SOME DE FINETTI-KOLMOGOROFF-NAGUMO TYPE INTEGRAL REPRESENTATION THEOREMS FOR MEANS ON MASSES (*)

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SOMMARIO.- *Introdotta la nozione di media per masse sull'asse reale si forniscono alcuni teoremi di caratterizzazione per importanti classi di medie. Si fa così vedere che le proprietà di consistenza, continuità e associatività per un funzionale reale, definito su un insieme di masse, sono le nozioni basilari per ottenere rappresentazioni integrali dello stesso del tipo de Finetti-Kolmogoroff-Nagumo. Si definisce e si studia inoltre un integrale del tipo S-integrale per funzioni illimitate (S-integrale improprio).*

SUMMARY.- *We introduce the notion of mean for masses on the real line and we give characterization theorems for some important classes of means. In this way we point out that the Consistency, Continuity and Associativity property of a real functional M on masses are the basic notions to get a de Finetti-Kolmogoroff-Nagumo type integral representation of M. Moreover we introduce and study a notion of improper S-integral for real functions, bounded or not.*

1. Introduction.

Some of the most used means in the applications (as arithmetic mean, geometric mean, harmonic mean and root-mean-square) are special cases of a mean of the type:

$$m_g(x_1,\dots,x_n) = g^{-1}\left[\frac{g(x_1)+\dots+g(x_n)}{n}\right],$$

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where g is a strictly monotone continuous function.

A characterization of this mean was given in 1930 by Kolmogoroff [5] and Nagumo [6], indipendently, by means of the notions of *consistency* (the value of the mean is equal to x_1 whenever $x_1 = ... = x_n$), *strict monotonicity* (the value of the mean increases whenever one of $x_1,..., x_n$ increases), *associativity* (the value of the mean does not change whenever we substitute some of $x_1,..., x_n$ by the mean of them) and *symmetry* (the value of the mean is invariant under any permutation of $x_1,..., x_n$).

A year later de Finetti [2] extended the first three notions to probability distribution functions F and he obtained a characterization of the g-means, i.e. means of the type:

$$m_g(F) = g^{-1}\left[\int_{-\infty}^{+\infty} g\, dF\right],$$

where g is a strictly monotone continuous function and the integral is a Riemann-Stieltjes integral.

This axiomatic treatment of g-means is known in literature as the *de Finetti-Kolmogoroff-Naguno Integral Representation Theorem*, which we state here only for the sake of completeness. First, we recall that, in this context, a probability distribution function is an increasing function F such that

$$F(-\infty) = 0, F(+\infty) = 1,$$

$$F(x) = \frac{F(x-0) + F(x+0)}{2} \quad \text{for all } x.$$

Moreover, a real map $\boldsymbol{m}$ on a set $\mathbf{D}$ of probability distribution functions is called:

- *consistent* iff $\boldsymbol{m}(I_{[x, +\infty[}) = x$ for all x;

- *strictly monotone* iff $\boldsymbol{m}(F) > \boldsymbol{m}(G)$ whenever $F < G$ in the usual pointwise ordering;

- *associative* iff $\boldsymbol{m}(\alpha_1 F_1 + \alpha_2 F_2) = \boldsymbol{m}(\alpha_1 G_1 + \alpha_2 G_2)$ whenever $\boldsymbol{m}(F_i) = \boldsymbol{m}(G_i)$, $\alpha_i \geq 0$ (i = 1,2) and $\alpha_1 + \alpha_2 = 1$.

That being recalled, we have the following:

THEOREM. *Let* $\mathbf{D}$ *be the set of all probability distribution functions with the closed interval* $[a,b]$ *as support. Moreover let* $\boldsymbol{m}$ *be a real map on* $\mathbf{D}$*. Then the following statements are equivalent:*

(i) m is consistent, strictly monotone and associative;
(ii) there is a strictly monotone continuous function g such that

$$m\,(F) = m_g\,(F)$$

for all $F \in \mathbf{D}$.
Moreover, $g\,|_{[a,b]}$ *is unique up to non degenerate affine transformations.*

In this paper, we extend the notion of g-mean to masses (i.e. bounded positive charges) on the real axis and we give some axiomatic characterizations of it. One of the reasons for this generalization to masses comes from difficulties with the use of distribution functions in a finitely additive setting (there is no one-to-one correspondence between masses and distribution functions (see [3], Theorem 3.3, p.4)). Therefore the monotonicity of the mean is not meaningfull in the context of masses and hence we replace it, in the characterization of g-means, by the notion of *continuity* of a mean with respect to the convergence in distribution of masses.

Now, we briefly describe the contents of the following sections.

In Section two, we give some useful notations and definitions used in the sequel.

In Section three, following the usual methods to introduce the improper Riemann-Stieltjes integral, we give the notion of S-integral for unbounded functions also. Moreover, we study the relationship between this integral and the S-integral, when both exist.

In Section four, we introduce the basic notion of a mean on an arbitrary set of masses by extending the Consistency Property. Moreover we introduce one notion of internality, two notions of continuity, three notions of associativity and we study the relationship between them. We also give the notion of σ-associativity and we link it and continuity and associativity. Finally, we conjecture that a suitable associative mean is not a σ-associative mean.

In section five, by means of the integral introduced in Section three, we define the g-mean directly and develope its properties from the definition.

In Section six, we prove a result which is the basic tool to study the characteristic properties of g-means. We can see it also as a generalization of the de Finetti-Kolmogoroff-Nagumo theorem to finitely additive probabilities on the same compact support.

In Section seven, five de Finetti-Kolmogoroff-Nagumo type theorems are given. The first three ones are related with means on the set of all non null masses on a compact interval of the real axis, the fourth one with means on the set of all non null masses having compact support and the fifth one with means on the set of all non null tight masses.

Finally, in Section eight, we give three results which are instrumental in proving some theorems of Section seven.

2. Notation and Definition.

Throughout this paper we adopt usual set theoretic and topological notations.

The set $\boldsymbol{R}$ is always understood to be the metric space of real numbers with the natural topology and $a, b, k, x, y, \alpha, \beta$, with or without indices, always denote real numbers. Moreover we denote by $\boldsymbol{R}^+$ the set of all non negative real numbers.

We denote by f, with or without indices, any real function on $\boldsymbol{R}$ and we put:

$$f(-\infty) = \lim_{x \to -\infty} f(x), \quad f(+\infty) = \lim_{x \to +\infty} f(x).$$

Moreover, given $A \subset \boldsymbol{R}$, we put

$$\Delta_A f = \operatorname*{Sup}_{x,y \in A} | f(x) - f(y) | = \operatorname*{Sup}_{x \in A} f(x) - \operatorname*{Inf}_{x \in A} f(x).$$

The letter $\mathbf{F}$ always denotes a field on $\boldsymbol{R}$ including the collection of all intervals in $\boldsymbol{R}$ and sets from $\mathbf{F}$ are denoted by F with or without indices.

A positive bounded charge on $\mathbf{F}$ is called a *mass* on $\mathbf{F}$; we denote it by μ with or without indices. As usual, $\|\mu\|$ denotes the norm of μ, i.e. $\|\mu\| = \mu(\boldsymbol{R})$, and $\mu^*(F) = \operatorname{Inf}\{\mu(U) : U$ open and $F \subset U \in \mathbf{F}\}$ for all F. Moreover, we call a *support* of μ every convex set F such that $\mu(F) = \|\mu\|$ and the *distribution function corresponding to* μ the function $F_\mu(x) = \mu(]-\infty, x])$, for all x. Finally, a mass μ is called a *tight mass* whenever $\mu(-\infty) = \lim_{x \to -\infty} \mu\big(]-\infty, x]\big) = 0$ and $\mu(+\infty) = \lim_{x \to +\infty} \mu\big([x, +\infty[\big) = 0$; i.e. whenever μ has no μ-adherence at $-\infty$ and $+\infty$, respectively.

We denote by $\mathbf{M}$, $\mathbf{M_o}$ and $\mathbf{M_{oo}}$ the set of all masses, the set of all tight masses and the set of all masses having bounded support, respectively; moreover we put $\mathbf{M_k} = \{\mu : \|\mu\| = k\}$, $\mathbf{M}[a,b] = \{\mu : \|\mu\| = \mu^*([a,b])\}$ and $\mathbf{M_k}[a,b] = \mathbf{M_k} \cap \mathbf{M}[a,b]$. Finally, the symbol $\mathbf{M'}$ always denotes an arbitraty subset of $\mathbf{M}$ and $(\mathbf{M'})^+$ denotes the set of all non null masses of $\mathbf{M'}$; moreover we put $\mathbf{M}^+[a,b] = (\mathbf{M}[a,b])^+$.

Now, given $k \geq 0$ we denote by k_x the following 0-k valued mass on **F**:

$$k_x(F) = k, \text{ if } x \in F,$$

$$= 0, \text{ if } x \notin F,$$

which is usually called a *degenerate mass.*

Given a bounded function f and a mass μ the symbol $S \int f\, d\mu$ always denotes a Stieltjes type integral, in the sense of S- integral ([1], Definition 4.5.5, p.116).

