$\mathcal{A}$-LOCALLY COMPACT SPACES(*)

by ALESSANDRO FEDELI (in L'Aquila)(**)

SOMMARIO.- *In un precedente lavoro si è usato un operatore di chiusura introdotto da Salbany, chiamato $\mathcal{A}$-chiusura, per introdurre il concetto di compattezza rispetto ad una classe $\mathcal{A}$ di spazi topologici (in breve $\mathcal{A}$-compattezza), e si è mostrato il ruolo dominante che gli spazi $\mathcal{A}$-compatti hanno nella classe $\mathcal{A}$. In questo lavoro si studiano gli spazi $\mathcal{A}$-localmente compatti, cioè gli spazi $(X, \tau) \in \mathcal{A}$ tali che la topologia $\tau_{\mathcal{A}}$ in X generata dalla $\mathcal{A}$-chiusura è una topologia localmente compatta e di Hausdorff. Tale approccio ci permette di provare, per molte classi $\mathcal{A}$ di spazi topologici, un analogo di un ben noto teorema di Whitehead sulle applicazioni quoziente.*

SUMMARY.- *In this paper we use a closure operator introduced by Salbany, called $\mathcal{A}$-closure, to introduce the $\mathcal{A}$-locally compact spaces, i.e. the spaces $(X, \tau) \in \mathcal{A}$ such that the topology $\tau_{\mathcal{A}}$ in X generated by the $\mathcal{A}$- closure is a locally compact Hausdorff topology. This approach allow us to prove, for many classes $\mathcal{A}$ of topology spaces, an analogous of a well known Whitehead theorem about quotient mappings.*

0. Introduction.

For each class $\mathcal{A}$ of topological spaces we have a closure operator $[\ ]_{\mathcal{A}} : P(X) \to P(X)$, called $\mathcal{A}$-closure, where X is a topological space and $P(X)$ is the power set of X, [15]. In the last years Dikranjan and Giuli ([2], [3]) characterized the closure operator $[\ ]_{\mathcal{A}}$ for many classes $\mathcal{A}$ of topological spaces.

If $\mathcal{A}$ is the class of Hausdorff spaces TOP_2 and $X \in TOP_2$ then $[\ ]_{\mathcal{A}} : P(X) \to P(X)$ coincides with the ordinary closure operator, [2].

In a previous paper we studied the $\mathcal{A}$-compact spaces, i.e. the spaces $(X, \tau) \in \mathcal{A}$ such that the topology $\tau_{\mathcal{A}}$ in X generated by the $\mathcal{A}$-closure is a compact topology [8], in this one we study (section 1) the $\mathcal{A}$-locally compact spaces, i.e. the spaces $(X, \tau) \in \mathcal{A}$ such that the topology $\tau_{\mathcal{A}}$ is a locally compact Hausdorff topology. In this section some relations between $\mathcal{A}$-locally compact, $\mathcal{A}$-compact and locally compact spaces are given. In section 2 we introduce the $k(\mathcal{A})$-spaces using the concept of $q(\mathcal{A})$ mapping defined in section 1, we establish some properties of

(*) Pervenuto in Redazione il 20 gennaio 1988.

(**) Indirizzo dell'Autore: Via Assergi 4, 67100 L'Aquila (Italy).

$k(\mathcal{A})$-spaces and prove that in general the class of $\mathcal{A}$-locally compact spaces is strictly smaller than the class of $k(\mathcal{A})$-spaces. We prove also an analogous of the following Michael theorem: if the cartesian product $X \times Y$, where X is a T_3-space, is a k-space for every κ-space Y, then the space X is locally compact.

NOTATION 0.1. The following categories are denoted as follows:

TOP	the category of topological spaces and continuous functions
TOPi	the category of topological spaces satisfying the T_i axiom $i = 0, 1, 2$.
URY	the category of Urysohn spaces (points are separated by disjoint closed neighborhoods)
TOP3	the category of regular Hausdorff spaces
Tych	the category of completely regular Hausdorff spaces
0-*dim*	the category of zero-dimensional spaces (i.e. Hausdorff spaces with a base of clopen sets).

Unless explicitly stated the topological terminology is that of [18].

A full and isomorphism-closed subcategory $\mathcal{A}$ of *TOP* is said to be epireflective if for each topological space X there exist $rX \in \mathcal{A}$ and an epimorphism $r_{\mathcal{A}}: X \to rX$ in *TOP* such that for each continuous function $f : X \to Y$, $Y \in \mathcal{A}$, there exists a (unique) continuous function $f' : rX \to Y$ such that $f' \circ r_{\mathcal{A}} = f$.

$\mathcal{A}$ is epireflective in *TOP* iff it is closed under the formation of products and subspaces [10]. Each class $\mathcal{B}$ of topological spaces has an epireflective hull $E(\mathcal{B})$ (i.e. there exists a smallest epireflective subcategory containing $\mathcal{B}$).

