

S^2 - TYPE MINIMAL SURFACES ENCLOSING MANY OBSTACLES IN \mathbb{R}^3 (*)

by ROBERTA MUSINA (in Trieste)(**)

SOMMARIO.- Sia Ω un sottoinsieme aperto e limitato di \mathbb{R}^3 con frontiera liscia. In questo lavoro si studia il problema dell'esistenza di mappe armoniche definite sulla sfera bidimensionale con immagine nel complementare di Ω . In particolare, nel caso in cui Ω non è connesso, si ottengono condizioni sufficienti ("Douglas criterion") per l'esistenza di sfere minime in $\mathbb{R}^3 \setminus \Omega$ appartenenti a classi di omotopia prescritte.

SUMMARY.- Let Ω be an open, smooth and bounded set in \mathbb{R}^3 . In this paper we deal with harmonic maps from the two-sphere into the complement of Ω . In case Ω is not connected, we state a "Douglas criterion" for the existence of many homotopically distinct minimal spheres in $\mathbb{R}^3 \setminus \Omega$.

In a joint paper with G. Mancini [8] we have discussed the problem of the existence of a closed surface having minimal area among all surfaces which are parametrized by S^2 and which enclose a given connected body Ω in \mathbb{R}^3 . Taking advantage of the fact that the second homotopy group of $\mathbb{R}^3 \setminus \Omega$ is not trivial, we can prove our existence result by showing that there exists a minimum for the energy functional on the class of maps from S^2 into $\mathbb{R}^3 \setminus \Omega$ which are not homotopic to a constant.

The argument above suggests the possibility of finding more solutions if, for example, we enrich the topology of the target space by letting the obstacle to be more complex.

In this paper we drop out the connectivity assumption on Ω , and we study the behaviour of minimizing sequences for the energy functional on the class of closed surfaces which "envelope" every connected component of Ω . The goal is to find a harmonic map from S^2 into $\mathbb{R}^3 \setminus \Omega$ which is not homotopic to a constant in the complement of each connected component of Ω .

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(**) Indirizzo dell'Autore: S.I.S.S.A. - Strada Costiera, 11 - 34014 Trieste (Italy).

We first go back to a minimization problem in \mathbb{R}^2 via composition with the stereographic projection. Using the same tools as in [8] we are able to define a suitable class X^e of maps $\mathbb{R}^2 \rightarrow \mathbb{R}^3 \setminus \Omega$ which “enclose”, in a weak sense, every connected component of the obstruction Ω , and we study the minimization problem

$$(0.1) \quad I := \text{Min}_{U \in X^e} \int_{\mathbb{R}^2} |\nabla U|^2 .$$

In the case of a connected obstacle, the infimum in (0.1) is always achieved, while in the general case the invariance of the Dirichlet integral and the constraint X^e with respect to translations and dilations in \mathbb{R}^2 may produce the phenomena of dichotomy for minimizing sequences (see Proposition 2.2). Consequently, there might exist minimizing sequences which are not compact up to translations and changes of scale.

In order to guarantee the compactness of all minimizing sequences we need a “Douglas criterion”, i.e. we require a family of strict inequalities holds true, the number of these inequalities depending on the number of connected component of Ω (see Theorem 2.3).

Similar hypothesis appear often in problems where the invariance with respect to the non compact group of dilations in \mathbb{R}^N might produce lack of compactness at some energy levels: see for example the collection of problems in [6], [7], [5].

Notations. $D_r(z)$ will denote the open disk of radius r and center z in \mathbb{R}^2 , and we will often write $D_r = D_r(0)$. With $|\cdot|$, \cdot , \wedge we denote respectively the norm, the scalar and the vector product in \mathbb{R}^3 , while $|\cdot|_2$, $|\cdot|_\infty$ are the L^2 , L^∞ norms respectively.