We say that the sequence (μ_n) is:

- *weakly convergent* to μ iff $S \int f\, d\mu = \lim_{n \to +\infty} S \int f\, d\mu_n$ for all bounded continuous functions f which are S-integrable with respect to μ, μ_n for all natural numbers n;

- *convergent in distribution* to μ iff $F_{\mu_n}(a) \to F_\mu(a)$ at any continuity point a of F_μ.

In the sequel, the first convergence will be denoted by $\mu_n \overset{w}{\to} \mu$ and the second one by $\mu_n \overset{d}{\to} \mu$.

Finally, given a function f, a mass μ and a subset A of R, we put $S \int_A f\, d\mu = S \int I_A f\, d\mu$, where I_A is the indicator function of A.

3. The Improper S-Integral.

We start with the following definition:

3.1 DEFINITION. Let f be such that $S \int_{[a,b]} f\, d\mu$ exists for all a,b. Then we put:

$$\int f\, d\mu = \lim_{a \to -\infty,\, b \to +\infty} S \int_{[a,b]} f\, d\mu$$

whenever the limit is a real number. This limit is called the *improper S-integral* of f with respect to μ. Moreover, we denote by $\int_A f\, d\mu$ the improper S-integral $\int I_A f d\mu$, whenever it exists.

3.2 REMARK. (i) Given a bounded function f, the improper S-integral and the S-integral of f with respect the mass μ are not always equal. For example, let **F** be the Borel σ-field on R and let $\mu \in \mathbf{M}^+$ such that $\mu(F) = 0$ for any bounded set F (1). Moreover let $f(x) = 1$ for all x. Then we have $S \int f\, d\mu = \|\mu\| > 0$ and $\int f\, d\mu = 0$.

(ii) Given a bounded function f, the improper S-integral may exist while the S- integral may not exist. For instance, let **F** be the smallest field containing the intervals and let μ be the mass considered in (i) on **F**. Moreover let $f(x) = \sin x$ for all x. Then we have $\int f\, d\mu = 0$. On the other hand, by Theorem 4.5.7 of [1], p.117, it is quite easy to verify that f is not S-integrable. It is interesting to point out that the non S-integrability of f is due to the choise of the field **F**. In fact any bounded continuous function is S-integrable whenever **F** includes all the open sets of R.

Now we give some definitions and properties, already considered in [3], to link S-integral and improper S-integral.

3.3 DEFINITION. A function f is called *regular at infinity* iff $f(-\infty)$ and $f(+\infty)$ are real numbers.

3.4 DEFINITION. Given a function f and a mass μ, the pair (f, μ) is called *regular at infinity* iff ($\mu(-\infty) = 0$ or $f(-\infty) \in R$) and ($\mu(+\infty) = 0$ or $f(+\infty) \in R$).

3.5 REMARK. Of course (f, μ) is regular at infinity whenever f is regular at infinity or μ is tight.

The following lemma may be easily proved.

3.6 LEMMA. *Let f be a bounded function. The pair (f, μ) is regular at infinity iff for all $\varepsilon > 0$ there are a,b such that the following two inequalities hold:*

$$\mu\,(]-\infty,a])\ \Delta_{]-\infty,a]} f < \varepsilon,\ \mu\,([b,+\infty[)\ \Delta_{[b,+\infty[} f < \varepsilon.$$

Now we are able to link S-integral and improper S-integral.

(1) Since the set of all bounded Borel subsets of R is an ideal, we can contruct a 0-1 valued mass of the type here considered, using the result that any ideal in **F** is contained in a maximal ideal in **F**.

3.7 THEOREM. *Let f be a bounded function and let (f, μ) be regular at infinity. Then the following statements are equivalent:*

(i) f is S-integrable with respect to μ;
(ii) f is improperly S-integrable with respect to μ.
Moreover if (i) holds then

$$S \int f \, d\mu = \int f \, d\mu + f(-\infty)\,\mu(-\infty) + f(+\infty)\,\mu(+\infty),$$

where the last two addends are equal to zero whenever they become meaningless (i.e. $f(-\infty)$ or $f(+\infty)$ do not exist).

Proof. Let $|f| \leq k$. (i) $\Rightarrow$ (ii). Given a, b we have

$$|S \int f \, d\mu - f(-\infty)\,\mu(-\infty) - f(+\infty)\,\mu(+\infty) - S \int_{[a,b]} f \, d\mu \,| \leq$$

$$\leq |S \int_{]-\infty,a[} f \, d\mu - f(-\infty)\,\mu(-\infty)\,| \, +$$

$$+ \, | \, S \int_{]b,+\infty[} f \, d\mu - f(+\infty)\,\mu(+\infty)\,|$$

In order to prove the implication and the desired equality, let $\varepsilon > 0$. Of course, it is sufficient to verify that each addend in the previous inequality is less than $k\varepsilon$, whenever $a < a'$ and $b > b'$ for some a', b'.

First we assume that $f(-\infty)$ does not exist. Since (f, μ) is regular at infinity we have $\mu(-\infty) = 0$ and hence there is a' such that $\mu(]-\infty, a'[) < \varepsilon$. Then we have

$$|S \int_{]-\infty,a[} f \, d\mu - f(-\infty)\,\mu(-\infty)\,| = |\, S \int_{]-\infty,a[} f \, d\mu \,| < k\varepsilon$$

for all $a < a'$.

Now let $f(-\infty) \in \boldsymbol{R}$. From the Mean Value Theorem for the S-integral we get:

$$\mu(]-\infty,a[) \operatorname*{Inf}_{x<a} f(x) \leq S \int_{]-\infty,a[} f \, d\mu \leq \mu(]-\infty,a[) \operatorname*{Sup}_{x<a} f(x).$$

Since $\lim\limits_{a\to-\infty} \operatorname*{Inf}_{x<a} f(x) = \lim\limits_{a\to-\infty} \operatorname*{Sup}_{x<a} f(x) = f(-\infty)$ we have

$\lim_{a \to -\infty} S \int_{]-\infty,a[} f \, d\mu = f(-\infty)\mu(-\infty)$, i.e. there is a' such that

$$| S \int_{]-\infty,a[} f \, d\mu - f(-\infty)\mu(-\infty) | < k\varepsilon,$$

for all $a < a'$.

Analogously, we get the other desired inequality with reference to $+\infty$. This proves the desired implication.

(ii) $\Rightarrow$ (i). Let $\varepsilon > 0$. Since (f, μ) is regular at infinity and f is a bounded function, by Lemma 3.6, there are a, b such that

$$\mu(]-\infty, a[) \, \Delta_{]-\infty, a[} f < \varepsilon,$$

$$\mu(]b, +\infty[) \, \Delta_{]b, +\infty[} f < \varepsilon.$$

Since the function $I_{[a,b]} f$ is S-integrable, by Theorem 4.5.7 of [1], p.117, there exists a partition $P^o = \left\{F_1^o, ..., F_m^o\right\}$ of R in F such that

$$\sum_{i=1}^{n} \Delta_{F_i} \left(I_{[a,b]} f\right) \mu(F_i) < \varepsilon$$

for any partition $P = \{F_1, ..., F_n\}$ of R in F finer than P^o (i.e. $P \geq P^o$).

Let $P^1 = \{F_i^o \cap [a,b] : i = 1, ..., m\} \cup \{F_i^o \cap]-\infty, a[: i = 1, ..., m\} \cup \{F_i^o \cap]b, +\infty[: i = 1, ..., m\}$; then $P^1 \geq P^o$. Moreover, given any partition $P = \{F_1, ..., F_n\}$ of R in F finer than P^1, we have

$$\sum_{i=1}^{n} \Delta_{F_i} f \, \mu(F_i) = \sum_{i \in \left\{i : F_i \subset [a,b]\right\}} \Delta_{F_i}\left(I_{[a,b]} f\right) \mu(F_i) + \sum_{i \in \left\{i : F_i \subset]-\infty,a[\right\}} \Delta_{F_i} f \, \mu(F_i) +$$

$$+ \sum_{i \in \left\{i : F_i \subset]b,+\infty[\right\}} \Delta_{F_i} f \, \mu(F_i) <$$

$$< \varepsilon + \Delta_{]-\infty,a[} f \sum_{i \in \left\{i : F_i \subset]-\infty,a[\right\}} \mu(F_i) + \Delta_{]b,+\infty[} f \sum_{i \in \left\{i : F_i \subset]b,+\infty[\right\}} \mu(F_i) \leq 3\varepsilon .$$

Hence, by Theorem 4.5.7 of [1], p.117, f is S-integrable. This proves the desired implication.

This completes the proof. ♦

The next corollaries follow quite easily from the previous theorem and Remark 3.5.

