EXISTENCE OF INVARIANT MASSES
An elementary treatment(*)

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To the Memory of Ugo Barbuti

SOMMARIO.- *Si prova con strumenti elementari che esiste una massa invariante rispetto ad una arbitraria famiglia commutante di trasformazioni su un insieme qualunque.*

SUMMARY.- *We prove by means of very elementary tools the existence of a finitely additive probability invariant under the elements of a commuting family of transformations.*

1. In [1] Chersi proves the existence of a finitely additive probability invariant under the elements of a commuting family of transformations. This result is obtained by means of some notions and results of Functional Analysis (for instance, weak * topology, isometric isomorphism between ba (Ω, F) and the dual of the space of uniform limits of sequences of F-simple functions, the Markoff-Kakutani fixed point theorem).

In this paper, we prove an equivalent formulation of Chersi's theorem by means of very elementary tools. To this purpose, we use the de Finetti coherent previsions ([3], vol. I, 3.3.5., p. 87) and an extension theorem for these ([3], vol. I, 3.10.7., p. 116) which may be proved by elementary properties of real numbers and the Zorn Lemma ([3], vol. II, p. 336-337; for a detailed proof in a more general context see [5], Lemma 3.3 and Theorem 3.4, p. 448-451).

2. Let Ω be a non-empty set and $\mathcal{S}$ a family of subsets of Ω. The letter S, with or without indices, always denotes an element from $\mathcal{S}$. Moreover the letter τ, with or without indices, always denotes a *transformation* of Ω, i.e. a map from Ω to Ω; τ is called $\mathcal{S}$-*measurable* iff $\tau^{-1}(S) \in \mathcal{S}$ for all S.

Given a family T of $\mathcal{S}$-measurable transformations, a coherent probability P on $\mathcal{S}$ is called T- *invariant* iff $P(S) = P(\tau^{-1}(S))$ for all S and for all $\tau \in T$. Moreover T is called a *commuting family* iff $\tau_1\tau_2 = \tau_2\tau_1$ for all

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$\tau_1, \tau_2 \in T$ (of course, the product denotes the usual composition of transformations).

Now, we are going to prove the following theorem.

THEOREM. *Let T be a commuting family of $\mathfrak{S}$- measurable transformations on Ω. Then there exists a T-invariant coherent probability on $\mathfrak{S}$.*

Proof. Let $\mathcal{X} = \{I_S - I_{\tau^{-1}(S)} : S \in \mathfrak{S} \text{ and } \tau \in T\}$, where I_A denotes the *indicator function* of an arbitrary subset A of Ω. Now, suppose there is a coherent prevision E on the set $B(\Omega)$ of all bounded real functions on Ω such that $E(X) = 0$ for all $X \in \mathcal{X}$. Then the map $P(S) = E(I_S)$ is a coherent probability on $\mathfrak{S}$. Since E is a linear functional and $E(I_S - I_{\tau^{-1}(S)}) = 0$ for all S and for all $\tau \in T$, P is a T-invariant coherent probability on $\mathfrak{S}$. Therefore to get the thesis it is sufficient to prove the existence of a coherent prevision on $B(\Omega)$ such that $\mathcal{X}$ is contained in its kernel.

Now, keeping in mind the de Finetti's extension theorem for coherent previsions, it is enough for us to prove that the null-function on $\mathcal{X}$ is a coherent prevision. Then, given $S_1, \ldots, S_n$ and $\tau_1, \ldots, \tau_n \in T$, let $\alpha_1, \ldots, \alpha_n$ be real numbers such that

$$\sum_{i=1}^{n} \alpha_i \, (I_{S_i} - I_{\tau_i^{-1}(S_i)}) \geq a.$$

Of course, in order to verify the coherence of the null-function, we must prove that $a \leq 0$.

