MULTIFUNCTIONS AND BLUMBERG SETS: A PROXIMITY APPROACH(*)

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SOMMARIO.- *Recentemente T. Neubrunn ha esteso alle multifunzioni le varie nozioni di insieme di Blumberg. I suoi risultati sono validi nell'ipotesi che le multifunzioni siano a valori chiusi e il codominio sia uno spazio normale. In questo lavoro mostriamo come i risultati di T. Neubrunn possono essere estesi a multifunzioni a valori arbitrari in uno spazio di Tychonoff. Lo stratagemma è l'introdurre opportune prossimità nel codominio.*

SUMMARY.- *Recently T. Neubrunn extended the various notions of Blumberg sets to multifunctions. In his main results the multifunctions are closed- valued and the range space is normal. In this paper it is shown how the results can be generalized to include arbitrary multifunctions and Tychonoff range. The trick is to use proximity on the range.*

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Let X and Y be topological spaces. $S(Y)$ denotes the family of all nonempty subsets of Y and a multifunction on X to Y is merely a function $F : X \to S(Y)$. We first generalize the definitions of upper and lower quasi-continuous multifunctions using proximity in the range and then derive generalizations of the results of T. Neubrunn [2]. We use a prime to distinguish our definitions and we refer to [1] for notations and results concerning with proximity spaces. The range space is assumed to be an R_0-space Y with a compatible $L0$-proximity. As usual, we write $A \ll B$ whenever $A \not\delta (X - B)$.

DEFINITION 1. $F : X \to S(Y)$ is *u'-quasi-continuous at* $x_0 \in X$ iff for

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any open set U containing x_0 and an open set V such that $F(x_0) \ll V$, there exists a nonempty open set $G \subset U$ such that for each $x \in G, F(x) \ll V$. It is said *u'- quasi-continuous*, if it is u'-quasi-continuous at any $x \in X$.

REMARK 1. If $\delta = \delta_0$, i.e. $A\delta B$ iff $\overline{A} \cap \overline{B} \neq \emptyset$, then u'-quasi-continuous is equivalent to u-quasi-continuous for all closed multifunctions. We also note that δ_0 is EF if and only if Y is normal.

DEFINITION 2. $F : X \to S(Y)$ is *l'-quasi-continuous at* $x_0 \in X$ iff for any open set U containing x_0 and an open set V such that $F(x_0)\delta V$, there exists a nonempty open set $G \subset U$ such that for each $x \in G, F(x)\delta V$. We say that F is *l'-quasi-continuous* if it is l'-quasi-continuous at any $x \in X$.

REMARK 2. If in the above definitions $\delta = \delta_0$, F closed-valued and Y regular, then l'-quasi-continuous is equivalent to l-quasi-continuous. Furthermore, if we require $x_0 \in G$, then we get the definitions of u'- continuous and l'-continuous respectively.

DEFINITION 3. $\mathcal{A} \subset S(Y)$ is *u'-dense* in $\mathcal{B} \subset S(Y)$ iff for every $B \in \mathcal{B}$, $B \ll V$, where V is open, there exists an $A \in \mathcal{A}$ such that $A \ll V$. $\mathcal{A}$ is *l'-dense* in $\mathcal{B}$ iff for every $B \in \mathcal{B}$, $B\delta V$ implies the existence of an $A \in \mathcal{A}$ such that $A\delta V$.

THEOREM 1. *F is u'-quasi-continuous if and only if for every dense set $D \subset X$ and open set $U \subset X$, $F(D \cap U)$ is u'-dense in $F(U)$.*

Proof: Suppose F is u'-quasi-continuous, D dense in X and $U \subset X$ is open, $F(x_0) \in F(U)$ and $F(x_0) \ll V$. Since F is u'-quasi-continuous, there exists a nonempty set $G \subset U$ such that for each $x \in G, F(x) \ll V$. Pick $x \in D \cap G$ and since $F(x) \ll V$ it follows that $F(D \cap U)$ is u'-dense in $F(U)$.

To prove the converse, suppose F is not u'-quasi-continuous at x_0. Then there is an open set V such that $F(x_0) \ll V$ and a dense set $D_1 \subset U$ such that $F(x) \not\ll V$ for each $x \in D_1$. Set $D = (X - U) \cup D_1$, then D is dense in X but $F(D \cap U) = F(D_1)$ is not u'-dense in X.

The following result is proved similarly.

THEOREM 2. *A function $F : X \to S(Y)$ is l'-quasi-continuous iff for every dense set $D \subset X$ and an open set $U \subset X$, the set $F(D \cap U)$ is l'-dense in $F(U)$.*

DEFINITION 4. $D \subset X$ is an *u'-Blumberg (l'-Blumberg) set* iff $\overline{D} = X$

and $F_{/D}$ is u'-continuous (l'-continuous). D is said to be a *full u'-Blumberg (full l'-Blumberg) set* iff it is u'-Blumberg (l'-Blumberg) and $F(U \cap D)$ is u'-dense (l'-dense) in $F(U)$ for every open set U in X.