3.8 COROLLARY. *Let f be a bounded function and let* $\mu \in \mathbf{M}_0$. *Then the following statements are equivalent:*

(i) f is S-integrable with respect to μ;
(ii) f is improperly S-integrable with respect to μ.
Moreover if (i) holds then $S \int f \, d\mu = \int f \, d\mu$

3.9 COROLLARY. *Let f be a bounded function regular at infinity. Then the following statements are equivalent:*

(i) f is S-integrable with respect to any μ;
(ii) f is improperly S-integrable with respect to any μ.
Moreover if (i) holds then

$$S \int f \, d\mu = \int f \, d\mu + f(-\infty)\, \mu(-\infty) + f(+\infty)\, \mu(+\infty).$$

3.10 COROLLARY. *Let* **F** *be the smallest field on* $\boldsymbol{R}$ *containing the intervals and let f be a bounded function. Then the following statements are equivalent:*

(i) f is S-integrable with respect to μ;
(ii) f is improperly S-integrable with respect to μ *and* (f, μ) *is regular at infinity. Moreover the equality in Theorem 3.7 holds as well.*

Proof. It is sufficient to observe that, by Theorem 4.5.7 of [1], p.117, (f, μ) is regular at infinity whenever f is S-integrable with respect to μ. This completes the proof. ♦

3.11 REMARK. Given a bounded function f, it is interesting to show that f may be S-integrable and improperly S-integrable with respect to a mass μ, even thought the pair (f, μ) is not regular at infinity. Let **F** be the Borel σ-field on $\boldsymbol{R}$ and let $\mu \in \mathbf{M}^+$ such that $\mu(F) = 0$ for any bounded set F. Moreover let

$$f(x) = 0, \text{ if } x \text{ is an irrational number,}$$

$$= 1, \text{ if } x \text{ is a rational number.}$$

Then we have $\int f \, d\mu = 0$, $S \int f \, d\mu = \mu(\mathrm{Q})$, where Q is the set of all rational numbers, and (f, μ) is not regular at infinity.

The following lemma paves the way to study the improper S-integrability of a continuous function on $\boldsymbol{R}$.

3.12 LEMMA. *Let f be such that $\int f\,d\mu$ exists. Then $\int f\,d\mu = \int_F f\,d\mu$ whenever $\mu(F) = \|\mu\|$.*

Proof. We have

$$\int f d\mu = \lim_{a\to-\infty, b\to+\infty} S\int_{[a,b]} f d\mu = \lim_{a\to-\infty, b\to+\infty}\left[S\int_F I_{[a,b]} f d\mu + S\int_{F^c} I_{[a,b]} f d\mu\right] =$$

$$= \lim_{a\to-\infty, b\to+\infty}\left[S\int I_{[a,b]} I_F f d\mu + S\int I_{F^c\cap[a,b]}\, f d\mu\right].$$

Since $\mu(F^c) = 0$ we get $\int f d\mu = \lim_{a\to-\infty, b\to+\infty} S\int_{[a,b]} I_F f d\mu = \int I_F f d\mu$. This proves the lemma. ♦

3.13 THEOREM. *Let f be a continuous function and let $\mu \in \mathbf{M}\,[a, b]$. Then $\int f\,d\mu$ exists and*

$$S\int_{[a',b']} f\,d\mu = \int_{[a',b']} f\,d\mu = \int f\,d\mu$$

for all $a' < a$ and $b' > b$. Moreover the equalities hold for $a' \leq a$ and $b' \geq b$ whenever $\|\mu\| = \mu([a, b])$.

Proof. It is easy to prove that $S\int_{[a',b']} f\,d\mu$ exists for all a', b' on noting that f is uniformly continuous on $[a', b']$.

Now, let $a'' < a' < a$ and $b'' > b' > b$. Then $\mu([a', b']^c) = 0$ and hence $S\int_{[a'',a'[\cup]b',b'']} f\,d\mu = 0$, i.e. $S\int_{[a'',b'']} f\,d\mu = S\int_{[a',b']} f\,d\mu$.

Therefore f is improperly S-integrable with respect to μ and whence, by Lemma 3.12, we get the desired equalities.

Of course, we can prove the last statement of the theorem in a similar way. This completes the proof. ♦

Since every mass of $\mathbf{M}_{oo}$ belongs to $\mathbf{M}\,[a, b]$ for some a, b, we get the following corollary.

3.14 COROLLARY. *Let f be a continuous function. Then f is improperly S-integrable with respect to every element of $\mathbf{M}_{oo}$.*

The following theorem gives a sufficient condition assuring the S-integrability of a bounded continuous function on $\boldsymbol{R}$.

3.15 THEOREM. *Let f be a bounded continuous function and let (f, μ) be regular at infinity. Then f is S-integrable and improperly S-integrable with respect to μ and the following equality holds:*

$$S \int f \, d\mu = \int f \, d\mu + f(-\infty)\, \mu(-\infty) + f(+\infty)\, \mu(+\infty).$$

Proof. On noting that, by Theorem 4.8 of [3], p.10, f is S-integrable with respect to μ, we get the thesis by Theorem 3.7. This completes the proof. ♦

The next corollaries follow quite easily from the previous theorem and Remark 3.5.

3.16 COROLLARY. *Let f be a continuous function regular at infinity. Then f is S-integrable and improperly S-integrable with respect to any μ. Moreover:*

$$S \int f \, d\mu = \int f \, d\mu + f(-\infty)\, \mu(-\infty) + f(+\infty)\, \mu(+\infty).$$

3.17 COROLLARY. *Let f be a bounded continuous function. Then f is S- integrable and improperly S-integrable with respect to any $\mu \in \mathbf{M_o}$. Moreover:*

$$S \int f \, d\mu = \int f \, d\mu.$$

Keeping in mind Corollary 3.14, we can say that for the existence of improper S-integral for all elements of $\mathbf{M_{oo}}$ it is sufficient to have a continuous function, bounded or not. The following theorem points out that this is not true if we consider the larger set $\mathbf{M_o}$ instead of $\mathbf{M_{oo}}$.

3.18 THEOREM. *Let f be a continuous function. Then the following statements are equivalent:*

(i) there exists $\int f \, d\mu$ for all $\mu \in \mathbf{M_o}$;
(ii) f is a bounded function.

Proof. (i) ⇒ (ii). Suppose f is not a bounded function. Let, for instance, $\operatorname{Sup} f = +\infty$. Then for every natural number n, there is x_n such that $f(x_n) \geq 2^n$. Now, we consider the mass

$$\mu = \sum_{i=1}^{+\infty} \frac{1}{2^n} \mathbf{1}_{x_n} .$$

We claim that $\int f\, d\mu$ does not exist. Let $a < b$. Then we have a finite number of elements $x_{n_1}, \ldots, x_{n_h}$ of (x_n) belonging to $[a, b]$. Hence

$$S \int_{[a,b]} f\, d\mu = \sum_{i=1}^{h} \frac{1}{2^{n_i}} f(x_{n_i}) \geq h$$

and so, carrying out the passage of the limit, $\int f\, d\mu$ does not exist. Since $\mu \in \mathbf{M}_0$, we get a contradiction. This proves the desired implication.

(ii) $\Rightarrow$ (i). This easily follows from Corollary 3.17.

This completes the proof. ♦

3.19 REMARK. Keeping in mind the previous proof we can weaken the tightness condition in (i) of the previous theorem by requiring that $\int f\, d\mu$ exist for all μ which are σ- mixtures of degenerate probabilities.

Now, in the following proposition we give a sufficient condition assuring the equality of improper S-integrals of continuous functions with respect to elements of $\mathbf{M}\,[a, b]$.

3.20 PROPOSITION. *Let $\mu \in \mathbf{M}\,[a, b]$. Moreover, let f_1, f_2 be two continuous functions such that $f_1|_{[a,\,b]} = f_2|_{[a,\,b]}$. Then $\int f_1\, d\mu = \int f_2\, d\mu$.*

Proof. Since the improper integral $\int f\, d\mu$ is a linear functional with respect to f, it is sufficient to prove that $\int f\, d\mu = 0$, whenever f is a continuous function such that $f|_{[a,\,b]}$ is the null function.

Therefore let f be a continuous function such that $f|_{[a,\,b]}$ is the null function. Let $\varepsilon > 0$. Then there exist a', b' such that $a' < a$, $b < b'$ and $|f(x)| < \varepsilon$ for all $x \in F = [a', a\,[\cup]\,b, b']$. Hence, by Theorem 3.13, we get:

$$\int f d\mu = S \int_{[a',b']} f d\mu = S \int_{[a,b]} I_{[a',b']}\, f d\mu + S \int_{[a,b]^c} I_{[a',b']}\, f d\mu = S \int I_F f\, d\mu$$

Therefore, $|\int f\, d\mu| \leq S \int I_F\, |f|\, d\mu \leq \varepsilon\, \|\mu\|$.