If $(U_n)_n \subseteq L^\infty(\mathbb{R}^2, \mathbb{R}^3)$, $\nabla U_n \in L^2(\mathbb{R}^2, \mathbb{R}^6)$, we will say that U_n converges (weakly) to U , if $U_n \rightarrow U$ a.e. and $\nabla U_n \rightarrow \nabla U$ in L^2 (respectively: weakly in L^2), and we shall write $U_n \rightarrow U$ (resp. $U_n \rightharpoonup U$).

1. The Variational Problem.

Let $(\Omega_i)_{i \in F}$ be a finite collection of disjoint subsets of \mathbb{R}^3 . We suppose that for each index $i \in F$, Ω_i is an open, bounded, connected set, and we define C as the closure of the unbounded connected component of $\mathbb{R}^3 \setminus \overline{\cup \Omega_i}$.

Throughout the paper we assume that

(1.1) *C is a lipschitz neighbourhood retract,*

that is there exists a lipschitz retraction of an open neighbourhood of C into C . We set

$$X(C) := \{U \in L^\infty(\mathbb{R}^2, \mathbb{R}^3) \mid U(z) \in C \text{ for a.e. } z \in \mathbb{R}^2, \nabla U \in L^2(\mathbb{R}^2, \mathbb{R}^6)\}.$$

Notice that $X(C)$ is closed with respect to weak convergence in Notations.

Assumption (1.1) allows us to prove, as in [11], the following density result:

LEMMA 1.1. *For every $U \in X(C)$, there exists a sequence $U_n \in X(C) \cap C^0(\mathbb{R}^2, \mathbb{R}^3)$ s.t.*

- (i) *for each n , U_n is constant far out;*
- (ii) *$U_n \rightarrow U$ a.e., $\nabla U_n \rightarrow \nabla U$ in $L^2(\mathbb{R}^2, \mathbb{R}^6)$.*

In order to give our definition of maps $U \in X(C)$ which enclose each of the Ω_i 's, we will use, as in [8], the Volume Functional

$$V(w) = \int w \cdot w_x \wedge w_y \text{ for } w \in L^\infty(\mathbb{R}^2, \mathbb{R}^3), \nabla w \in L^2(\mathbb{R}^2, \mathbb{R}^6)$$

(see for example [12], [1], [2]). For every $i \in F$, we fix a point $\xi_i \in \Omega_i$, and we define the map

$$p_i: C \rightarrow S^2, \quad p_i \mu = \frac{u - \xi_i}{|u - \xi_i|}.$$

Since p_i is lipschitz continuous on C , it turns out that for every $U \in X(C)$, $V(p_i U)$ is well defined by Hölder inequality.

Moreover, if $U \in X(C)$ is continuous and regular at infinity, that is

$$\text{there exists } \lim_{|z| \rightarrow \infty} U(z) = :U(\infty)$$

then $V(p_i U)/4\pi$ is an integer, and gives the degree of $p_i U \circ \Pi : S^2 \rightarrow S^2$, where Π is the stereographic projection (see [9], [1]). In particular, since Ω_i is connected, $V(p_i U)$ does not depend on the choice of the point ξ_i in Ω_i . In addition, if $V(p_i U) \neq 0$ then $U \circ \Pi$ is not homotopic to a constant as a map $S^2 \rightarrow \mathbb{R}^3 \setminus \Omega_i$.

Thanks to the density Lemma, the set

$$X^c(c) := \{U \in X(C) \mid V(p_i U) \neq 0 \quad \forall i \in F\}$$

can be regarded as the class of admissible functions which "enclose" each obstacle Ω_i . In fact, if $(U_n)_n \subseteq X(C)$, $U_n \rightarrow U$ (see Notations in the Introduction), then

$$V(p_i U_n) \rightarrow V(p_i U) \quad \text{for every } i \in F.$$

REMARK 1.2. Since $V(p_i U)/4\pi \in Z \quad \forall U \in X(C)$ and $\forall i \in F$, Hölder inequality implies:

$$4\pi \leq |V(p_i U)| \leq \frac{1}{2d(\xi_i, \partial\Omega_i)^2} |\nabla U|_2^2 \quad \text{if } V(p_i U) \neq 0.$$