Plainly we have $I_{\tau^{-1}(A)} = I_A \, \tau$ for all subset A of Ω and for all τ. Then, adopting the following notations:

$$f_i = \alpha_i I_{S_i} \quad , \quad f_i(\tau) = f_i \, \tau \quad \text{for all } \tau \quad (i = 1, \ldots, n),$$

we have $\sum_{i=1}^{n} [f_i - f_i(\tau_i)] \geq a$ and hence $\{\sum_{i=1}^{n} [f_i - f_i(\tau_i)]\} \, \tau \geq a$ for all τ. Therefore, the following inequality

$$(*) \qquad \sum_{i=1}^{n} [f_i(\tau) - f_i(\tau_i \tau)] \geq a$$

holds for all τ.

Now, we are going to make use of an argument suggested by Dixmier's proof of Theorem 2 of [4], p. 216. Let p be any natural number and let Φ be the set of all maps from $\{1, ..., n\}$ to $\{1, ..., p\}$; moreover let $\tau(\varphi) = \prod_{i=1}^{n} \tau_i^{\varphi(i)}$ for all $\varphi \in \Phi$. Then, given $i \in \{1, ..., n\}$, we consider the following sum

$$\sum_{\varphi \in \Phi} [f_i(\tau(\varphi)) - f_i(\tau_i \tau(\varphi))].$$

Since T is a commuting family, the terms of this sum cancel each other except possibly those $f_i(\tau(\varphi))$ with $\varphi(i) = 1$ and those $f_i(\tau_i \tau(\varphi))$ with $\varphi(i) = p$. On noting that the number of these terms is $2p^{n-1}$, we have

$$\sum_{\varphi \in \Phi} [f_i(\tau(\varphi)) - f_i(\tau_i \tau(\varphi))] \leq 2kp^{n-1},$$

where $k = \max(|\alpha_1|, ..., |\alpha_n|)$.

Since the set Φ has p^n elements, by the inequality (*), we get

$$a p^n \leq \sum_{\varphi \in \Phi} \sum_{i=1}^{n} [f_i(\tau(\varphi)) - f_i(\tau_i \tau(\varphi))] =$$

$$= \sum_{i=1}^{n} \sum_{\varphi \in \Phi} [f_i(\tau(\varphi)) - f_i(\tau_i \tau(\varphi))] \leq 2nkp^{n-1}.$$

On noting that $k \geq 0$ and that p can be choosen arbitrarily, we have $a \leq 0$. This completes the proof.♦

COROLLARY. *Let T be a commuting family of injective transformations on Ω. Moreover let $\mathfrak{S}$ be a family of subsets of Ω such that $\tau(S) \in \mathfrak{S}$ for all S and for all $\tau \in T$. Then there is a coherent probability P on $\mathfrak{S}$ such that $P(S) = P(\tau(S))$ for all S and for all $\tau \in T$.*

Proof. From the previous theorem (choose $\mathfrak{S}$ as the power set of Ω) there is a T-invariant coherent probability on the power set of Ω. Then, by the injectivity of any element of T, we have $P(S) = P(\tau^{-1}(\tau(S))) = P(\tau(S))$ for all S and for all $\tau \in T$. This completes the proof.♦

Since any coherent probability on a field is a finitely additive probability and viceversa ([2], p. 77 or, in a more general context, [5], Theorem 5.6, p. 455), we get immediately the following corollary.

COROLLARY. *Let $\mathcal{S}$ be a field on Ω. Moreover let T be a commuting family of $\mathcal{S}$-measurable transformations on Ω. Then there is a T-invariant finitely additive probability on $\mathcal{S}$.*

REMARK. (i) If $\mathcal{S}$ is a σ-field, then, by the previous Corollary, we get Chersi's theorem of [1], p. 178. On the other hand, let T be a commuting family of $\mathcal{S}$-measurable transformations on Ω. Now, if we consider in the Chersi's theorem the σ-field of all subsets of Ω, we obtain a T-invariant finitely additive probability P on this σ-field. Since P is also a coherent probability, the restriction of P on $\mathcal{S}$ is a T-invariant coherent probability. So we get our theorem.

Therefore we can claim that our theorem and Chersi's theorem are equivalent.

(ii) Obviously, given a positive real number k, the theorem also assures the existence of a T-invariant mass (i.e. T-invariant positive charge) on $\mathcal{S}$ with norm k.

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