This proves the proposition. ♦

3.21 COROLLARY. *Let μ, μ_n be a sequence in* M $[a, b]$ *such that* $\mu_n \overset{w}{\rightarrow} \mu$. *Then we have* $\int f\, d\mu_n \rightarrow \int f\, d\mu$ *for all continuous functions f.*

Proof. If f is a bounded function, the thesis immediately follows from the definition of the weak convergence. If f is an unbounded function, the thesis easily follows from Proposition 3.20. This completes the proof. ♦

The next Helly type theorems for the improper S-integral are related to the elements of $\mathbf{M}_{oo}$. To express the first one we need the following definition.

3.22 DEFINITION. A sequence (μ_n) of masses is called a *sequence with bounded support* iff there is a bounded set which is a support for all μ_n.

3.23 THEOREM. *Let (μ_n) be a sequence with bounded support such that* $\mu_n \overset{w}{\rightarrow} \mu$. *Then* $\int f\, d\mu_n \rightarrow \int f\, d\mu$ *for all continuous functions f.*

Proof. By the hypothesis there is an interval $[a', b']$ such that $\|\mu_n\| = \mu_n([a', b'])$ for all n. Moreover, by Remark 3.7, (i), of [3], p. 6, there are a, b such that $\mu^*(\{a\}) = \mu^*(\{b\}) = 0$ and $a < a'$, $b' < b$. Therefore $\mu^*([a, b]) = \mu([a, b])$.

We claim that $\mu \in$ M $([a, b])$. Suppose this is not true, i.e. $\mu([a,b]) < \|\mu\|$. Hence $\mu([a,b]^c) > 0$. Let $\varepsilon > 0$ such that $\varepsilon < \mu([a, b]^c)$. Then there exists an open interval $U \supset [a, b]$ such that $\mu(U) - \mu([a, b]) < \varepsilon$. Now, let $F = U^c$. Then we can consider a continuous function $g: \mathbf{R} \rightarrow [0, 1]$ with

$$g(x) = 0, \text{ if } x \in [a, b]$$

$$= 1, \text{ if } x \in F.$$

Then

$$S \int g\, d\mu \geq \mu(F) > \|\mu\| - (\mu([a, b]) + \varepsilon) > 0,$$

and $S \int g\, d\mu_n = 0$ for all n. Since $\mu_n \overset{w}{\rightarrow} \mu$, we get a contradiction. This proves the claim.

On noting that μ_n, $\mu \in$ M $([a, b])$, the thesis easily follows from Corollary 3.21. This completes the proof. ♦

3.24 PROPOSITION. *Let f be an unbounded continuous function. Then there are* μ, $\mu_n \in M^+_{00}$ *for all n such that* $\mu_n \overset{w}{\to} \mu$ *and the sequence* ($\int f\, d\mu_n$) *is unbounded (hence the sequence* ($\int f\, d\mu_n$) *does not converge to* $\int f\, d\mu$).

Proof. For instance, let Sup $f = +\infty$. Then there exists x_n such that $f(x_n) \geq 2^n$ for all n. Let

$$\mu_n = \left(1 - \frac{1}{n}\right) \mathbf{1}_0 + \frac{1}{n} \mathbf{1}_{x_n}$$

for all n. Obviously $\mu_n \in \mathbf{M}^+ [0, x_n] \subset \mathbf{M}^+_{00}$.

We claim that $\mu_n \overset{w}{\to} \mathbf{1}_0$. Let g be a bounded continuous function. Then we have

$$S \int g\, d\mu_n = \left(1 - \frac{1}{n}\right) g(0) + \frac{1}{n} g(x_n) \to g(0) = S \int g\, d\mathbf{1}_0.$$

This proves the claim.

On the other hand, by Theorem 3.13, we have

$$\int f\, d\mu_n = S \int_{[-1, x_n + 1]} f\, d\mu_n = \left(1 - \frac{1}{n}\right) f(0) + \frac{1}{n} f(x_n) \to +\infty.$$

This completes the proof. ♦

3.25 REMARK. Since, by Corollary 4.13 of [3], p.15, weak convergence and convergence in distribution are equivalent in $\mathbf{M}_0$, we can replace the former by the latter in all previous statements.

4. Means on Masses and Related Topics.

To begin with, we introduce the basic notion of mean on masses.

4.1 DEFINITION. A functional m from M' to R is called a *mean* on M' iff the following *consistency property*:

$$m(k_x) = x$$

holds for all $k \in \boldsymbol{R}^+$ and for all x such that $k_x \in$ **M'**.

4.2 REMARK. The set of all means on **M'** is a convex set, closed respect to the pointwise convergence of sequences.

Now, we are going to give the basic notions regarding means.

4.3 DEFINITION. A mean m is called an *internal mean* iff $m\,(\mu) \in \overline{F}$, whenever F is a support of μ.

4.4 REMARK. (i) Plainly a real functional m on **M'** satisfying internality condition is a mean.

(ii) The set of all internal means on **M'** is a convex set, closed with respect to the pointwise convergence of sequences.

Since in the next section we are going to study an important class of internal means, we give here only an example of a non internal mean.

4.5 EXAMPLE. Let **F** be the Borel σ-field on $\boldsymbol{R}$ and let **M'** $=$ **M** $^+[0, 1]$. Moreover let λ be the Lebesgue measure on **F**. Now let

$$m\,(\mu) = \frac{1}{\|\mu\|}\left[\int x\,\mu(dx) + \|\mu\| - \sum_x \mu(\{x\})\right].$$

Plainly m is a mean on **M'**. Let $\lambda'\,(F) = \lambda\,(F \cap [0, 1])$ for all F. Of course $\lambda' \in$ **M'**. Since λ' is a diffuse mass, we get

$$m(\lambda') = \int x\,\lambda'\,(dx) + 1 = \int_{[0,1]} x\,\lambda\,(dx) + 1 > 1$$

and hence m is not an internal mean on **M'**.

The following definition introduces the notion of continuity for a mean, with respect to the convergence in distribution of masses.

4.6 DEFINITION. Let m be a mean on **M'**. Then m is called a *continuous mean* iff $m\,(\mu_n) \to m\,(\mu)$ whenever $\mu_n \overset{d}{\to} \mu$.

Now we give two examples of non continuous means; the former is a non internal mean, the latter is an internal one.

4.7 EXAMPLES. (i) Keeping in mind Example 4.5, let

$$\mu_n(F) = n\lambda\left(F \cap \left[0, \frac{1}{n}\right]\right)$$

for all F and for all n. Plainly $m(\mu_n) \to 1$ and $\mu_n \xrightarrow{d} \mathbf{1}_0$. Then the mean m is not a continuous mean on noting that, by the consistency property, $m(\mathbf{1}_0) = 0$.

(ii) Let $\mathbf{M'} = \mathbf{M}^+_{oo}$ and let

$$m(\mu) = \frac{1}{\|\mu\|}\int x\,\mu(dx).$$

for all $\mu \in \mathbf{M'}$. Plainly m is a mean on $\mathbf{M'}$. By Proposition 3.24 and Remark 3.25, there are $\mu_n,\ \mu \in \mathbf{M'}$ such that $\mu_n \xrightarrow{d} \mu$ and $\lim_{n->+\infty} \int x\,\mu_n(dx) \neq \int x\,\mu(dx)$. On noting that $\|\mu_n\| \to \|\mu\|$, we can say that m is not a continuous mean. The internality of this mean will be proved in the following section.

In the previous example (ii), the continuity of the mean does not hold for a sequence of masses with supports invading the set $\boldsymbol{R}$. This is not surprising because, by Theorem 3.23, the same mean is continuous with respect to every sequence of masses with bounded support. All that suggests the following definition.

4.8 DEFINITION. Let m be a mean on $\mathbf{M'}$. Then m is called a *conditionally continuous mean* iff $m(\mu_n) \to m(\mu)$ whenever $\mu_n \xrightarrow{d} \mu$ and (μ_n) is a sequence with bounded support.

Now, we prove a lemma which is instrumental in proving some results in this section and in the following ones.

4.9 LEMMA. *Let $k_{nh} \geq 0$ for all n and for all $h = 1, \ldots, m$. Moreover let $k_{nh} \to k_h$ and $x_{nh} \to x_h$ for all $h = 1, \ldots, m$. Then we have*

$$\sum_{h=1}^{m} (k_{nh})_{x_{nh}} \xrightarrow{w} \sum_{h=1}^{m} (k_h)_{x_h}.$$

Proof. Let f be a bounded continuous function. Then by well known properties of S-integral we have:

$$\alpha = S \int f d \left[\sum_{h=1}^{m} (k_{nh})_{x_{nh}}\right] = \sum_{h=1}^{m} S \int f d\left[(k_{nh})_{x_{nh}}\right] = \sum_{h=1}^{m} f(x_{nh})\, k_{nh}$$

and hence, by hypothesis,

$$\alpha = \lim_{n\to+\infty} \sum_{h=1}^{m} f(x_h) k_h = S \int f d \left[\sum_{h=1}^{m} (k_h)_{x_h}\right].$$

This completes the proof. ♦

The following proposition shows that we cannot define a conditionally continuous mean on **M** $[a, b]$ whenever $a \neq b$. Hence, a fortiori, there is no conditionally continuous mean on **M**.