Thus, there exists $\delta^* > 0$ such that

$$|\nabla U|_2^2 \geq \delta^* \quad \text{if } U \in X(C), V(p_i U) \neq 0 \text{ for some } i \in F.$$

The aim of this paper is to study the minimization Problem:

$$(1) \quad \begin{cases} \text{find } U_\infty \in X^c(C) \text{ such that} \\ \int_{\mathbb{R}^2} |\nabla U_\infty|^2 = I^F(C) := \inf_{U \in X^c(C)} \int_{\mathbb{R}^2} |\nabla U|^2. \end{cases}$$

Before ending this Section we notice that a solution to Problem 1 corresponds to a harmonic map $S^2 \rightarrow C$. In fact, it turns out that the map $U_\infty \circ \Pi \in H^1(S^2, C)$ minimizes the energy functional on its homotopy class.

In addition we can prove as in [8], Theorem 2.1, that in case ∂C is regular enough, then U_∞ is a closed, regular surface and it has minimal area in the class $X^c(C)$.

We summarize these results in the following

THEOREM 1.3. *Suppose $U_\infty \in X^c(C)$ solves Problem 1. Then*

- (i) $U_\infty \circ \Pi$ is harmonic as a map $S^2 \rightarrow C$.
- (ii) If ∂C is a class C^2 , then $U_\infty \in C^{1,\alpha}(\mathbb{R}^2, \mathbb{R}^3)$, U_∞ is regular at infinity and conformal on \mathbb{R}^2 :

$$|(U_\infty)_x|^2 - |(U_\infty)_y|^2 = 0 = (U_\infty)_x \cdot (U_\infty)_y.$$

Moreover, U_∞ minimizes the area functional on $X^c(C)$:

$$\int |(U_\infty)_x \wedge (U_\infty)_y| \leq \int |U_x \wedge U_y| \quad \forall U \in X^e(C) .$$

2. The Existence Result.

This Section is divided into two parts: in the first one we describe our “Douglas Criterion”, and in the second one we state and prove the existence result.

The Douglas Criterion.

In this Section we only require that the constraint C satisfies (1.1). If $A \subseteq F$ is any subset of F , we set

$$I^A(C) := \begin{cases} \text{Inf} \left\{ \int |\nabla U|^2 \mid U \in X(C), V(p_i U) \neq 0 \quad \forall i \in A \right\} & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset . \end{cases}$$

Clearly, $I^F(C) \geq I^A(C)$ and $I^A(C) > 0$ if $A \neq \emptyset$ (see Remark 1.2). We shall prove our existence result whenever the following “Douglas Criterion” holds true:

$$(D) \quad I^F(C) \leq \sum_{i=1}^m I^{A_i}(C) \quad \text{for any family of } m \geq 2 \text{ not empty subsets of } F, \text{ s.t. } \bigcup_i A_i = F .$$

Notice that in case of one connected obstacle no restrictions are required, i.e. (D) is automatically verified.

Condition (D) has a simpler form when $|F| \leq 3$, by means of the following

PROPOSITION 2.1. *Suppose $|F| \leq 3$. Then*

$$(2.1) \quad I^{A \cup B} \leq I^A + I^B \quad \forall A, B \subseteq F .$$

From (2.1) it follows that $I^F \leq I^{F \setminus A} + I^A \quad \forall A \subset F$ and thus condition (D) is equivalent to

$$(D') \quad I^F < I^{F \setminus A} + I^A \quad \forall A \subset F, A \neq \emptyset .$$

It is quite likely that 2.1 holds for any number of obstacles; we limit ourselves to the case $|F| \leq 3$ since the proof in the general case seems to be more complicated.

The construction we will use in the proof of Proposition 2.1 shows that the Douglas criterion (D) is a necessary condition for compactness, up to translations and changes of scale in \mathbb{R}^2 , of all minimizing sequences:

PROPOSITION 2.2. *Suppose $|F| \leq 3$, and suppose that there exists $A \subset F$ not empty, s.t.*

$$I^F = I^{F \setminus A} + I^A.$$

Then there exists a minimizing sequence for Problem 1 such that none of those obtained from it by translations and dilations in \mathbb{R}^2 is compact.

