ON SELF-INJECTIVE AND QUASI-FROBENIUSEAN RINGS(*)

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SOMMARIO.- *Si sviluppano proprietà di anelli auto-iniettivi e p- iniettivi. Si studiano poi anelli di von Neumann regolari ed associati. Infine si considerano anelli pseudo-Frobeniani e quasi-Frobeniani.*

SUMMARY.- *Properties of self-injective and p-injective rings are developed. Von Neumann regular and associated rings are studied. Pseudo- Frobeniusean and quasi-Frobeniusean rings are also considered.*

Introduction. Standard references like [2], [3] and [10] have motivated many authors to study von Neumann regular rings and associated rings. Some time ago, we introduced p-injective modules to study von Neumann regular rings, V-rings and certain generalizations (cf. for example, [12], [13]). Recall that a left A-module M is p-injective if, for any principal left ideal P of A, any left A- homomorphism of P into M extends to A. Call A a left p-injective ring if ${}_AA$ is p-injective. A theorem of M. IKEDA-T. NAKAYAMA [5] guarantees that A is left p- injective iff every principal right ideal of A is a right annihilator ideal. P-injectivity has also been studied in connection with semi- groups and torsion theories (cf. [4], [8], [11]). In this note, the following property of p-injective rings will play a key role: if A is left p-injective, then any projective left A-module is p-injective.

This will lead to new results on regular rings and quasi-Frobeniusean or pseudo-Frobeniusean rings. Among the results proved are the following: (1) If A is semi-prime, then A is left self-injective with non-zero socle iff A contains an injective maximal left ideal; (2) If A is left p-injective such that any left ideal is either a maximal left annihilator or finitely generated projective, then A is quasi- Frobeniusean; (3) A is quasi-Frobeniusean if every p-injective faithful left A-module is an injective generator of the category of left A-modules. An interesting characterization of simple Artinian rings is the following: A is simple Artinian iff A is a prime ring

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containing an injective maximal left ideal and an injective maximal right ideal.

Throughout, A denotes an associative ring with identity and A- modules are unital. J, Z will stand respectively for the Jacobson radical and the left singular ideal of A. A left (right) ideal of A is called reduced if it contains no non-zero nilpotent element. An ideal of A will always mean a two-sided ideal. Recall that (1) A is ELT (resp. MELT) iff every essential (resp. maximal essential, if it exists) left ideal of A is an ideal; (2) A is left V-ring (resp. p-V-ring) iff every simple left A-module is injective (resp. p-injective). ELT rings are left bounded [1, p. 49]. MELT or ELT rings generalize semi-simple Artinian rings, left duo rings and rings whose left ideals are quasi-injective. Left p-V-rings generalize effectively left V-rings and von Neumann regular rings; (3) A-Mod denotes the category of left unital A-modules; (4) ${}_AG$ is a generator iff for any $N \in A$-Mod, there exists an epimorphism of a direct sum of copies of G onto N; (5) ${}_AC$ is a cogenerator iff for any $M \in A$- Mod, there exists a monomorphism of M into a direct product of copies of C (cf. [7]).

Note that left p-injective rings generalize non-trivially left self- injective rings, von Neumann regular rings and right pseudo-Frobeniusean rings.

LEMMA P. *Let A be a left p-injective ring. Then any projective left A-module is p-injective.*

Proof. Let M be a projective left A-module. Since ${}_AA$ is a generator of A-Mod, there exist D, a direct sum of copies of ${}_AA$, and an epimorphism $g : {}_AD \to {}_AM$. Since D is a direct sum of p- injective left A-modules, then ${}_AD$ is p-injective. Now $D/\ker g \approx M$ wich implies that $\ker g$ is a direct summand of ${}_AD$, whence $D \approx \ker g \oplus (D/\ker g)$. This proves that $D/\ker g$ is a p-injective left A-module which yields ${}_AM$ p-injective.

If A is left p-injective with maximum condition on left annihilators, then A is left perfect which implies that A/J is Artinian. Call a left A-module M Σ-cyclic if M is a direct sum of cyclic left A- modules. Then [1, Theorem 24.20. and Corollary 24.22.] together with the proof of Lemma P yield

PROPOSITION 1. *The following conditions are equivalent:*

(1) *A is quasi-Frobeniusean*;
(2) *Every faithful projective left A-module is injective*;
(3) *Every Σ-cyclic projective left A-module is injective*;
(4) *A is left p-injective such that every p-injective projective left A- module is injective*;
(5) *A is a left p-injective left cogenerator with maximum condition on left annihilators*;
(6) *A is a left p-injective right cogenerator with maximum condition on left annihilators.*

PROPOSITION 2. *If A is right pseudo-Frobeniusean, then every projective left (or right) A-module is p-injective.*

Proof. If A is right pseudo-Frobeniusean, then A is right self-injective and every right ideal of A is a right annihilator. By [5, Theorem 1], A is left p-injective. The proposition then follows from Lemma P.