4.10 PROPOSITION. *Let* m *be a conditionally continuous mean on* **M'**. *Moreover let* $k_n \downarrow 0$ *and let* $x_1 \neq x_2$ *such that* $(k_n)_{x_1}, (k_n)_{x_2} \in$ **M'** *for all* n. *Then the null mass does not belong to* **M'**.

Proof. Assume this is not true, i.e. the null mass $\mu_0 \in$ **M'**. Since $k_n \downarrow 0$, by Lemma 4.9 and Remark 3.25, we get $(k_n)_{x_i} \xrightarrow{d} 0_{x_i} = \mu_0$, and so, by the conditional continuity of $m, x_i = m((k_n)_{x_i}) \to m(\mu_0)$, i.e. $m(\mu_0) = x_1 = x_2$ (Contradiction!). This completes the proof. ♦

Now, we introduce some notions of the associativity in order to pave the way for integral representation theorems which will be proved in the sequel.

4.11 DEFINITION. Let m be a mean on **M'**. Then we may consider the following *replacement property:*

$m(\alpha\mu_1 + \beta\mu_2) = m(\alpha\mu_1^0 + \beta\mu_2^0)$ whenever $m(\mu_i) = m(\mu_i^0)$ (i = 1,2) and $\alpha, \beta \geq 0$.

That being stated, the mean m is called a:

(i) *weakly associative mean* iff the replacement property holds whenever $\alpha + \beta = 1$ and $\|\mu_1\| = \|\mu_2\| = \|\mu_1^0\| = \|\mu_2^0\|$;

(ii) *associative mean* iff the replacement property holds whenever $\|\mu_1\| = \|\mu_1^0\|$ and $\|\mu_2\| = \|\mu_2^0\|$;

(iii) *strongly associative mean* iff the replacement property holds in any case.

4.12 REMARK. Plainly, all the notions of associativity are equivalent for means on $\mathbf{M'} = \mathbf{M_k}\,[a, b]$ for some a, b and k.

In the following proposition we link weakly associative means with associative ones.

4.13 PROPOSITION. *Let* m *be a mean on* $\mathbf{M}$. *Then the following statements are equivalent:*

(i) m *is an associative mean;*

(ii) m *is a weakly associative mean which satisfies the following multiplicative property:*

$$m\,(\alpha\mu) = m\,(\mu) \text{ for all } \alpha \in R^+ \text{ and for all } \mu.$$

Proof. (i) ⇒ (ii). Obviously m is a weakly associative mean. Moreover we have

$$m\,(\alpha\mu) = m\,[\alpha\,(\|\mu\|)_{m(\mu)}] = m\,[(\alpha\,\|\mu\|)_{m(\mu)}] = m\,(\mu).$$

This proves the desired implication.

(ii) ⇒ (i). Let $\|\mu_i\| = \|\mu_i^0\|$ and $m\,(\mu_i) = m\,(\mu_i^0)$ $(i = 1,2)$. Then, by the multiplicative property, we have

$$m\,(\alpha\mu_1 + \beta\mu_2) = m\left[\frac{\alpha\,\|\mu_1\|}{\alpha\,\|\mu_1\| + \beta\,\|\mu_2\|}\,\frac{\mu_1}{\|\mu_1\|} + \frac{\beta\,\|\mu_2\|}{\alpha\,\|\mu_1\| + \beta\,\|\mu_2\|}\,\frac{\mu_2}{\|\mu_2\|}\right]$$

and

$$m\left[\frac{\mu_i}{\|\mu_i\|}\right] = m\left[\frac{\mu_i^0}{\|\mu_i^0\|}\right] \quad (i = 1,2)\,.$$

Hence, by the weakly associative property, we get

$$m\,(\alpha\mu_1 + \beta\mu_2) = m\left[\frac{\alpha\,\|\mu_1^0\|}{\alpha\,\|\mu_1^0\| + \beta\,\|\mu_2^0\|}\,\frac{\mu_1^0}{\|\mu_1^0\|} + \frac{\beta\,\|\mu_2^0\|}{\alpha\,\|\mu_1^0\| + \beta\,\|\mu_2^0\|}\,\frac{\mu_2^0}{\|\mu_2^0\|}\right] =$$

$$= m\,(\alpha\mu_1^0 + \beta\mu_2^0)\,.$$

This proves the desired implication.

This completes the proof. ♦

4.14 REMARK. Keeping in mind the proof of the previous proposition we can say that the above equivalence also holds if we replace **M** by a set **M'** such that:

(i) $\mathbf{M'} \supset \mathbf{M_k}$ for some $k \in \boldsymbol{R}^+$;

(ii) $\|\mu\|_x \in \mathbf{M'}$ for all x whenever $\mu \in \mathbf{M'}$.

Moreover, we can also replace **M** by **M** $[a, b]$ whenever $m\,(\mu) \in [a, b]$ for all $\mu \in$ **M** $[a, b]$. Finally, we can also replace **M** by the set of all positive measures on the Borel σ-field on $\boldsymbol{R}$.

Some of the following examples show that the three notions of associativity are not pairwise equivalent and that not every mean is weakly associative.

4.15 EXAMPLES. (i) Let **F** be the Borel σ-field on $\boldsymbol{R}$ and let $\mathbf{M'} = \mathbf{M}^+\,[0,1]$. Moreover, let

$$m\,(\mu) = \frac{\int x\,\mu\,(dx)}{\|\mu\|} = \|\mu\| - \sum_x \mu\,\left(\{x\}\right)\,.$$

Plainly m is a weakly associative mean on **M'**. Since the multiplicative property does not hold for m, by the previous proposition and Remark 4.14, we get that m is not an associative mean and, a fortiori, m is not a strongly associative mean.

(ii) Keeping in mind Example 4.5, it is easy to prove that the mean m there considered is a weakly associative mean which satisfies the multiplicative property. Hence, by the previous proposition, m is an associative mean. To see that m is not a strongly associative mean it is sufficient to consider the following masses:

$$\mu_1 = \mathbf{1}_{\frac{1}{2}}\;,\; \mu_1^0 = \mathbf{2}_{\frac{1}{2}}\;,\; \mu_2 = \mu_2^0 = \mathbf{2}_1\,;$$

in fact we have $m\,(\mu_1 + \mu_2) \neq m\,(\mu_1^0 + \mu_2^0)$.

(iii) Let $\mathbf{M'} = \mathbf{M_1}\,[0,1] \cup \mathbf{M_2}\,[0,1]$. Moreover let

$$m(\mu) = \frac{1}{\|\mu\|} \int x \mu(dx)$$

for all $\mu \in$ **M'**. Plainly m is a mean on **M'**. Now, it is tedious but not difficult to prove that m is a strongly associative mean on **M'**.

(iv) Let **M'** = $\mathbf{M}^{+}[1,3]$. Moreover let

$$m(\mu) = \frac{\int x^2 \mu(dx)}{\int x \mu(dx)}$$

for all $\mu \in$ **M'**. Plainly m is a mean on **M'**. Now, we claim that m is not a weakly associative mean. Let

$$\mu_1 = \frac{1}{2}\left(\mathbf{1}_1 + \mathbf{1}_2\right), \mu_1^0 = \mathbf{1}_{\frac{5}{3}}, \mu_2 = \mu_2^0 = \mathbf{1}_3 .$$

On noting that $\|\mu_1\| = \|\mu_2\| = \|\mu_1^0\| = \|\mu_2^0\| = 1$, $m(\mu_1) = m(\mu_1^0)$ and $m\left(\frac{2}{3}\mu_1 + \frac{1}{3}\mu_2\right) \neq m\left(\frac{2}{3}\mu_1^0 + \frac{1}{3}\mu_2^0\right)$, we obtain the claim.

(v) Let **M'** = $\mathbf{M}^{+}_{oo}$ and let $m(\mu) = \mathrm{Inf}\,\{x : \mu(]x, +\infty[) = 0\}$, for all $\mu \in$ **M'**. Plainly m is a mean on **M'**, which satisfies the multiplicative property.

Now, we claim that

$$m(\mu_1 + \mu_2) = \max(m(\mu_1), m(\mu_2)).$$

Let x be such that $(\mu_1 + \mu_2)(]x, +\infty[) = 0$. Then we have $\mu_1(]x, +\infty[) = \mu_2(]x, +\infty[) = 0$ and hence $m(\mu_1), m(\mu_2) \leq x$. Therefore we get

$$m(\mu_1 + \mu_2) \geq \max(m(\mu_1), m(\mu_2)).$$

Now, by reductio ad absurdum, assume the converse inequality does not hold. Let, for instance, $m(\mu_1 + \mu_2) > m(\mu_1) \geq m(\mu_2)$. Then there exists x such that

$$m(\mu_1) \leq x < m(\mu_1 + \mu_2),\ \mu_1(]x, +\infty[) = \mu_2(]x, +\infty[) = 0.$$

Hence we get $(\mu_1 + \mu_2)(]x, +\infty[) = 0$ which is a contradiction. This proves the claim.