All subcategories listed in 0.1 are epireflective subcategories of *TOP*.

We define, now, a closure operator introduced by Salbany [15], and studied by Dikranjan and Giuli in [2], [3].

DEFINITION 0.2. Let $\mathcal{A}$ be a (non empty) class of topological spaces, let X be a topological space and F a subset of X.

A point x of X is said to be a point of $\mathcal{A}$-closure of F in X if for each $f, g : X \to A \in \mathcal{A}$, such that $f|F = g|F$ (where $f|F$ denotes the restriction of f to F), $f(x) = g(x)$.

The set of all points of $\mathcal{A}$-closure of F in X is said to be the $\mathcal{A}$-closure of F in X and it is denoted by $[F]^X_{\mathcal{A}}$.

For every $X \in TOP$ and $M \subset X$ and every $\mathcal{A} \subset TOP$ $[M]^X_{\mathcal{A}} = [M]^X_{E(\mathcal{A})}$ holds (prop. 1.4, [2]) hence in the sequel we consider exclusively epireflective subcategories of TOP.

DEFINITION 0.3. Let $\mathcal{A} \subset TOP, X \in TOP$ and $F \subset X$:

(a) F is said to be $\mathcal{A}$-closed in X if $[F]^X_{\mathcal{A}} = F$

(b) F is said to be $\mathcal{A}$-dense in X if $[F]^X_{\mathcal{A}} = X$

(c) F is said to be $\mathcal{A}$-open in X if X-F is $\mathcal{A}$-closed in X

(d) A function $f : X \to Y$, $X, Y \in \mathcal{A}$, is said to be $\mathcal{A}$-continuous if $f([F]^X_{\mathcal{A}}) \subset [f(F)]^Y_{\mathcal{A}}, F \subset X$.

Every continuous function $f : X \to Y, X, Y \in \mathcal{A}$, is $\mathcal{A}$-continuous (prop. 1.2 (x), [3]).

(e) A function $f : X \to Y, X, Y \in \mathcal{A}$, is said to be $\mathcal{A}$-closed if for every $\mathcal{A}$-closed set $F \subset X$ the image $f(X)$ is $\mathcal{A}$-closed in Y.

(f) The coarsest topology in X which contains all $\mathcal{A}$-closed subsets as closed sets is said to be the $\mathcal{A}$-closure topology of X and, if τ is the topology of X, it is denoted by $\tau_{\mathcal{A}}$

$F_{\mathcal{A}}: TOP \to TOP$ will denote the functor which assigns to $(X, \tau) \in TOP$ the space $(X, \tau_{\mathcal{A}})$. For each continuous map $f : (X, \tau) \to (Y, \sigma)$ in TOP the continuity of $f = F_{\mathcal{A}}(f) : (X, \tau_{\mathcal{A}}) \to (Y, \sigma_{\mathcal{A}})$ follows from 1.2 (x) of [2].

The functor $F_{\mathcal{A}}$ is said to be finitely multiplicative if it preserves finite products, i.e. $(\prod_I \tau_i)_{\mathcal{A}} = \prod_I (\tau_i)_{\mathcal{A}}, I = 1, ..., k$.

The $\mathcal{A}$-closure is not in general a Kuratowski operator (remark 1.3 (a) [2]). If the $\mathcal{A}$-closure is a Kuratowski operator then is easy to see that a function $f : (X, \tau) \to (Y, \sigma)$ is $\mathcal{A}$-continuous ($\mathcal{A}$- closed) iff $f = F(f) : (X, \tau_{\mathcal{A}}) \to (Y, \sigma_{\mathcal{A}})$ is continuous (closed). In this paper we consider only the $\mathcal{A}$-closure that are Kuratowski operators.

The $\mathcal{A}$-closure is said to be hereditary [6], if for $M \subset Y \subset X$ we have $[M]^Y_{\mathcal{A}} = [M]^X_{\mathcal{A}} \cap Y$ for all M, Y and X. For all categories $\mathcal{A}$ listed in 0.1, except $\mathcal{A} = Ury$, and for every $(X, \tau) \in \mathcal{A}$ the $\mathcal{A}$-closure is hereditary in (X, τ).

In that case $i = F_{\mathcal{A}}(i) : (Y, \sigma_{\mathcal{A}}) \to (X, \tau_{\mathcal{A}})$ is an embedding, where τ is the topology of X, σ the relative topology of Y, and i is the embedding map of the subspace (Y, σ) in the space (X, τ).

The following results can be found in [2], [3].

1) $\tau_{\mathcal{A}} \leq \tau$ for all $(X, \tau) \in \mathcal{A}$ iff $\mathcal{A} \subset TOP_2$.