Proof of Proposition 2.1. We can suppose that A, B are not empty, and

$$(2.2) \quad I^A(C) < I^{A \cup B}(C), I^B(C) < I^{A \cup B}(C).$$

Let $(U_n)_n, (W_n)_n \subset X(C)$ be minimizing sequences for $I^A(C), I^B(C)$ respectively, that is

$$(2.3) \quad \begin{cases} V(p_i U_n) \neq 0 \quad \forall i \in A, & \int |\nabla U_n|^2 = I^A + o(1); \\ V(p_i W_n) \neq 0 \quad \forall i \in B, & \int |\nabla W_n|^2 = I^B + o(1). \end{cases}$$

Because of the density Lemma, and the invariance of the Dirichlet integral and the Volume functional with respect to dilations in \mathbb{R}^2 , we can choose U_n, W_n to be continuous and constant outside the unit disk. Using the sequences $(U_n)_n, (W_n)_n$, we will construct a sequence $(\Phi_n)_n \subset X(C)$ such that

$$(2.4) \quad \int |\nabla \Phi_n|^2 = I^A(C) + I^B(C) + o(1);$$

$$(2.5) \quad V(p_i \Phi_n) \neq 0 \quad \forall i \in A \cup B$$

and this will conclude the proof of Proposition 2.1.

Hypotheses (2.2) imply:

$$\exists a \in A, \exists b \in B \text{ s.t. } a \neq b \text{ and (for a subsequence)}$$

$$V(p_a W_n) = 0, V(p_b U_n) = 0,$$

otherwise $I^B(C) = I^{A \cup B}(C)$ or $I^A(C) = I^{A \cup B}(C)$. Let us define the sets

$$N := \{i \in A \cup B \mid V(p_i U_n) = V(p_i W_n)\}$$

$$\tilde{N} := \{i \in A \cup B \mid V(p_i U_n) = -V(p_i W_n)\}.$$

Since $|V(p_i U_n)| + |V(p_i W_n)| \neq 0 \forall i \in A \cup B$, then N, \tilde{N} are disjoint (eventually empty), and $a, b \notin N \cup \tilde{N}$. Since $\{a, b\} \cup N \cup \tilde{N} \subset A \cup B$ and $A \cup B$ contains at most three indexes, than either $N = \emptyset$ or $N \neq \emptyset$ and in this case $\tilde{N} = \emptyset$.

Now, suppose $N = \emptyset$, and define

$$\Phi_n(z) := \begin{cases} U_n(n^2 z) & \text{if } |z| \leq 1/n \\ \gamma\left(\frac{\lg n |z|}{2 \lg n}\right) & \text{if } 1/n < |z| < n \\ W_n\left(\frac{n^2 z}{|z|^2}\right) & \text{if } |z| \geq n \end{cases}$$

where $\gamma : [0,1] \rightarrow C$ is a smooth (lipschitz) path joining $U_n(\infty)$ with $W_n(\infty)$. Notice that Φ_n is well defined and continuous on R^2 . Easy computations show that $\Phi_n \in X(C)$ and

$$\int_{\{|z| < 1/n\}} |\nabla \Phi_n|^2 = \int_{\{|z| < n\}} |\nabla U_n|^2 = I^A(C) + o(1);$$

$$\int_{\{|z| > n\}} |\nabla \Phi_n|^2 = \int_{\{|z| < n\}} |\nabla W_n|^2 = I^B(C) + o(1);$$

because U_n, W_n are constant outside the unit disk. Moreover,

$$(2.6) \quad \int_{\{1/n < |z| < n\}} |\nabla \Phi_n|^2 = o(1) \text{ as } n \rightarrow \infty,$$

and (2.4) follows immediately. Using again (2.6), we can prove that (for a subsequence)

$$V(p_i \Phi_n) = V(p_i U_n) - V(p_i W_n) \quad \forall i \in F$$

and thus $(\Phi_n)_n$ verifies (2.5), since by hypotheses $N = \emptyset$.