From the equality just proved, it follows that m is a strongly associative mean. Since $m(\mu)$ belongs to every bounded closed support of μ, we can say that m is an internal mean. Finally we observe that m is not a continuous mean. To see this it is sufficient to consider the following sequence of masses:

$$\mu_n = \left(1-\frac{1}{n}\right) 1_0 + \frac{1}{n} 1_1 .$$

(vi) Let M' = M $[a, b]$. Keeping in mind the previous example, let

$$m(\mu) = \text{Inf}\,\{x : \mu\,(]x, +\infty[) = 0\}, \text{ if } \mu \in \mathbf{M}^+$$

$$= a, \text{ otherwise,}$$

for all $\mu \in$ M'. Since the equality proved in the previous example and the multiplicative property also hold, we can say that m is an internal strongly associative mean which is not continuous. It is interesting to observe that we can also prove the non continuity of m by Proposition 4.10.

4.16 Remark. (i) Obviously the set of all continuous means on M' is a convex set. Moreover this set is not closed with respect to the pointwise convergence of sequences. To see this, let M' = $\{\mu \in \mathbf{M}^+ [0,1]$: μ positive measure$\}$. Moreover, let

$$m_n(\mu) = \left[\frac{\int x^n \mu\,(dx)}{\|\mu\|}\right]^{\frac{1}{n}}$$

for all $\mu \in$ M' and for all n. Plainly m_n is a mean on M' for all n.

Now, given $\mu \in$ M', it is well known that the sequence

$$\left(\int x^n \left[\frac{\mu}{\|\mu\|}\right](dx)\right)$$

converges to the essential supremum of the identity. Hence $m_n(\mu) \to m(\mu)$, where m is the mean considered in Examples 4.15, (v), which is not a continuous mean on M'.

Since, in the following section we are going to prove that m_n is continuous for all n, we can conclude that not every limit of continuous means is a continuous mean as well.

(ii) The set of all associative means on M' is not a convex set. To see this, let $\mathbf{M'} = \mathbf{M}^+[0,1]$ and let

$$m_1(\mu) = \frac{\int x\,\mu\,(dx)}{\|\mu\|}, \quad m_2(\mu) = \left[\frac{\int x^2\mu\,(dx)}{\|\mu\|}\right]^{\frac{1}{2}}$$

for all $\mu \in \mathbf{M'}$. As we shall see later, m_1 and m_2 are associative means on $\mathbf{M'}$. Now, let $m = \frac{1}{2}(m_1 + m_2)$ and let

$$\mu_1 = \frac{1}{2}(\mathbf{1}_1 + \mathbf{1}_2), \mu_1^0 = \mathbf{1}_{\frac{3+\sqrt{10}}{4}}, \mu_2 = \mu_2^0 = \mathbf{1}_3.$$

On noting that $\|\mu_1\| = \|\mu_2\| = \|\mu_1^0\| = \|\mu_2^0\| = 1, m(\mu_1) = m(\mu_1^0)$ and $m\left(\frac{2}{3}\mu_1 + \frac{1}{3}\mu_2\right) \neq m\left(\frac{2}{3}\mu_1^0 + \frac{1}{3}\mu_2^0\right)$, m is not a weakly associative mean.

(iii) We do not known if the set of all associative means on $\mathbf{M'}$ is a closed set with respect to the pointwise convergence of sequences.

Now we give the last definition of the section.

4.17 DEFINITION. Let m be a mean on $\mathbf{M'}$. Then m is called a *σ-associative mean* iff

$$m\left(\sum_{n=1}^{+\infty} \alpha_n \mu_n\right) = m\left(\sum_{n=1}^{+\infty} \alpha_n \mu_n^0\right)$$

whenever $\alpha_n \geq 0$, $\|\mu_n\| = \|\mu_n^0\|$ and $m(\mu_n) = m(\mu_n^0)$ for all n.

4.18 REMARK. Obviously every σ-associative mean is also associative. We do not know any example of an associative mean which is not a σ-associative mean. Now we conjecture that the following associative mean is not σ-associative. Let $\mathbf{F}$ be the Borel σ-field on $[0,1]$ and let λ be the Lebesgue measure on $[0,1]$. Moreover let $\mathbf{M'}$ be the set of all positive measures μ having the derivative $\frac{d\mu}{d\lambda}$ at the point $x_o \in [0,1]$. Then we can consider the following associative mean on $\mathbf{M'}$:

$$m(\mu) = \int x\,\mu\,(dx) + \frac{d\mu}{d\lambda}(x_0).$$

Keeping in mind Fubini's convergence theorem we think that this associative mean is not σ-associative.

The following two theorems link the associative property with the σ-associative property by means of the continuity property.

4.19 THEOREM. *Let* $\boldsymbol{m}$ *be a conditionally continuous associative mean on* $\mathbf{M}_{oo}^{+}$. *Then* $\boldsymbol{m}$ *is a* σ*-associative mean.*

Proof. Let $(\mu_n), (\mu_n^0)$ be two sequences in $\mathbf{M}_{oo}^{+}$ such that, $\mu = \sum_{n=1}^{+\infty} \alpha_n \mu_n$, $\mu^0 = \sum_{n=1}^{+\infty} \alpha_n \mu_n^0 \in \mathbf{M}_{oo}^{+}$, with $\alpha_n \geq 0$, $\|\mu_n\| = \|\mu_n^0\|$ and $\boldsymbol{m}(\mu_n) = \boldsymbol{m}(\mu_n^0)$ for all n.

Since

$$\|\mu\| = \sum_{n=1}^{+\infty} \alpha_n \|\mu_n\| = \|\mu^0\| = \sum_{n=1}^{+\infty} \alpha_n \|\mu_n^0\|,$$

the sequences above considered are sequences with bounded support. Therefore there are a, b such that $\mu, \mu^0, \mu_n, \mu_n^0 \in \mathbf{M}^{+}[a, b]$ for all n.

Let

$$\mu_n^1 = \sum_{i=1}^{n} \alpha_i \mu_i^0 + \sum_{i=n+1}^{+\infty} \alpha_i \mu_i .$$

Then $\mu_n^1 \in \mathbf{M}^{+}[a, b]$ for all n and, by the associative property, we get $\boldsymbol{m}(\mu_n^1) = \boldsymbol{m}(\mu)$ for all n. Now we claim that $\mu_n^1 \xrightarrow{d} \mu^0$. We have:

$$|\mu^0(F) - \mu_n^1(F)| = \left|\sum_{i=n+1}^{+\infty} \alpha_i \left(\mu_i^0(F) - \mu_i(F)\right)\right| \leq$$

$$\leq \sum_{i=n+1}^{+\infty} \alpha_i \left(|\mu_i^0(F)| + |\mu_i(F)|\right) \leq \sum_{i=n+1}^{+\infty} \alpha_i \left(\|\mu_i^0\| + \|\mu_i\|\right) =$$

$$= 2 \sum_{i=n+1}^{+\infty} \alpha_i \|\mu_i\| \to 0 .$$

Therefore $\mu_n^1 (F) \to \mu^0 (F)$ for all F and so we get the claim.

Whence, by the conditional continuity of $\boldsymbol{m}$, we obtain $\boldsymbol{m}(\mu) = \boldsymbol{m}(\mu_n^1) \to \boldsymbol{m}(\mu^0)$. This completes the proof. ♦

4.20 THEOREM. *Let $\boldsymbol{m}$ be a continuous associative mean on $\mathbf{M}_o^+$ (or $\mathbf{M}^+$). Then $\boldsymbol{m}$ is a σ-associative mean.*

Proof. Keeping in mind the previous proof it is sufficient to observe that $\mu_n^1 \in \mathbf{M}_o^+$ for all n. From the continuity of $\boldsymbol{m}$ we get the thesis. This completes the proof. ♦

4.21 REMARK. (i) Two similar theorems also hold if we replace the associative property by the weakly associative one or by the strongly associative one.

(ii) Keeping in mind Examples 4.15, (v), it is easy to prove that

$$\boldsymbol{m}\left[\sum_{n=1}^{+\infty} \mu_n\right] = \operatorname{Sup} \boldsymbol{m}(\mu_n) .$$

Therefore $\boldsymbol{m}$ is a σ-associative mean which is not conditionally continuous. Hence the converse of Theorem 4.19 does not hold.

Finally, in the next statements, we link associativity and continuity with internality.

4.22 THEOREM. *Let $\boldsymbol{m}$ be a weakly associative continuous mean on $\mathbf{M}_\mathbf{k}[a, b]$, with $k > 0$. Then $\boldsymbol{m}$ is an internal mean.*

Proof. The proof is carried out in the following steps.