2) For $\mathcal{A} = TOP_2, TOP_3, Tych$, 0- dim, $\tau_{\mathcal{A}} = \tau$ for each $(X, \tau) \in \mathcal{A}$.
3) The TOP_0-closure is the front-closure defined in [14]: $FrCL(A) = \{x \in X$: for each open $n$hood $\mathcal{U}$ of x, $\overline{\{x\}} \cap \mathcal{U} \cap A \neq 0\}$.
4) The TOP_1-closure is the identity for all T_1-spaces.
5) For $\mathcal{A} = Ury$ let $X \in \mathcal{A}$ and $M \subset X$, we define $cl_\theta(M) = \{x \in X$: for each $n$hood V of x, $\overline{V} \cap M \neq 0\}$, this is the θ-closure introduced by Velichko [16]. For $X \in Ury$ and $M \subset X$ we have $cl_\theta M \subset [M]^X_{Ury}$ and $M = cl_\theta(M)$ iff $M = [M]^X_{Ury}$, thus the Ury- closure is the idempotent hull of cl_θ.

I am much indebted to Professors D. Dikranjan and E. Giuli for many useful suggestions.

1. $\mathcal{A}$ - Locally Compact spaces.

DEFINITION 1.1. Let $\mathcal{A}$ be an epireflective subcategory of TOP. $(X, \tau) \in \mathcal{A}$ is said to be $\mathcal{A}$-locally compact iff $(X, \tau_{\mathcal{A}})$ is a locally compact Hausdorff space.

DEFINITION 1.2. [8]. Let $\mathcal{A}$ be an epireflective subcategory of TOP. $(X, \tau) \in \mathcal{A}$ is said to be $\mathcal{A}$-compact iff $(X, \tau_{\mathcal{A}})$ is compact.

The following classes are denoted as follows:

$\mathcal{A}Comp$	the class of compact spaces X such that $X \in \mathcal{A}$
$\mathcal{A}LocComp$	the class of locally compact spaces X such that $X \in \mathcal{A}$
$K_{\mathcal{A}}$	the class of $\mathcal{A}$-compact spaces
$LK_{\mathcal{A}}$	the class of $\mathcal{A}$-locally compact spaces.

Let $LM\text{-}T_2$ be the category of Lawson-Madison spaces (a topological space X is $LM\text{-}T_2$ iff every compact subspace of X is T_2, [11], [12]). If (X, τ) is $\mathcal{A}$- compact and $(X, \tau_{\mathcal{A}}) \in LM\text{-}T_2$ then (X, τ) is $\mathcal{A}$-locally compact.

EXAMPLES 1.3.

(a) For each $(X, \tau) \in TOP_0$, (X, τ_{TOP_0}) is a T_2-space [3] hence $K_{TOP_0} \subset LK_{TOP_0}$.

If (X, τ) is a T_D-space (every point is the intersection of a closed and an open set) then (X, τ_{TOP_0}) is a discrete space [3], hence (X, τ) is a TOP_0-locally compact space.

If X is an infinite T_D-space then it is not TOP_0-compact.

(b) For each $(X, \tau) \in TOP_1$, (X, τ_{TOP_1}) is discrete hence $LK_{TOP_1} = TOP_1$

(c) Let X_j denote a T_1-space with cofinite topology and infinite cardinality j. If $A = Haus(\{X_j\}) = \{X \in TOP$ such that every continuous map $f : X_j \to X$ is constant$\}$ [11], then τ_A is discrete for every $(X, \tau) \in \mathcal{A}$ ([5], prop. 1.11), hence $LK_{Haus(\{X_j\})} = Haus(\{X_j\})$.

(d) Let $\mathcal{A} = LM\text{-}T_2$, if $(X, \tau) \in K_{\mathcal{A}}$ then $(X, \tau_{\mathcal{A}}) \in LM\text{-}T_2Comp = TOP_2Comp$ hence $(X, \tau) \in LK_{\mathcal{A}}$. Now let (X, τ) be an uncountable space with the cocountable topology (i.e. a proper subset is closed if and only if it is countable), (X, τ) is a $LM\text{-}T_2$ space (since every compact subset is finite [11]). By 1.11 in [5] it follows that $(X, \tau_{\mathcal{A}})$ is an uncountable discrete space, hence $(X, \tau) \in LK_{\mathcal{A}}$ but it is not $\mathcal{A}$-compact, therefore $K_{\mathcal{A}} \underset{\neq}{\subset} LK_{\mathcal{A}}$.

(e) $LK_{\mathcal{A}} = \mathcal{A}LocComp$ for $\mathcal{A} = TOP_2, TOP_3, Tych$, $0\text{-}dim$.

We will denote by FT_2 the class of functionally Hausdorff spaces (points are separated by continuous real valued maps).