If $N \neq \emptyset$, and hence $\tilde{N} = \emptyset$, we replace W_n with

$$\tilde{W}_n(x, y) := W_n(y, x)$$

in the definition of $(\Phi_n)_n$. Since

$$\int |\nabla \tilde{W}_n|^2 = \int |\nabla W_n|^2, \quad V(p_i \tilde{W}_n) = -V(p_i W_n) \quad \forall i \in F,$$

and $\tilde{N} = \emptyset$, we can prove as before that the new sequence $(\Phi_n)_n$ satisfies (2.4), (2.5), and the proof is complete.

Proof of Proposition 2.2. Let us go back to the proof of Proposition 2.1, with $B = F \setminus A$; if equality holds in (2.1) for some not empty subset $A \subset F$, then the sequence $(\Phi_n)_n$ is minimizing for the minimum problem $I^F(C)$. Suppose by contradiction that there exist sequences $(z_n)_n \subset \mathbb{R}^2$, $(t_n)_n \subset]0, +\infty[$ s.t.

$$(2.7) \quad \tilde{\Phi}_n \rightarrow \Phi, \quad \nabla \tilde{\Phi}_n \rightarrow \nabla \Phi \text{ in } L^2(\mathbb{R}^2, \mathbb{R}^6)$$

for some $\Phi \in X(C)$, where

$$\tilde{\Phi}_n(z) = \Phi_n \left(\frac{z - z_n}{t_n} \right).$$

From (2.7), and from the continuity properties of the Volume Functional it follows in particular that $\Phi \in X^e(C)$, and hence

$$\int_{\mathbb{R}^2} |\nabla \Phi|^2 = I^A(C) + I^{F \setminus A}(C) = I^F(C).$$

Taking into account the definition of the sequence $(\Phi_n)_n$ we easily find

$$\int_{\{|z - z_n| < t_n/n\}} |\nabla \tilde{\Phi}_n|^2 = I^A + o(1), \quad \int_{\{|z - z_n| > t_n/n\}} |\nabla \tilde{\Phi}_n|^2 = I^B + o(1).$$

We can immediately exclude the case $(z_n)_n$ bounded, since in this case (2.7) would imply $t_n/n \geq \varepsilon > 0$, $t_n n \leq \text{const.} < \infty$, which is impossible. On the other hand, by simple computations we see that in case $|z_n| \rightarrow \infty$, then both the sequences $|z_n| - t_n/n$, $|z_n| - t_n n$ must be bounded, and this is again an absurd.

The Existence Theorem.

Proposition 2.2 shows that the Douglas criterion (D) is a necessary condition for compactness of all minimizing sequences. Actually, (D) it is also sufficient to prevent the phenomena of dichotomy:

THEOREM 2.3 $I^F(C)$ is achieved in $X^e(C)$, provided (D) holds.

REMARK 2.4. In case of one connected obstacle: $|F| = 1$, no conditions are required. Thus, Theorem 2.3 includes the existence Theorem in [8].

The proof of Theorem 2.3 is based on the analysis, via a blow-up technique, of sequences $(U_n)_n \subset X^e(C)$ whose weak limit (see Notations) does not belong to $X^e(C)$. This technique was introduced in this framework by Sacks and Uhlenbeck [10].

To perform the blow-up technique, we will use a Lemma in [4] (see also [13]), which applies to sequences $(U_n)_n \subset X(C)$, s.t. $(U_n)_n$ is bounded in L^∞ and $(|\nabla U_n|)_n$ is bounded in L^2 . For such a sequence, we can find a subsequence $(U_n)_n$, a map $U \in X(C)$, and for any index j we can find a finite set of distinct points $d_1^j, \dots, d_{k_j}^j$ in R^2 , and a finite set of integers $d_1^j, \dots, d_{k_j}^j$ such that

$$(2.8) \quad U_n \rightarrow U \text{ a.e.}, \nabla U_n \rightharpoonup \nabla U \text{ weakly in } L^2,$$

$$(2.9) \quad \det(p_j U_n, \nabla(p_j U_n)) \rightarrow \det(p_j U, \nabla(p_j U)) + 4\pi \sum_{h=1}^{k_j} d_h^j \delta_{a_h^j} \quad \forall j \in F,$$

weakly in the sense of measures.