COROLLARY 2.1. *If A is right pseudo-Frobeniusean, then any finitely generated projective left (or right) ideal of A is generated by an idempotent.*

We now turn to a condition for an idempotent to be central.

PROPOSITION 3. *Let K be a left ideal of A such that there exists no non-zero nilpotent left ideal of A contained in K. If* $K = Ae$, $e = e^2 \in A$, *and K is an ideal of A, then e is central in A.*

Proof. Let $A = K \oplus V$, where $V = Au$, $u = 1 - e$. If $K \cap uA = R \neq 0$, then $R^2 = 0$ which implies that $(AR)^2 = AR^2 = 0$, contradicting our hypothesis on K. Therefore $K \cap uA = 0$ which implies that $uAe \subseteq K \cap uA = 0$, yielding $ae = eae$ for all $a \in A$. Also, $AeAu \subseteq K \cap V = 0$ which implies that $ea = eae$ for all $a \in A$. Thus $ae = ea$ for all $a \in A$ which proves that e is central in A.

COROLLARY 3.1. *If U is a minimal non-nilpotent left ideal of A which is an ideal of A, then* $_AU$, U_A *are injective and U is generated by a central idempotent.*

Proof. By Proposition 3., $U = Ae$, where e is a central idempotent. If $A = M \oplus U$, $M = A(1 - e)$, then M is a maximal left ideal which is an ideal of A and hence a maximal right ideal. By [14, Lemma 1], $_AA/M$ and A/M_A are injective. The corollary then follows.

Since any simple projective left A-module is isomorphic to a minimal left ideal generated by an idempotent, we have

COROLLARY 3.2. *Suppose that every simple left A-module is either injective or projective. If any minimal left ideal of A is an ideal of A, then A is a left V-ring.*

THEOREM 4. *The following conditions are equivalent for a semi-prime ring A:*

(1) *A is left self-injective with non-zero socle;*
(2) *A contains an injective maximal left ideal.*

Proof. (1) implies (2) evidently.

Assume (2). Let M be an injective maximal left ideal of A. Then $A = M \oplus U$, where U is a minimal left ideal. Suppose that ${}_AU$ is not injective. Then there exists a left ideal I and a left A-homomorphism $f : I \to U$ which does not extend to A. Therefore f is non-zero and $f(I) = U$, whence $I/K \approx U$, where $K = \ker f$. Since ${}_AU$ is projective, $I = K \oplus V$, where V is a minimal left ideal of A and ${}_AV \approx {}_AU$. If $MV = 0$, then M must be an ideal of A (otherwise, $V = 0$, a contradiction!). Since A is semi-prime, if $M = Ae$, $e = e^2 \in A$, then e is central in A by Proposition 3. Therefore ${}_AU$ is injective by Corollary 3.1., which contradicts our hypothesis! Thus $MV \neq 0$. Let $v \in V$ such that $Mv \neq 0$. Then $V = Av$ and if $p : A \to AV$ is the map $a \to av$ for all $a \in A$, $i : Av \to I$ the inclusion map, set $h = fip$. Then $h : A \to U$ and if g is the restriction of h to M, $g : M \to U$ such that $g(M) = h(M)$. Now $h(M) = f(Mv) \neq 0$ (otherwise, $Mv \subseteq K \cap V = 0$, a contradiction!). It follows that $M/\ker g \approx U$ and since ${}_AU$ is projective, $M \approx \ker g \oplus (M/\ker g)$ which implies that ${}_AM/\ker g$ is injective. Thus ${}_AU$ is injective which again contradicts our original hypothesis! This proves that ${}_AU$ must be injective, whence $A = M \oplus U$ is a left self-injective ring. We have shown that (2) implies (1).

Following [9], A is called semi-simple iff $J = 0$. Also, A is left non-singular iff $Z = 0$.

COROLLARY 4.1. *The following conditions are equivalent:*

(1) *A is left self-injective regular with non-zero socle;*
(2) *A is a semi-prime ring containing a non-singular injective maximal left ideal of A;*

(3) *A is a left p-V-ring containing an injective maximal left ideal*;
(4) *A is a right p-V-ring containing an injective maximal left ideal*;
(5) *A is a semi-simple ring containing an injective maximal left ideal.*

Proof. Obviously, (1) implies (2) through (4).