PROPOSITION 1.4. If $(X, \tau) \in LK_{\mathcal{A}}$ and $\tau_{\mathcal{A}} \leq \tau$, then whenever F is an $\mathcal{A}$-closed set in (X, τ) and $x \notin F$ there is a continuous function $g : (X, \tau) \to [0,1]$ such that $g(x) = 0$ and $g(F) \subset \{1\}$.
In particular $\mathcal{A} \subset TOP_2$ yields $LK_{\mathcal{A}} \subset FT_2$.
Conversely if $FT_2 \subset \mathcal{A} \subset TOP_2$ then $K_{\mathcal{A}}$ and FT_2 yield $LK_{\mathcal{A}}$.

Proof. Let F an $\mathcal{A}$-closed set in (X, τ) and $x \notin F$, then F is a closed set in $(X, \tau_{\mathcal{A}}) \in TOP_2LocComp \subset Tych$, hence there exists a continuous function $f : (X, \tau_{\mathcal{A}}) \to [0, 1]$ such that $f(x) = 0$ and $f(F) \subset \{1\}$. If $i : (X, \tau) \to (X, \tau_{\mathcal{A}})$ is the identity then $g = foi$ is a continuous function from (X, τ) in $[0, 1]$ such that $g(x) = 0$ and $g(F) \subset \{1\}$. Conversely if $(X, \tau_{\mathcal{A}})$ is compact and $(X, \tau) \in FT_2$ then $\tau \geq \tau_{\mathcal{A}} \geq \tau_{FT_2}$ by virtue of $FT_2 \subset \mathcal{A} \subset TOP_2$, now $(X, \tau_{FT_2}) \in TOP_2$ (cf. [4]), so $(X, \tau_{\mathcal{A}})$ is compact Hausdorff, therefore $(X, \tau) \in LK_{\mathcal{A}}$.

We recall that a topological space $(X, \tau) \in \mathcal{A}$ is called $\mathcal{A}$-minimal if $\tau' \leq \tau$ and $(X, \tau') \in \mathcal{A}$ imply $\tau' = \tau$.

THEOREM 1.5. If $Tych \subset \mathcal{A} \subset TOP_2$ then for every $\mathcal{A}$-minimal space (X, τ) the following conditions are equivalent:

a) (X, τ) is a locally compact Hausdorff space
b) (X, τ) is $\mathcal{A}$-locally compact

c) (X, τ) is a compact Hausdorff space
d) (X, τ) is $\mathcal{A}$-compact.

Proof. By the above proposition b) (and obviously all the other conditions) imply $(X, \tau) \in FT_2$. Hence $(X, \tau_{Tych}) \in Tych \subset \mathcal{A}$ and $\tau \geq \tau_{\mathcal{A}} \geq \tau_{Tych}$ yield the coincidence of all three topologies.

Thus a) $\Leftrightarrow$ b) $\Leftarrow$ c) $\Leftrightarrow$ d). Finally the compactness of (X, τ) follows by the well known fact that every *Tych*-minimal space is compact (cf. [1]).

THEOREM 1.6. Let $\mathcal{A}$ be such that for every $(X, \tau) \in \mathcal{A}$ the $\mathcal{A}$-closure is hereditary in (X, τ) then:

1) In an $\mathcal{A}$-locally compact space (X, τ) the intersection of an $\mathcal{A}$-closed with an $\mathcal{A}$-open set is $\mathcal{A}$-locally compact.
2) An $\mathcal{A}$-locally compact subset B of a space $(X, \tau) \in \mathcal{A}$ such that $(X, \tau_{\mathcal{A}}) \in TOP_2$ is the intersection of an $\mathcal{A}$-open set and an $\mathcal{A}$-closed set.

Moreover the following conditions are equivalent:

a) An $\mathcal{A}$-dense subset D of an $\mathcal{A}$-compact space (X, τ) such that $(X, \tau_{\mathcal{A}}) \in LM\text{-}T_2$ is $\mathcal{A}$-locally compact.
b) $X\text{-}D$ is $\mathcal{A}$-closed in (X, τ).

Proof. 1) Let $(X, \tau) \in LK_{\mathcal{A}}$ if F is $\mathcal{A}$-closed in (X, τ) and E is $\mathcal{A}$-open in (X, τ) then F is closed in $(X, \tau_{\mathcal{A}})$ and E is open in $(X, \tau_{\mathcal{A}})$. Since $(X, \tau_{\mathcal{A}})$ is a locally compact Hausdorff space then $F \cap E$ is locally compact in $(X, \tau_{\mathcal{A}})$, hence $F \cap E$ is an $\mathcal{A}$-locally compact subset of (X, τ).