If a is any point in R^2 , we can consider the set of indexes j such that $p_j U_n$ "concentrates" at a :

$$A_a = \{j \in F \mid d_{h_j}^j = a, d_{h_j}^j \neq 0 \text{ for some } h_j\}.$$

Notice that if $A_a \neq \emptyset$, then $\forall j \in A_a$ there exists a unique h_j s.t. $d_{h_j}^j = a$, $d_{h_j}^j \neq 0$.

Similarly, $A_\infty \subset F$ will be the set of indexes j such that $p_j U_n$ "concentrates" at infinity:

$$A_\infty = \{j \in F \mid c_j \neq 0\}, \text{ where}$$

$$c_j := \frac{1}{4\pi} \lim [V(p_j U_n) - V(p_j U)] - \sum_{h=1}^{k_j} d_h^j \in \mathbb{Z}.$$

The sets A_a, A_∞ could possibly be empty.

The following Proposition is a crucial step in the proof of Theorem 2.3:

PROPOSITION 2.5. *Let $(U_n)_n \subset X(C)$ be as above. Then*

- (i) $\liminf_n \int_{D_\rho(a)} |\nabla U_n|^2 \geq \int_{D_\rho(a)} |\nabla U|^2 + I^A_a(C) \quad \forall a \in \mathbb{R}^2, \forall \rho \text{ small};$
- (ii) $\liminf_n \int_{\mathbb{R}^2 \setminus D_R} |\nabla U_n|^2 \geq \int_{\mathbb{R}^2 \setminus D_R} |\nabla U|^2 + I^A_\infty(C) \quad \forall R \text{ large}.$

Proof. We follow here the outline of the proofs of Propositions 2.3, 2.4 in [8]. Remark that in case $A_a = \emptyset$, or $A_\infty = \emptyset$ the estimates in Proposition follow directly from the weak lower semicontinuity of the Dirichlet integral.

(i). Suppose $A_a \neq \emptyset$ for some $a \in \mathbb{R}^2$. Fix $\rho > 0$ s.t. $D_{2\rho}(a)$ does not contain any concentration point different from a . Passing eventually to a subsequence, we can suppose $\int_{D_\rho(a)} |\nabla U_n|^2$ has a limit as $n \rightarrow \infty$. For almost every $r < \rho$, we can find a subsequence $(U_{n_m})_m$ for which the sequence of traces on $\partial D_r(a)$ is bounded in $H^1(\partial D_r(a), \mathbb{R}^3)$. Thus, by using Proposition A.1. in [8] we can prove that (for a subsequence)

$$\exists h_m \in H^1(D_r(a), C), h_m = U_{n_m} \text{ on } \partial D_r(a), \text{ such that}$$

$$(2.10) \quad \lim_m \int_{D_r(a)} |\nabla h_m|^2 \leq \int_{D_r(a)} |\nabla U|^2.$$

Next, we define the map

$$\hat{U}_m(z) := \begin{cases} U_{n_m}(z) & \text{if } |z-a| < r \\ h_m \left(\frac{z-a}{|z-a|^2} r^2 + a \right) & \text{if } |z-a| \geq r. \end{cases}$$