Theorems 1 and 2 can be rewritten as

THEOREM 3. *If $F : X \to S(Y)$ is u'-quasi- continuous (l'-quasi-continuous) and D u'- Blumberg (l'-Blumberg), then D is full u'- Blumberg (full l'-Blumberg).*

Our next result is a generalization of theorem 1 of T. Neubrunn [2].

THEOREM 4. *Suppose $F : X \to S(Y)$, Y Tychonoff with a compatible EF-proximity δ. If there exists a full u'-Blumberg set D for F and $F(U \cap D)$ is l'-dense in $F(U)$ for each open set $U \subset X$, then F is u'-quasi-continuous.*

Proof: Suppose V is open in Y and $F(x_0) \ll V$. Then, since δ is an EF-proximity and by the Strong Axiom (lemma 3.2 and theorem 3.9 of [1], there exists an open set V_1, such that

$$F(x_0) \ll V_1 \subset \overline{V}_1 \ll V.$$

Since $F(U \cap D)$ is u'-dense in $F(U)$, there exists an $x_1 \in U \cap D$ such that

$$F(x_1) \ll V_1.$$

Since $F_{/D}$ is u'-continuous, there is a nonempty open set $G \subset U$ such that

$$(*) \text{ for each } x \in G \cap D, F(x) \ll V_1.$$

Claim: $F(x) \ll V$ for each $x \in G$.

If the claim is not true, there exists $x^* \in G$ such that $F(x^*) \not\ll V$. Hence $F(x^*)\delta(X\text{-}V)$ and consequently,

$$(\#)\ F(x^*)\delta(X - \overline{V}_1).$$

Since $F(G \cap D)$ is l'-dense in $F(G)$, there is an $x' \in G \cap D$ such that $F(x')\delta(X - \overline{V}_1)$. But $(X - \overline{V}_1) \subset X - V_1$ and both

$$F(x')\delta(X-\overline{V}_1) \text{ and } F(x')\delta(X-V_1),$$

which contradicts (*).

REMARK 3. T. Neubrunn's result (Theorem 1 [2]) follows from above by taking F to be closed valued, Y normal and $\delta = \delta_0$.

COROLLARY 1. *Suppose $F : X \to S(Y)$ is a function from a topological space X to $S(Y)$, where Y is Tychonoff. If there is a set $D \subset X$ which is both u'-Blumberg and full l'-Blumberg then F is u'-quasi-continuous.*

COROLLARY 2. *Suppose $F : X \to S(Y)$ is a function, where Y is Tychonoff. If there exists a full u'-Blumberg set for F and F is l'-quasi-continuous, then F is u'- quasi-continuous.*

THEOREM 5. *Suppose $F : X \to S(Y)$ is a function with Y Tychonoff equipped with a compatible EF-proximity δ. If there is a set $D \subset X$ which is a full l'- Blumberg for F and such that $F(D \cap G)$ is u'-dense in $F(G)$ for every open set $G \subset X$, then F is l'-quasi-continuous.*

Proof: Let $x_0 \in X$, U an open set containing x_0 and $V \subset Y$ an open set such that $F(x_0)\delta V$. Since D is a full l'-Blumberg set, there exists an $x_1 \in D \cap U$ such that $F(x_1)\delta V$. From the l'- continuity of $F_{/D}$ at x_1, we have a nonempty open set $G \subset U$ and $F(x)\delta V$ for each $x \in G \cap D$.
Claim: for each $x \in G$, $F(x)\delta V$.

If the claim is not true, there is an $x^* \in X$ such that $F(x^*)\not\delta V$, i.e. $F(x^*) \ll X-V$. Since δ is EF, by the Strong Axiom, there is an open set V_1 such that

$$F(x^*) \ll V_1 \subset \overline{V}_1 \ll X-V.$$

Since $F(G \cap D)$ is u'-dense in $F(G)$, it follows that there is an $x \in D \cap G$ such that

$$F(x)\delta(X-V_1).$$

But for every $x \in G \cap D$, $F(x)\delta V$ which contradicts the above.

COROLLARY 3. *Let $F : X \to S(Y)$ be a function with Y Tychonoff. If F admits a set D which is simultaneously full u'-Blumberg and full l'-Blumberg, then F is l'-quasi-continuous.*

COROLLARY 4. *Suppose $F : X \to S(Y)$ is a function with Y Tychonoff and F has a full l'-Blumberg set. If F is u'-quasi-continuous, then F is l'-quasi- continuous.*

COROLLARY 5. *Suppose $F : X \to S(Y)$ is a function with Y Tychonoff and there is a set $D \subset X$ which is full u'-Blumberg and l'-Blumberg for F. Then F is both u'-quasi-continuous and l'-quasi- continuous.*

REFERENCES

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[2] NEUBRUNN, T.: *On Blumberg sets for multifunctions*, Bull. de l'Acad. Polonaise des Sciences vol. XXX (1982), 109-113.