2) If B is an $\mathcal{A}$-locally compact subset of (X, τ) then it is a locally compact subset of the Hausdorff space $(X, \tau_{\mathcal{A}})$ then $B = F \cap E$ where F is closed in $(X, \tau_{\mathcal{A}})$ (hence $\mathcal{A}$-closed in (X, τ)) and E is open in $(X, \tau_{\mathcal{A}})$ (hence $\mathcal{A}$-open in (X, τ)).

a) $\Rightarrow$ b) Let D be an $\mathcal{A}$-locally compact subset of $(X, \tau) \in K_{\mathcal{A}}$ such that $(X, \tau_{\mathcal{A}}) \in LM\text{-}T_2$, then $(X, \tau_{\mathcal{A}}) \in LM\text{-}T_2\,Comp = TOP_2Comp \subset TOP_2$ from 2) above it follows that $D = F \cap E$, where F is $\mathcal{A}$-closed in (X, τ) and E is $\mathcal{A}$-open in (X, τ).

Since D is $\mathcal{A}$-dense in (X, τ) we have that $X = [D]^X_{\mathcal{A}} = [F \cap E]^X_{\mathcal{A}} \subset [F]^X_{\mathcal{A}} \cap [E]^X_{\mathcal{A}} = F \cap [E]^X_{\mathcal{A}}$ hence $F = [E]^X_{\mathcal{A}} = X$, therefore $D = E$ i.e. D is $\mathcal{A}$-open hence $X\text{-}D$ is $\mathcal{A}$-closed in (X, τ).

b) $\Rightarrow$ a) Let $X\text{-}D$ be $\mathcal{A}$-closed in (X, τ) then D is $\mathcal{A}$-open, hence it is the intersection of an $\mathcal{A}$-closed set with an $\mathcal{A}$-open set, i.e. $D = D \cap [D]^X_{\mathcal{A}}$, but (X, τ) is $\mathcal{A}$-locally compact hence from 1) above it follows that D is an $\mathcal{A}$-locally compact subset of (X, τ).

REMARKS 1.7. (a) Let $(X, \tau) \in LK_{\mathcal{A}}$ let $(Y, \sigma) \in \mathcal{A}$ such that $(Y, \sigma_{\mathcal{A}}) \in TOP_2$. If $f : (X, \tau) \to (Y, \sigma)$ is $\mathcal{A}$-continuous, $\mathcal{A}$-open and onto then $(Y, \sigma) \in LK_{\mathcal{A}}$

(b) Let $\mathcal{A}$ be such that $F_{\mathcal{A}}$ is finitely multiplicative, then $\prod_{\alpha=1}^{k} (X_\alpha, \tau_\alpha) \in LK_{\mathcal{A}}$ if and only if $(X_\alpha, \tau_\alpha) \in LK_{\mathcal{A}}$ for each α.

THE WHITEHEAD THEOREM. For every locally compact Hausdorff space X and any quotient mapping $g : Y \to Z$, the Cartesian product $f = \mathrm{id}_X \times g : X \times Y \to X \times Z$ is a quotient mapping.

DEFINITION 1.8. Let $(X, \tau), (Y, \sigma) \in \mathcal{A}$, a mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be $q(\mathcal{A})$ if $f = F_{\mathcal{A}}(f) : (X, \tau_{\mathcal{A}}) \to (Y, \sigma_{\mathcal{A}})$ is a quotient mapping.

REMARKS 1.9. (a) Let $f : (X, \tau) \to (Y, \sigma)$ be an $\mathcal{A}$-continuous and onto mapping, then the following conditions are equivalent:

(1) f is a $q(\mathcal{A})$ mapping

(2) $f^{-1}(F)$ is $\mathcal{A}$-closed in (X, τ) iff F is $\mathcal{A}$-closed in (Y, σ).

(b) If $f : X \to Y$ is $\mathcal{A}$-continuous, $\mathcal{A}$-closed and onto then it is a $q(\mathcal{A})$ mapping.

THEOREM 1.10. Let $\mathcal{A}$ be such that $F_{\mathcal{A}}$ is finitely multiplicative.

For every $\mathcal{A}$-locally compact space (X, τ) and any $q(\mathcal{A})$ mapping $g : (Y, \sigma) \to (Z, \rho)$, the cartesian product $f = \mathrm{id}_X \times g : (X, \tau) \times (Y, \sigma) \to (X, \tau) \times (Z, \rho)$ is $q(\mathcal{A})$.

Proof. If (X, τ) is an $\mathcal{A}$-locally compact space then $(X, \tau_{\mathcal{A}})$ is a locally compact Hausdorff space. Since $g : (Y, \sigma) \to (Z, \rho)$ is $q(\mathcal{A})$ then $g = F_{\mathcal{A}}(g) : (Y, \sigma_{\mathcal{A}}) \to (Z, \rho_{\mathcal{A}})$ is a quotient mapping.