1°. We claim that $\boldsymbol{m}$ is an internal mean on the set of all mixtures of two degenerate masses in $\mathbf{M}_\mathbf{k}[a, b]$. Assume this is not true. Then there are x, y, t_o such that $x < y$, $t_o \in [0,1]$ and $f(t_o) \notin [x, y]$, where

$$f(t) = \boldsymbol{m}\,[(1 - t)k_x + tk_y]$$

for all $t \in [0,1]$. Plainly $f(0) = x$ and $f(1) = y$.

We claim that f is a continuous function. Let $t_n \to t$. From Lemma 4.9 and Remark 3.25, it follows that $(1 - t_n)\, k_x + t_n k_y \xrightarrow{d} (1 - t)\, k_x + t\, k_y$.

Hence, by the continuity of m, we get $f(t_n) \to f(t)$, i.e. f is continuous. This proves the claim.

Now, for instance, let $f(t_o) > y$. Then, by the continuity of f, there is $t_1 \in]0, t_o[$ such that $f(t_1) = y$. Hence $t_o = (1 - \alpha) t_1 + \alpha 1$. Therefore, by the weakly associative property, we have:

$$y = m(k_y) = m\,[(1-\alpha)\,k_{f(t_1)} + \alpha k_y] = m\,\{(1-\alpha)\,[(1-t_1)\,k_x + t_1\,k_y] + a\,k_y\} = m\,[(1-\alpha+\alpha t_1 - t_1)\,k_x + (\alpha - \alpha t_1 + t_1)\,k_y] = f((1-\alpha)\,t_1 + \alpha) = f(t_o),$$

i.e. a contradiction.

Of course, in the same way, we get a contradiction if $f(t_o) < x$. This proves the claim.

2°. We claim that m is an internal mean on the set of all mixtures of degenerate masses in $\mathbf{M_k}\,[a, b]$. For simplicity we prove here the internality only for the mixtures of three degenerate masses. Given $a \le x_1 < x_2 < x_3 \le b$, by the weakly associative property, we have:

$$m\left(\alpha_1 k_{x_1} + \alpha_2 k_{x_2} + \alpha_3 k_{x_3}\right) =$$

$$= m\left[\alpha_1 k_{x_1} + (\alpha_2+\alpha_3)\left(\frac{\alpha_2}{\alpha_2+\alpha_3} k_{x_2} + \frac{\alpha_3}{\alpha_2+\alpha_3} k_{x_3}\right)\right] =$$

$$= m\left[\alpha_1 k_{x_1} + (\alpha_2+\alpha_3) k_{m\left(\frac{\alpha_2}{\alpha_2+\alpha_3} k_{x_2} + \frac{\alpha_3}{\alpha_2+\alpha_3} k_{x_3}\right)}\right],$$

and hence, by 1°,

$$m\left(\alpha_1 k_{x_1} + \alpha_2 k_{x_2} + \alpha_3 k_{x_3}\right) \in \left[x_1, m\left(\frac{\alpha_2}{\alpha_2+\alpha_3} k_{x_2} + \frac{\alpha_3}{\alpha_2+\alpha_3} k_{x_3}\right)\right]$$

and

$$m\left(\frac{\alpha_2}{\alpha_2+\alpha_3} k_{x_2} + \frac{\alpha_3}{\alpha_2+\alpha_3} k_{x_3}\right) \in [x_2, x_3].$$

This proves the claim.

3°. We claim that every mass of $\mathbf{M_k}\,[x, y]$, with $a \le x < y \le b$, is a weak limit of a sequence of mixtures of degenerated masses in $\mathbf{M_k}\,[x, y]$. Given n, let $P_n = \left\{F_1^{(n)}, \ldots, F_{h_n}^{(n)}\right\}$ be a partition of $\left[x - \frac{y-x}{2n}, y + \frac{y-x}{2n}\right]$ in

F such that the diameter of $F_i^{(n)}$ is equal to $\frac{y-x}{n}$ for all i. Moreover, let $x_1^{(n)} = x, x_{h_n}^{(n)} = y$ and $x_i^{(n)} \in F_i^{(n)}$ for all i.

Given a mass $\mu \in \mathbf{M_k}\,[x, y]$, we can consider the following sequence of masses:

$$\mu_n = \frac{\mu\,(F_1^{(n)})}{\|\mu\|} k_{x_1^{(n)}} + \dots + \frac{\mu\,(F_{h_n}^{(n)})}{\|\mu\|} k_{x_{h_n}^{(n)}}$$

Now, we claim that $\mu_n \overset{w}{\rightarrow} \mu$. Given a bounded continuous function f_1, f_1 is an uniformly continuous function on $\left[x - \frac{y-x}{2}, y + \frac{y-x}{2}\right]$. Let $\varepsilon > 0$. Then there exists a real number $\delta > 0$ such that $|f_1\,(x_1)\text{-}f_1\,(x_2)| < \varepsilon$ for all $x_1, x_2 \in \left[x - \frac{y-x}{2}, y + \frac{y-x}{2}\right]$ with $|x_1 - x_2| < \delta$. Then we have:

$$|S \int f_1\, d\mu - S \int f_1\, d\mu_n\,| = |S \int f_1\, d\mu - \sum_{i=1}^{h_n} f_1\,(x_i^{(n)})\,\mu\,(F_i^{(n)})| =$$

$$= |\sum_{i=1}^{h_n} S \int_{F_i^{(n)}} \left[f_1 - f_1(x_i^{(n)})\right] d\mu| \le \sum_{i=1}^{h_n} S \int_{F_i^{(n)}} |f_1 - f_1(x_i^{(n)})|\, d\mu <$$

$$< \sum_{i=1}^{h_n} \varepsilon\mu(F_i^{(n)}) = \varepsilon\|\mu\| = \varepsilon k$$

for all n such that $n > \frac{y-x}{\delta}$. This proves the claim.

4°. We claim that $\boldsymbol{m}$ is an internal mean on $\mathbf{M_k}\,[a, b]$. Let $\mu \in \mathbf{M_k}\,[a, b]$ and let F be a closed support of μ. Then there are $a \le x < y \le b$ such that $[x, y] \subset F$ and $\mu \in \mathbf{M_k}\,[x, y]$. Therefore it is sufficient to prove $\boldsymbol{m}\,(\mu) \in [x, y]$.

Now, let (μ_n) be the sequence considered in 3°. Since, by Remark 3.25, $\mu_n \overset{d}{\rightarrow} \mu$, by the continuity of $\boldsymbol{m}$, we have $\boldsymbol{m}\,(\mu_n) \rightarrow \boldsymbol{m}\,(\mu)$. Moreover, since, by 2°, $\boldsymbol{m}\,(\mu_n) \in [x, y]$ for all n, we get $\boldsymbol{m}\,(\mu) \in [x, y]$.

This completes the proof. ♦

The following corollary follows quite easily from the previous theorem.

4.23 COROLLARY. *Let m be a weakly associative conditionally continuous mean on* $M^+_{0\sigma}$. *Then m is an internal mean.*

Since, in general, convergence in distribution is weaker than weak convergence, we end this section with the following open problem: find a non continuous mean m which is continuous with respect to the weak convergence, i.e. $m(\mu_n) \to m(\mu)$ whenever $\mu_n \overset{w}{\to} \mu$.

5. The G-Mean.

In what follows, we assume that g, with or without indices, is a strictly monotone continuous real function on R.

To begin with, we introduce the following definition.

5.1 DEFINITION. Let $M' \subset M^+$ such that $\frac{\int g\, d\mu}{\|\mu\|} \in g(R)$ for all $\mu \in M'$.

Then the following mean

$$m_g(\mu) = g^{-1}\left[\frac{\int g\, d\mu}{\|\mu\|}\right]$$

on M', is called a *g-mean* on M'.

5.2 REMARK. (i) Let g be a bounded function. Then g is regular at infinity and hence, by Corollary 3.16, g is improperly S-integrable with respect to any μ. Now, by an example, we show that the condition

$$\frac{\int g\, d\mu}{\|\mu\|} \in g(R)$$

may not hold. Keeping in mind the mass μ considered in Remark 3.2, (i), it is sufficient to note that $\int g\, d\mu = 0$ for all g.

(ii) Let f be a continuous function such that $f > 0$. Moreover let $\mu \in M^+_0$. Then there is a bounded set F such that $\mu(F) > 0$. Therefore $S\int_F f\, d\mu > 0$ and hence $\int f\, d\mu > 0$.

Now, let $\operatorname{Inf} g(R) = a$ and $\operatorname{Sup} g(R) = b$. Therefore $g - a > 0$ and $b - g > 0$. Hence

$$a\,\|\mu\| < \int g\, d\mu < b\,\|\mu\|$$

for all $\mu \in M_o^+$. Since, by Theorem 3.18, $\int g\, d\mu$ exists for all $\mu \in M_o^+$, the previous inequalities assure that the g-mean may be defined on M_o^+, whenever g is bounded.