Assume (2). Since A contains a non-singular maximal left ideal, then A is left non-singular. Therefore (2) implies (1) by Theorem 4.

If A is either a left or a right p-V-ring, then $J = 0$ by [13, Lemma 1]. Thus either (3) or (4) implies (5).

Finally, (5) implies (1) by Theorem 4.

Since a semi-prime ELT ring is left non-singular, the next corollary then follows from [15, Theorem 1].

COROLLARY 4.2. *If A is semi-prime* ELT *with an injective maximal left ideal, then A is a left and right self-injective regular, left and right V-ring of bounded index.*

Since a prime ring with non-zero socle is left (and right) non-singular, the next corollary follows immediately.

COROLLARY 4.3. *A is primitive left self-injective regular with non- zero socle iff A is a prime ring containing an injective maximal left ideal.*

We may now have some nice characterizations of simple Artinian rings.

COROLLARY 4.4. *The following conditions are equivalent*:

(1) *A is simple Artinian*;
(2) *A is a prime* MELT *ring with an injective maximal left ideal*;
(3) *A is a prime* ELT *ring with an injective maximal left ideal*;
(4) *A is a prime* ELT *ring with an injective maximal right ideal*;
(5) *A is a prime ring containing an injective maximal left ideal and an injective maximal right ideal.*

Proof. Obviously, (1) implies (2) and (4).

Since a MELT fully left idempotent ring is ELT [14, Proposition 2], then (2) implies (3) by Corollary 4.3. Assume (3). Then A is left self-injective regular with non-zero socle by Corollary 4.3., which implies that A is right self-injective [15, Theorem 1]. Thus (3) implies (5). Assume (4). Then

A is ELT right self-injective regular with non-zero socle by Corollary 4.3. which implies that every essential right ideal of A is an ideal. By [15, Theorem 1], A is left self-injective and hence (4) implies (5). (5) implies (1) by [6, Corollary 2.2.] and Corollary 4.3.

Since a finitely generated p-injective left ideal of A is a direct summand of ${}_AA$, the proof of Theorem 4. yields

THEOREM 5. *If A is semi-prime with a finitely generated p-injective maximal left ideal, then A is a left p-injective ring.*

As usual, A is called a left *p.p.* ring if every principal left ideal of A is a projective left A-module. Lemma P and Theorem 5. yield

COROLLARY 5.1. *The following conditions are equivalent*:

(1) *A is regular with non-zero socle*;
(2) *A is a semi-prime left p.p.ring containing a finitely generated p- injective maximal left ideal.*

[16, Lemma 2.] and Theorem 5. lead to

COROLLARY 5.2. *A is strongly regular with non-zero socle iff A contains a reduced finitely generated p-injective maximal left ideal.*

A left ideal I of A is called a maximal left annihilator iff I is a proper left annihilator ideal such that for any left annihilator J containing I, either $J = I$ or $J = A$.

PROPOSITION 6. *Let A be a left p-injective ring whose left ideals are either maximal left annihilators or finitely generated projective. Then A is quasi-Frobeniusean.*

Proof. Suppose that $Z = 0$. Let I be a proper left ideal of A. If I is a maximal left annihilator, since $Z = 0$, then I is not essential. Let N be a non-zero left ideal such that $I \cap N = 0$. Then $K = I \oplus N$ cannot be a maximal left annihilator which implies that ${}_AK$ is finitely generated projective. By Lemma P, I is a finitely generated p-injective left ideal which is therefore a direct summand of ${}_AA$. In case I is finitely generated projective, then I is again a direct summand of ${}_AA$ by Lemma P. This proves that A is semi-simple Artinian in this case. Now suppose that $Z \neq 0$. If $0 \neq z \in Z$, Az cannot be projective (because Z contains no non-zero idempotent) which implies that Az is a maximal left annihilator. For any $u \in A$, $u \notin Az$, $B = Az + Au$ cannot be a maximal left annihilator and hence B is finitely generated p-injective by Lemma P and therefore a direct summand of ${}_AA$.

Thus $B = A$ which proves that Az is a maximal left ideal of A, yielding $Az = Z$ is also a minimal left ideal of A. Since Z contains no non-zero idempotent, then Z is an essential left ideal of A. For any non-zero proper left ideal V, $V \cap Z \neq 0$ which implies $V \cap Z = Z$, whence $V = Z$. Therefore A is left self-injective Artinian which is therefore quasi-Frobeniusean. We see that A is then quasi-Frobeniusean in any case.

