_0 = \alpha_1(\nu_c) = 0,$$

$$\alpha'_0 = (d\alpha_1/d\nu)(\nu_c) = \nu_c^{-1}[\cos(\pi/n)]^n \neq 0, \tag{4.5}$$

while, for $k = 2, ..., n - 1$,

$$Re\lambda_k(\nu_c) = n\{[\cos(2k - 1)\pi/n)]/[\cos(\pi/n)] - 1\} < 0.$$

Moreover,

$$\omega_1(\nu) = Im\lambda_1(\nu) = n\{(\log(\nu) - 1)^{1/n} \sin(\pi/n)\},$$

$$\omega_0 = \omega_1(\nu_c) = n \tan(\pi/n), \tag{4.6}$$

$$\omega'_0 = (d\omega_1/d\nu)(\nu_c) = \nu_c^{-1}[\tan(\pi/n)][\cos(\pi/n)]^n. \tag{4.7}$$

Observe that these eigenvalues are numbered in such a way that the pair on the imaginary axis is λ_1 and $\lambda_n = \bar{\lambda}_1$ ($\nu > e$). Thus a Hopf bifurcation occurs at $\nu = \nu_c$.

For example for $n = 50$, $\nu_c = 8.197$, in agreement with the data in [12], $\alpha'_0 = 0.110$, $\omega_0 = 3.145$, and the period at the bifurcation is $T_0 = 2\pi/\omega_0 = 1.997$ ([12] reports exactly 2).

Let us reduce now the jacobian matrix $A(\nu_c) = Df(x^*(\nu_c); \nu_c)$ to its real Jordan form, computing the transition matrix P.

LEMMA 4.1. *Any eigenvalue $\lambda = n(\rho + i\sigma)$ of $A(\nu_c)$ has a corresponding eigenvector*

$$u = (1, z^{-1}, z^{-2}, \ldots, z^{-n+1})$$

where $z = 1 + \rho + i\sigma$ ($\rho, \sigma \in R$).

Proof. Remark first than $-n$ is not an eigenvalue of $A(\nu_c)$, so that $z \neq 0$. The system to be solved is

$$-n u_1 + n\{-[\cos(\pi/n)^{-n}\} u_n = n(\rho + i\sigma) u_1,$$

$$n(u_{j-1} - u_j) = n(\rho + i\sigma) u_j, \quad j = 2, \ldots, n.$$

Now $u_1 = 0$ is not allowed for an eigenvector, because this would imply by a recursive argument that $u = 0$. Therefore we can choose $u_1 = 1$, and we find recursively that $u_j = u_{j-1}/(1 + \rho + i\sigma)$.

Letting $\lambda_k(\nu_c) = n(\rho_k + i\sigma_k)$, we get from (4.4)

$$z_k = 1 + \rho_k + i\sigma_k = [\cos(\pi/n)]^{-1}[\cos((2k - 1)\pi/n) + i \sin((2k - 1)\pi/n)].$$

Let us define for any integer $p \geq 0$ the real constant numbers

$$C_p = \cos(p\pi/n), S_p = \sin(p\pi/n).$$

Then ($k = 1, ..., n; j = 1, ..., n - 1$)

$$z_k^{-j} = C_1^j [C_{(2k-1)j} - i S_{(2k-1)j}],$$

and the entries of the transition matrix P are

$$P_{hm} = C_1^{h-1} C_{(2k-1)(h-1)} \quad \text{if } m + 1 = 2k, \text{ i.e. if } m \text{ is odd,}$$

$$= C_1^{h-1} S_{(2k-1)(h-1)} \quad \text{if } m + 1 = 2k + 1, \text{ i.e. if } m \text{ is even} \tag{4.8}$$

$(h, m = 1, ..., n)$.