By a simple computation, it turns out that the map \hat{U}_m belongs to $X(C)$, and using (2.10) we get that for every $j \in A_a$ and k large enough

$$\begin{aligned} |V(p_j \hat{U}_m)| &\geq \left| \int_{D_r(a)} \det(p_j U_{n_m}, \nabla(p_j U_{n_m})) \right| - \left| \int_{D_r(a)} \det(p_j h_m, \nabla(p_j h_m)) \right| \\ &\geq |4\pi d_{h_j}^j| - \text{const.} \int_{D_r(a)} |\nabla U|^2 + o(1) \geq 4\pi - \varepsilon \geq 0 \end{aligned}$$

where h_j is the unique index s.t. $a_{h_j}^j \neq 0$, $a_{h_j}^j = a$ (see the definition of the set A_a). Again from (2.10), and from the definition of $I^A_a(C)$, we infer that

$$I^A_a(C) \leq \int_{R^2} |\nabla \hat{U}_m|^2 = \int_{D_r(a)} |\nabla U_{n_m}|^2 + \int_{D_r(a)} |\nabla h_m|^2.$$

Thus, we have proved that for a.e. $r < \rho$

$$\begin{aligned} \lim \int_{D_\rho(a)} |\nabla U_n|^2 &= \lim \left[\int_{D_\rho \setminus D_r} |\nabla U_{n_m}|^2 + \int_{D_r} |\nabla U_{n_m}|^2 \right] \\ &\geq \int_{D_\rho(a)} |\nabla U|^2 + I^A_a(C) - 2 \int_{D_r(a)} |\nabla U|^2 \end{aligned}$$

and (i) follows by letting r go to zero.

(ii). We can apply the same argument to the sequence $U_n^*(z) := U_n(z/|z|^2)$. Notice that $U_n^* \rightarrow U^*$, where $U^*(z) = U(z/|z|^2)$. Using again the Lemma in [4] we get:

$$(2.11) \quad \det(p_j U_n^*, \nabla(p_j U_n^*)) \rightarrow \det(p_j U^*, \nabla(p_j U^*)) - 4\pi \sum_{\substack{h=1 \\ a_h^j \neq 0}}^{k_j} a_h^j \delta b_h^j - 4\pi c_j \delta_0$$

where $b_h^j = a_h^j / |a_h^j|^2$ if $a_h^j \neq 0$, and c_j are the integers defined above. In particular $c_j \neq 0$ iff $j \in A_\infty$, that is, using notations above, $A_\infty = A_0^*$. Thus, from the first part of the Proposition we infer

$$\liminf_n \int_{D_\rho} |\nabla U_n^*|^2 \geq \int_{D_\rho} |\nabla U^*|^2 + I^{A_\infty}(C) \quad \forall \rho \text{ small,}$$

which immediately implies (ii), with $R = 1/\rho$.

Applying Proposition 2.5 to a minimizing sequence $(U_n)_n$ we get the following

PROPOSITION 2.6. *Suppose that the Douglas criterion (D) is satisfied. Let $(U_n)_n$ be a minimizing sequence for Problem 1, and suppose $U_n \rightarrow U$ for some $U \in X(C)$. If $U \notin X^e(C)$ then U is constant and*

either $\exists! a \in \mathbb{R}^2$ s.t. $\liminf \int_{D_\rho(a)} |\nabla U_n|^2 = I^F(C) \quad \forall \rho > 0$,
 or $\nabla U_n \rightarrow 0$ in $L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^6)$.

Proof. By hypothesis, the set

$$B := \{j \in F \mid V(p_j U) = 0\}$$

is not empty. Comparing (2.9), (2.11), we get the existence of a finite (maybe empty) set of points a_1, \dots, a_m in \mathbb{R}^2 such that

$$A_\infty \cup \bigcup_{i=1}^m A_{a_i} = B.$$

If r is small enough, from Proposition 2.5 we infer

$$\begin{aligned} I^F(C) &= \lim \left[\int_{D_{1/r} \setminus \bigcup_i D_r(a_i)} |\nabla U_n|^2 + \int_{\mathbb{R}^2 \setminus D_{1/r}} |\nabla U_n|^2 + \sum_{i=1}^m \int_{D_r(a_i)} |\nabla U_n|^2 \right] \geq \\ &\geq \int_{\mathbb{R}^2} |\nabla U|^2 + I^{A_\infty}(C) + \sum_{i=1}^m I^{A_{a_i}}(C) \end{aligned}$$

that is, using the definition of $I^{F \setminus B}(C)$,

$$\begin{aligned} (2.12) \quad I^F(C) &\geq \int_{\mathbb{R}^2} |\nabla U|^2 + I^{A_\infty}(C) + \sum_{i=1}^m I^{A_{a_i}}(C) \geq \\ &\geq I^{F \setminus B}(C) + I^{A_\infty}(C) + \sum_{i=1}^m I^{A_{a_i}}(C). \end{aligned}$$