We add another result on quasi-Frobeniusean rings.

PROPOSITION 7. *If every p-injective faithful left A-module is an injective generator of A-Mod, then A is quasi-Frobeniusean.*

Proof. Let Q denote the injective hull of ${}_AA$. In as much as Q is a left generator, there exist C, a direct sum of copies of Q, and an epimorphism $g : {}_AC \to {}_AA$. Now $C/\ker g \approx A$ implies that $C \approx \ker g \oplus (C/\ker g)$. Since C is a p-injective faithful left A-module, by hypothesis, ${}_AC$ is injective which implies that ${}_AA$ is injective. Let P be a projective left A-module. Then there exist D, a direct sum of copies of ${}_AA$, and an epimorphism $h : D \to P$. Since ${}_AD$ is p-injective faithful, then ${}_AD$ is injective which implies that ${}_AP$ is injective. By [1, Theorem 24.20.], A is quasi- Frobeniusean.

PROPOSITION 8. *If A is left p-injective such that every complement left ideal of A is finitely generated projective, then A is left continuous (in the sense of UTUMI [10]).*

Proof. By Lemma P, every complement left ideal of A is finitely generated p-injective which is therefore a direct summand of ${}_AA$. Since A is left p-injective, then any left ideal which is isomorphic to a direct summand of ${}_AA$ is itself a direct summand of ${}_AA$. A is therefore left continuous.

We now consider a generalization of quasi-Frobeniusean rings.

PROPOSITION 9. *Let A be a left p-injective left perfect ring. Then any flat left ideal of A is generated by an idempotent.*

Proof. Let I be a flat left ideal of A. Since A is left perfect, then ${}_AI$ is projective [7, p. 294] which implies that ${}_AI$ is p-injective (Lemma P).

Therefore A/I is a flat left A-module [15, p. 277] which implies that ${}_AA/I$ is projective. It follows that I is a direct summand of ${}_AA$.

COROLLARY 9.1. *If A is left self-injective left perfect, then any flat left ideal of A is injective.*

In the next result, condition (6) extends [13, Theorem 7.].

THEOREM 10. *The following conditions are equivalent:*

(1) *A is semi-simple Artinian;*
(2) *A is left p-injective with maximum condition on left annihilators such that every ideal of A is a flat left A-module;*
(3) *A is left p-injective such that every maximal left ideal of A is finitely generated projective;*
(4) *A is semi-prime with maximum condition on left annihilators containing a finitely generated p-injective maximal left ideal;*
(5) *A is semi-prime left hereditary with an injective maximal left ideal;*
(6) *A is left p-injective satisfying the maximum condition on left annihilators such that the left socle of A is a flat left A-module.*

Proof. It is easily seen that (1) implies (2) through (5). (2) implies (6) evidently.

Assume (3). Then every maximal left ideal is finitely generated p-injective (Lemma P) which is therefore a direct summand of ${}_AA$. Therefore (3) implies (1).

(4) implies (6) by Theorem 5.

(5) implies (1) by Theorem 4.

Assume (6). Since A satisfies the descending chain condition on right annihilators and every principal right ideal of A is a right annihilator, then A is left perfect which implies that every flat left A-module is projective [7, p. 294]. Since A is left p- injective, then $Z = J$ and since A satisfies the maximum condition on left annihilators, J is therefore nilpotent. In as much as A/J is semi-simple Artinian (because A is left perfect), then A is semi-primary which implies that A is also right perfect, whence S, the left socle of A, is an essential left ideal of A. Since ${}_AS$ is flat and A is left p-injective left perfect, then ${}_AS$ is a direct summand of ${}_AA$ by Proposition 9. This, together with the fact that ${}_AS$ is essential in ${}_AA$, yields $S = A$. Thus (6) implies (1).

We give two remarks and conclude with a question.

REMARK 1. A is strongly regular iff A is a MELT ring whose simple right modules are flat such that any minimal left ideal (if it exists) is an ideal of A.

REMARK 2. The following conditions are equivalent for a ring A satisfying the maximum condition on left annihilators:

(a) A is quasi Frobeniusean; (b) Every injective left A-module is flat; (c) Every flat left A-module is injective.

QUESTION: Is A von Neumann regular if A is a left semi- hereditary ring such that every simple left A-module is flat or every simple right A-module is flat?

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