To write the linear systems (2.2) and (2.3) we observe that $D^2 f(x^*(\nu_c))$ [resp. $D^3 f(x^*(\nu_c))$] has just only one coordinate different from 0, namely

$$\partial_n \partial_n f_1 (x^*(\nu_c)) = C_1^{-n} - 1 \text{ [resp. } \partial_n \partial_n \partial_n f_1 (x^*(\nu_c)) = 2 - C_1^{-n}].$$

We obtain

$$D^2 f(x^*)[P_i][P_j] = \Sigma_{msr} \partial_s \partial_r f_m (x^*(\nu_c)) P_{si} P_{rj} e_m$$

$$= (C_1^{-n} - 1) P_{ni} P_{nj} e_1,$$

$$D^3 f(x^*)[P_i][P_j][P_k] = \Sigma_{msrt} \partial_s \partial_r \partial_t f_m (x^*(\nu_c)) P_{si} P_{rj} P_{tk} e_m$$

$$= (2 - C_1^{-n}) P_{ni} P_{nj} P_{nk} e_1.$$

It is therefore sufficient to solve only once the linear system $P\zeta = e_1$ for the n-dimensional vector $\zeta = (\zeta_1, \zeta_2, ..., \zeta_n)$ (which is nothing but the first column of P^{-1}), and we have our 2-nd and 3-rd order derivatives:

$$\xi_{ijm} = \partial_i \partial_j F_m (0) = (C_1^{-n} - 1) P_{ni} P_{nj} \zeta_m, \tag{4.9}$$

$$\eta_{ijkm} = \partial_i \partial_j \partial_k F_m (0) = (2 - C_1^{-n}) P_{ni} P_{nj} P_{nk} \zeta_m. \tag{4.10}$$

From now on we fix $n = 50$. The straightforward routine LSARG of the IMSL, Inc. MATH/LIBRARY, carried out on a CDC CYBER 170/730,

computes the vector $X = P^{-1} e_1$ (P from (4.8)), and then we follow the HKW procedure using (4.9), (4.10) for the partial derivatives of F.

At last we get that the computed value of the coefficient μ_2 whose sign determines the direction of the bifurcation is 0.0107, so that the bifurcating periodic solutions exist for $\nu > \nu_c$. Moreover, their period and the exponent determining their stability are respectively

$$T = 1.997\,(1 + 0.775\,\delta^2 + O(\delta^4)),$$

$$\beta = -0.221\,\delta^2 + O(\delta^4),$$

with $\delta^2 = (\nu - \nu_c) + O([\nu - \nu_c]^2)$, $\nu_c = 8.197$. Thus, being $\beta < 0$ for ν near ν_c, these solutions are asymptotically orbitally stable (with asymptotic phase).

REFERENCES

[1] D.J. ALLWRIGHT, *Harmonic balance and the Hopf bifurcation*, Math. Proc. Camb. Phil. Soc., *82* (1977), 453-467.

[2] D.J. ALLWRIGHT, *The Hopf bifurcation in some biochemical control loops*, J. Math. Biology, *11* (1981), 85-93.

[3] R. ABRAHAM, J.E. MARSDEN and T. RATIU, *Manifolds, tensor analysis and applications*, Reading, MA, Addison-Wesley 1983.

[4] S.B. CHAE, *Holomorphy and calculus in normed spaces*, New York, Marcel Dekker inc., 1985.

[5] B.C. GOODWIN, *Temporal Organization in Cells*, New York Academic Press, 1963.

[6] B.D. HASSARD, N.D. KAZARINOFF and Y-H. WAN, *Theory and applications of Hopf bifurcation*, London Mathematical Society Lecture Note Series, 41, Cambridge, Cambridge University Press, 1981.

[7] S. INVERNIZZI and A. MEDIO, *On lags and chaos in economic dynamic models*, J. Math. Economics, to appear.

[8] S. INVERNIZZI and G. TREU, *Quantitative analysis of the Hopf bifurcation in the n-dimensional Goodwin system*, to appear.

[9] R.M. MAY, *Nonlinear problems in ecology and resource management*, in: "Chaotic Behavior of Deterministic Systems", G. Iooss, R.H.G. Helleman and R. Stora, eds., Amsterdam, North-Holland, 1983.

[10] N. MACDONALD, *Time lags in biological models*, Lecture Notes in Biomathematics, 27, Berlin/Heidelberg/New York, Springer Verlag, 1978.

[11] J.E. MARSDEN and M. MCCRACKEN, *The Hopf bifurcation and its applications*, Applied Math. Sciences, vol. 19, New York, Springer Verlag, 1978.

[12] C. SPARROW, *Bifurcation and chaotic behavior in simple feedback systems*, J. Theor. Biol., *83* (1980), 93-105.

NOTE ADDED IN PROOFS

The use of a numerical routine solving $PX = e_1$ is unnecessary. Infact, using the ideas in [8], one can formally prove that $X_i = 2/50$ for i odd, white $X_i = 0$ for i even.