From (2.12) and hypothesis (D) it follows that at most one of the sets $F \setminus B$, A_∞ , A_{a_i} is not empty. Since $B \neq \emptyset$ we first get

$$F = B \text{ and } U = \text{constant.}$$

Moreover, either there exists a unique $a \in \mathbb{R}^2$ s.t. $A_a \neq \emptyset$, and in this case $A_a = F$, or $A_\infty = F$. In view of Proposition 2.5 this concludes the proof of Proposition 2.6.

Proof of Theorem 2.3. Let (U_n) be a minimizing sequence for Problem 1. Since projections on convex sets reduce the Dirichlet integral and since ∂C is bounded, we can suppose that $(U_n)_n$ is bounded in L^∞ . Hence, passing eventually to a subsequence, we may assume $(U_n)_n$ verifies (2.8), (2.9) and (2.11) for some $U \in X(C)$. If $U \in X^e(C)$ then

$$I^F(C) \leq \int |\nabla U|^2 \leq \liminf \int |\nabla U_n|^2 = I^F(C)$$

and we are done.

In case $U \notin X^e(C)$, we can use the “concentration function” (see for example [6], [7], [3]):

$$Q_n(t) := \sup_{z \in \mathbb{R}^2} \int_{D_t(z)} |\nabla U_n|^2$$

in order to find a sequence $(t_n)_n$ of positive real numbers, and a sequence $(z_n)_n$ of points in \mathbb{R}^2 , such that

$$\delta \leq \int_{D_{t_n}(z_n)} |\nabla U_n|^2 \leq Q_n(t_n) \leq I^F(C) - \delta$$

where $\delta \in [0, I^F/2]$ is a fixed number. We claim that the minimizing sequence

$$\tilde{U}_n(z) := U_n(t_n z + z_n)$$

converges (up to subsequences) to a solution of Problem 1. Notice that $\tilde{U}_n \in X^e(C)$, $\int |\nabla \tilde{U}_n|^2 = \int |\nabla U_n|^2 = I^F(C) + o(1)$, and moreover $(\tilde{U}_n)_n$ is bounded in L^∞ . Thus, we can assume

$$\tilde{U}_n \rightarrow U_\infty \text{ a.e.}, \nabla \tilde{U}_n \rightarrow \nabla U_\infty \text{ weakly in } L^2$$

for some $U_\infty \in X(C)$. If $U_\infty \notin X^e(C)$, arguments above show that U_∞ is constant, and $(\tilde{U}_n)_n$ satisfies one of the possibilities in Proposition 2.6. But since for every large R , for every small r , and for every $a \in \mathbb{R}^2$,

$$\int_{D_R} |\nabla \tilde{U}_n|^2 \geq \int_{D_{t_n R}(z_n)} |\nabla U_n|^2 \geq \delta > 0, \text{ and}$$

$$\int_{D_r(a)} |\nabla \tilde{U}_n|^2 \leq \int_{D_{t_n r}(t_n a + z_n)} |\nabla U_n|^2 \leq Q_n(t_n) \leq I^F(C) - \delta,$$

we see that those two possibilities cannot occur. Thus, we have proved that $U_\infty \in X^e(C)$, and the conclusion follows from weak semicontinuity of the Dirichlet integral.

REMARK 2.7. The method above, and in particular Proposition 2.5, allows us to prove a multiplicity result, similar to Theorem 3.5 in [8], for the Plateau problem for disk-type minimal surfaces with many obstacles.

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