Since $F_{\mathcal{A}}$ is finitely multiplicative we have that $F_{\mathcal{A}}[(X, \tau) \times (Y, \sigma)] = (X, \tau_{\mathcal{A}}) \times (Y, \sigma_{\mathcal{A}})$ and $F_{\mathcal{A}}[(X, \tau) \times (Z, \rho)] = (X, \tau_{\mathcal{A}}) \times (Z, \rho_{\mathcal{A}})$, then the mapping $f = F_{\mathcal{A}}(f) = F_{\mathcal{A}}(\mathrm{id}_X \times g) = F_{\mathcal{A}}(\mathrm{id}_X) \times F_{\mathcal{A}}(g) : (X, \tau_{\mathcal{A}}) \times (Y, \sigma_{\mathcal{A}}) \to (X, \tau_{\mathcal{A}}) \times (Z, \rho_{\mathcal{A}})$ is a quotient mapping (by Whitehead theorem), therefore $f : (X, \tau) \times (Y, \sigma) \to (X, \tau) \times (Z, \rho)$ is a $q(\mathcal{A})$ mapping.

2. k($\mathcal{A}$)-space.

We recall that a Hausdorff space is a k-space if it is an image of a locally compact Hausdorff space under a quotient mapping.

DEFINITION 2.1. Let $\mathcal{A}$ be an epireflective subcategory of *TOP*. $(X,\tau) \in \mathcal{A}$ is said to be a $k(\mathcal{A})$-space if (X,τ) is an image of an $\mathcal{A}$-locally compact space under a $q(\mathcal{A})$ mapping.

We will denote by $k(\mathcal{A})$ the class of $k(\mathcal{A})$-spaces.

EXAMPLES 2.2.

(a) $LK_{TOP_1} = k(TOP_1) = TOP_1$.

(b) For $\mathcal{A} = Haus(\{X_j\})$ we have $LK_{\mathcal{A}} = k(\mathcal{A}) = \mathcal{A}$.

(c) $k(TOP_2) = k$-spaces.

(d) For $\mathcal{A} = TOP_3$, *Tych*, *0-dim* every $k(\mathcal{A})$-space is a k-space.

(e) For $\mathcal{A} = LM\text{-}T_2$ if (X,τ) is a k-space then $\tau = \tau_{\mathcal{A}}$ (corollary 4.2.(b), [9]) Let (X,τ) be a non locally compact k-space, obviously (X,τ) is not $\mathcal{A}$- locally compact, since (X,τ) is a k-space there exist $(Z,\rho) \in TOP_2LocComp$ and a quotient mapping $f : (Z,\rho) \to (X,\tau)$, but (Z,ρ) is a k- space hence $(Z,\rho) = (Z,\rho_{\mathcal{A}})$, therefore $f = F_{\mathcal{A}}(f) : (Z,\rho) \to (X,\tau)$ is a quotient mapping, hence $f : (Z,\rho) \to (X,\tau)$ is $q(\mathcal{A})$ and $(Z,\rho) \in LK_{\mathcal{A}}$ therefore (X,τ) is a $k(\mathcal{A})$-space and $LK_{\mathcal{A}} \underset{\neq}{\subset} k(\mathcal{A})$.

(f) For $\mathcal{A} = Ury$ if (X,τ) is a T_3-space then $\tau = \tau_{\mathcal{A}}$ [2].
If (X,τ) is a regular and not locally compact k- space then a similar argument to (e) shows that (X,τ) is $k(Ury)$ but it is not *Ury*-locally compact hence $LK_{\mathcal{A}} \underset{\neq}{\subset} k(\mathcal{A})$.

We don't know if there exists a space $(X,\tau) \in k(TOP_0)$ which is not TOP_0-locally compact; if such a space exists then (X,τ_{TOP_0}) is a k-space not locally compact.

PROPOSITION 2.3. Suppose $(X,\tau) \in \mathcal{A}$ and $(X,\tau_{\mathcal{A}}) \in TOP_2$.

1. If (X,τ) is a $k(\mathcal{A})$-space then $(X,\tau_{\mathcal{A}})$ is a k-space. Conversely if $(X,\tau_{\mathcal{A}})$ is a quotient space of a locally compact Hausdorff space Y and there is $Z \in \mathcal{A}$ with $F_{\mathcal{A}}(Z) = Y$ then (X,τ) is a $k(\mathcal{A})$-space.
   If, in addition, for each $(X,\tau) \in \mathcal{A}$ the $\mathcal{A}$-closure is hereditary in (X,τ), we have
2. If (X,τ) is a $k(\mathcal{A})$-space then for $F \subset (X,\tau)$ the set F is $\mathcal{A}$-closed in (X,τ) provided that the intersection of F with any $\mathcal{A}$-compact subspace Z of the space (X,τ) is $\mathcal{A}$-closed in Z.
3. If $(X,\tau_{\mathcal{A}})$ is a space such that for each quotient mapping $f : Y \to (X,\tau_{\mathcal{A}})$, $Y \in TOP_2LocComp$, there exists $Z \in \mathcal{A}$ such that $F_{\mathcal{A}}(Z) = Y$ then the converse of 2. holds.

Proof. 1. Let (X, τ) be a $k(\mathcal{A})$-space then there exists a $q(\mathcal{A})$ mapping $f: (Y, \sigma) \to (X, \tau)$ such that (Y, σ) is an $\mathcal{A}$- locally compact space, hence $f = F_{\mathcal{A}}(f) : (Y, \sigma_{\mathcal{A}}) \to (X, \tau_{\mathcal{A}})$ is a quotient mapping, $(X, \tau_{\mathcal{A}}) \in TOP_2$ and $(Y, \sigma_{\mathcal{A}})$ is a locally compact Hausdorff space, therefore $(X, \tau_{\mathcal{A}})$ is a k-space.

If $(X, \tau_{\mathcal{A}})$ is a k-space such that there exists a quotient mapping $f = F_{\mathcal{A}}(f) : Y \to (X, \tau_{\mathcal{A}})$ with Y locally compact Hausdorff space and $Y = F_{\mathcal{A}}(Z)$ where $Z \in \mathcal{A}$ then $f: Z \to (X, \tau)$ is a $q(\mathcal{A})$ mapping and Z is an $\mathcal{A}$-locally compact space, therefore (X, τ) is a $k(\mathcal{A})$-space.

2. Let (X, τ) be a $k(\mathcal{A})$-space then by 1. it follows that $(X, \tau_{\mathcal{A}})$ is a k-space. Let $F \subset (X, \tau)$ such that $F \cap Z$ is $\mathcal{A}$-closed in $Z \subset (X, \tau)$ for each $\mathcal{A}$-compact Z, then $F \cap Z$ is closed in $Z \subset (X, \tau_{\mathcal{A}})$ for each compact Z, but $(X, \tau_{\mathcal{A}})$ is a k-space hence F is closed in $(X, \tau_{\mathcal{A}})$ (th. 3.3.18, [7]), i.e. F is $\mathcal{A}$-closed in (X, τ).

3. Now let us prove that $(X, \tau_{\mathcal{A}})$ is a k- space, let $F \subset (X, \tau_{\mathcal{A}})$ such that $F \cap Z$ is closed in Z for each compact $Z \subset (X, \tau_{\mathcal{A}})$, then $F \cap Z$ is $\mathcal{A}$-closed in $Z \subset (X, \tau)$ for each $\mathcal{A}$-compact Z, hence by hypothesis F is $\mathcal{A}$-closed in (X, τ) therefore it is closed in $(X, \tau_{\mathcal{A}})$, hence $(X, \tau_{\mathcal{A}})$ is a k-space (th. 3.3.18, [7]), and by 1. it follows that (X, τ) is a $k(\mathcal{A})$-space.

Theorem 1.10 implies

PROPOSITION 2.4. Let $\mathcal{A}$ be such that $F_{\mathcal{A}}$ is finitely multiplicative.
The Cartesian product $X \times Y$ of an $\mathcal{A}$- locally compact space X and a $k(\mathcal{A})$-space Y is a $k(\mathcal{A})$-space.

THEOREM (Michael [13]). Suppose X is a T_3-space; if the Cartesian product $X \times Y$ is a k-space for every k-space Y, then the space X is locally compact.

THEOREM 2.5. Let $\mathcal{A}$ be such that $F_{\mathcal{A}}$ is finitely multiplicative and let us suppose that for each k-space Z there exists $(Y, \sigma) \in \mathcal{A}$ such that $F_{\mathcal{A}}(Y, \sigma) = Z$.

If the Cartesian product $(X, \tau) \times (Y, \sigma)$, where $(X, \tau) \in \mathcal{A}$ and $(X, \tau_{\mathcal{A}})$ is a T_3-space, is a $k(\mathcal{A})$-space for every $k(\mathcal{A})$-space (Y, σ), then the space (X, τ) is $\mathcal{A}$-locally compact.

Proof. If (X, τ) is not an $\mathcal{A}$-locally compact space then $(X, \tau_{\mathcal{A}})$ is not locally compact hence from the Michael theorem follows that there exists a k- space Z such that $(X, \tau_{\mathcal{A}}) \times Z$ is not a k-space.

By hypothesis there exists a space $(Y, \sigma) \in \mathcal{A}$ such that $(Y, \sigma_{\mathcal{A}}) = Z$ then (Y, σ) is a $k(\mathcal{A})$-space and by $F_{\mathcal{A}}[(X, \tau) \times (Y, \sigma)] = (X, \tau_{\mathcal{A}}) \times Z$ it follows that $(X, \tau) \times (Y, \sigma)$ is not a $k(\mathcal{A})$-space.

PROPOSITION 2.6. Let $\mathcal{A}$ be such that for each $(X, \tau) \in \mathcal{A}$ the $\mathcal{A}$-closure is hereditary in (X, τ). An $\mathcal{A}$-continuous map $f: (X, \tau) \to (Y, \sigma)$ of a space $(X, \tau) \in \mathcal{A}$ to a $k(\mathcal{A})$-space (Y, σ) such that $(Y, \sigma_{\mathcal{A}}) \in TOP_2$ is $\mathcal{A}$-closed iff for every $\mathcal{A}$-compact subspace $Z \subset (Y, \sigma)$ the restriction $f_Z : f^{-1}(Z) \to Z$ is $\mathcal{A}$-closed.

Proof. If $f : (X, \tau) \to (Y, \sigma)$ is $\mathcal{A}$-closed then $f = F_{\mathcal{A}}(f) : (X, \tau_{\mathcal{A}}) \to (Y, \sigma_{\mathcal{A}})$ is continuous, closed and $(Y, \sigma_{\mathcal{A}})$ is a k-space therefore $f_Z = F_{\mathcal{A}}(f_Z) : f^{-1}(Z) \to Z$ is closed for every compact subspace $Z \subset (Y, \sigma_{\mathcal{A}})$ (th. 3.3.22, [7]), i.e. $f_Z : f^{-1}(Z) \to Z$ is $\mathcal{A}$-closed for every $\mathcal{A}$-compact subspace $Z \subset (Y, \sigma)$.

If for every $\mathcal{A}$-compact subspace $Z \subset (Y, \sigma)$ $f_Z : f^{-1}(Z) \to Z$ is $\mathcal{A}$-closed then $f_Z = F_{\mathcal{A}}(f_Z) : f^{-1}(Z) \to Z$ is closed for every compact subspace $Z \subset (Y, \sigma_{\mathcal{A}})$ hence $f = F_{\mathcal{A}}(f) : (X, \tau_{\mathcal{A}}) \to (Y, \sigma_{\mathcal{A}})$ is closed (th. 3.3.22, [7]), i.e. $f : (X, \tau) \to (Y, \sigma)$ is $\mathcal{A}$-closed.

REFERENCES

[1] M. P. BERRI, J. R. PORTER and R. M. STEPHENSON, *A survey of minimal topological spaces*, in: General Topology ad its Relations to Modern Analysis and Algebra (Proc. Kampur Top. Conf., Acad. Press, 1970), 93-114.

[2] D. DIKRANJAN and E. GIULI, *Closure operators induced by topological epireflections*, Coll. Math. Soc. J. Bolyai, *41* (1983), 233-246.

[3] D. DIKRANJAN and E. GIULI, *Epimorphisms and cowell poweredness of epireflective subcategories of TOP*, Rend. Circolo Mat. Palermo, Suppl. *6* (1984), 121-136.

[4] D. DIKRANJAN and E. GIULI, *Urysohn-closed spaces-old and new*, preprint.

[5] D. DIKRANJAN and E. GIULI, *Ordinal invariants and epimorphisms in some categories of weak Hausdorff spaces*, Comment. Math. Univ. Carolinae 27, *2* (1986), 395-417.

[6] D. DIKRANJAN and E. GIULI, *Closure operator I*, Topology Appl. *27* (1987), 129-143.

[7] R. ENGELKING, *General Topology*, Polish Scientific Publishers, Warszawa (1977).

[8] A. FEDELI, *$\mathcal{A}$-compact spaces*, Serdica, Bulg. Acad. Sci., to appear.

[9] E. GIULI and M. HUSEK, *A diagonal theorem for epireflective subcategories of TOP and cowellpoweredness*, Ann. Mat. Pura Appl. IV, *CXLV* (1986), 337-346.

[10] H. HERRLICH, *Categorical Topology*, Gen. Topology Appl. *1* (1971), 1-15.

[11] R.-E. HOFFMANN, *On weak Hausdorff spaces*, Arch. Math. *32* (1979), 487-504.

[12] J. LAWSON and B. MADISON, *Comparisons of notions of weak Hausdorffness*, Topology Proc. Memphis State Univ., Conference 1975, Ed. by S. P. Franklin and B. V. Smith Thomas (New York-Basel, 1976), 207-215.

[13] E. MICHAEL, *Local compactness and Cartesian products of quotient maps and k-spaces*, Ann. Inst. Fourier *18* (1968), 281-286.

[14] L. NEL and R. G. WILSON, *Epireflections in the category of T_0-spaces*, Fund. Math. *75* (1972), 69-74.

[15] S. SALBANY, *Reflective subcategories and closure operators*, in: Categorical Topology, Lecture Notes in Math. n. *540* (Springer, Berlin, 1976), 548-565.

[16] H. V. VELICHKO, *H-closed topological spaces*, Mat. Sb. (N. S.), *70* (112) (1966), 98-112; Amer. Math. Soc. Transl. *78* (series 2) (1969), 103-118.

[17] J. H. C. WHITEHEAD, *A note on a theorem due to Borsuk*, Bull. Amer. Math. Soc., *54* (1948), 1125-1132.

[18] S. WILLARD, *General Topology*, (Addison Wesley, Reading, 1970).