(iii) Let g be an unbounded function. Then $\int g\, d\mu$ does not exist for all $\mu \in M_o^+$, as we can see by Theorem 3.18. Therefore there is no g-mean on $M' \supset M_o^+$.

(iv) Let g be a bounded function. Then, by Corollary 3.16, g is S-integrable and hence it is possible to consider the mean

$$g^{-1}\left[\frac{S\int g\, d\mu}{\|\mu\|}\right]$$

whenever $\dfrac{S\int g\, d\mu}{\|\mu\|} \in g\,(\boldsymbol{R})$.

Keeping in mind Corollary 3.16, we can note that this mean is different from m_g, in general. On the other hand, by Corollary 3.17, the two means are equal on $M' \cap M_o^+$.

Now, we are going to look into the basic properties of g-means.

5.3 THEOREM. *Every g-mean is:*

(i) internal;

(ii) conditionally continuous. Moreover m_g is a continuous mean on $M' \cap M_o^+$ whenever g is a bounded function;

(iii) associative;

(iv) σ-associative on M_{oo}^+. Moreover m_g is a σ-associative mean on M_o^+ whenever g is a bounded function. Finally, no g-mean is strongly associative on M' if $k_x, h_x, k_y, h_y, k_x + h_y, h_x + k_y \in M'$ for some $x \neq y$ and $k \neq h$ with $h, k \in R^+$.

Proof. (i) For instance, let g be an increasing function. Let F be a support of $\mu \in M'$ such that $b = \operatorname{Sup} F$. Since $\mu\,(\overline{F}) = \|\mu\|$, by Lemma 3.12, we obtain:

$$\int g\, d\mu = \int_{\overline{F}} g\, d\mu = \lim_{a' \to -\infty,\, b' \to +\infty} S\int_{[a',b']} I_{\overline{F}}\, g\, d\mu =$$

$$= \lim_{a' \to -\infty,\, b' \to +\infty} S\int_{\overline{F}} I_{[a',b'] \cap \overline{F}}\; g\, d\mu\,.$$

On noting that $[a', b'] \cap \overline{F} = [a', b]$ for all $b' > b$, we get

$$\int g\, d\mu = \lim_{a' \to -\infty} S \int_{\overline{F}} I_{[a', b]}\, g\, d\mu .$$

From the Mean Value Theorem for the S-integral, we obtain

$$S \int_{\overline{F}} I_{[a', b]}\, g\, d\mu \leq g(b) \, \|\mu\| ;$$

hence $\int g\, d\mu \leq g(b) \|\mu\|$, and so $m_g(\mu) \leq b$.

In a similar way we can prove that $m_g(\mu) \geq \operatorname{Inf} F$. This proves the internality of m_g.

(ii) First we prove the conditional continuity of m_g. Let (μ_n) be a sequence with bounded support such that $\mu_n \xrightarrow{d} \mu$. Then there are a, b such that $\mu_n, \mu \in \mathrm{M}\,[a, b]$.

Now let $\mu' \in \mathrm{M}\,[a, b]$. Then, by Theorem 3.13, we have

$$\int g\, d\mu' = \int_{[a', b']} g\, d\mu'$$

with $a' < a$ and $b' > b$. Let g' be a bounded function such that $g'|_{[a', b']} = g|_{[a', b']}$. Hence, by Theorem 3.13 and Proposition 3.20, we get

$$\int g\, d\mu' = \int g' d\mu' .$$

Since $\mu_n, \mu \in \mathrm{M}\,[a, b]$ and $\mu_n \xrightarrow{d} \mu$, by Corollary 3.8, Remark 3.25 and the previous equality, we have

$$\int g\, d\mu_n = S \int g' d\mu_n \to S \int g' d\mu = \int g\, d\mu .$$

On noting that $\|\mu_n\| \to \|\mu\|$ and g^{-1} is a continuous function, we can say that m_g is conditionally continuous.

The last part of (ii) easily follows from Corollary 3.8.

(iii) Let $\|\mu_i\| = \|\mu_i^o\|$ and $m_g(\mu_i) = m_g(\mu_i^o)$ $(i = 1,2)$. Since g is an injective function it is sufficient to prove that

$$\int g\, d(\alpha\mu_1 + \beta\mu_2) = \int g\, d(\alpha\mu_1^o + \beta\mu_2^o) .$$

This equality easily follows from the bilinearity of the improper S-integral.

(iv) The σ-associativity of m_g on M_{oo}^+ easily follows from Theorem 4.19 and (ii), (iii). To prove the σ-associativity of m_g on M_o^+ let g be a bounded function and let (μ_n) be a sequence in M_o^+ such that $\mu = \sum_{n=1}^{+\infty} \alpha_n \mu_n \in M_o^+$ with $\alpha_n \geq 0$ for all n. Since

$$\mu_n^o(F) = \sum_{i=1}^{n} \alpha_i \mu_i(F) \to \mu(F)$$

for all F, we have $\mu_n^o \overset{d}{\to} \mu$ and so, by (ii), we get

$$\sum_{i=1}^{n} \alpha_i \int g\, d\mu_i = \int g\, d\mu_n^o \to \int g\, d\mu\,,$$

i.e. $\sum_{n=1}^{+\infty} \alpha_n \int g\, d\mu_n = \int g\, d\mu$.

On noting that $\|\mu\| = \sum_{n=1}^{+\infty} \alpha_n \|\mu_n\|$, we can say that m_g is a σ- associative mean. This completes the proof of (iv).

Finally, we prove the last statement of the theorem. Let $x \neq y$ and $h \neq k$ such that $h, k \in R^+$ and $k_x, k_y, h_x, h_y, k_x + h_y, h_x + k_y \in$ M'. Assume, by reductio ad absurdum, that m_g is strongly associative. Then we have $m_g(k_x + h_y) = m_g(h_x + k_y)$ and so $k\,g(x) + h\,g(y) = h\,g(x) + k\,g(y)$. Therefore we have $g(x) = g(y)$ (Contradiction!).

This completes the proof.♦

5.4 Remark. Keeping in mind Remark 4.16, (ii), we can say that the set of all g-means on M' is not a convex set. Moreover, keeping in mind Remark 4.16, (i), we can see that this set is not closed with respect to the pointwise convergence of sequences.

The following two theorems characterize the equality of two g-means, by means of non degenerate affine transformations.

5.5 Theorem. *Let* M' $=$ $M_k[a, b]$, *with* $k > 0$. *In order that* $m_{g_1} = m_{g_2}$ *it is necessary and sufficient that* $g_2|_{[a,b]} = \alpha\, g_1|_{[a,b]} + \beta$, *with* $\alpha \neq 0$.

Proof. Since the sufficiency is easy to verify, we prove here only the necessity. Let $m_{g_1} = m_{g_2}$. Moreover let

$$\mu_t = \frac{t-a}{b-a} k_a + \frac{b-t}{b-a} k_b ,$$

for all $t \in [a, b]$.

Since $m_{g_1}(\mu_t) = m_{g_2}(\mu_t)$ for all t, by properties of the improper S-integral, we have

$$g_1^{-1}\left[\frac{t-a}{b-a} g_1(a) + \frac{b-t}{b-a} g_1(b)\right] = g_2^{-1}\left[\frac{t-a}{b-a} g_2(a) + \frac{b-t}{b-a} g_2(b)\right] = x(t)$$

for all $t \in [a, b]$. Now, adapting the proof of Theorem 83 of [4], p.66, we get easily the thesis. This completes the proof. ♦

5.6 THEOREM. *Let* $\mathbf{M}' \supset \mathbf{M}_{oo}^{+}$. *In order that* $m_{g_1} = m_{g_2}$ *it is necessary and sufficient that* $g_2 = \alpha g_1 + \beta$, *with* $\alpha \neq 0$.

Proof. Since the sufficiency is obvious, we prove here only the necessity. Let $m_{g_1} = m_{g_2}$. Then given $a < b$, by the previous theorem, we have $g_2|_{[a,b]} = \alpha g_1|_{[a,b]} + \beta$ for some $\alpha \neq 0$ and β. Now, we claim that $g_2 = \alpha g_1 + \beta$. Let $x \notin [a, b]$. Let, for instance, $x > b$. Then, from the same theorem, we get $g_2|_{[a,x]} = \alpha' g_1|_{[a,x]} + \beta'$ for some $\alpha' \neq 0$ and β'. Hence

$$g_2|_{[a,b]} = \alpha g_1|_{[a,b]} + \beta$$

$$= \alpha' g_1|_{[a,b]} + \beta',$$

and so $(\alpha - \alpha') g_1|_{[a,b]} = \beta' - \beta$. Since g_1 is a strictly monotone function, we get the thesis. This completes the proof. ♦

5.7 REMARK. (i) Keeping in mind the proofs of the previous two theorems, we can say that:

- the hypothesis $\mathbf{M}' = \mathbf{M}_{\mathbf{k}}[a, b]$ in Theorem 5.5 can be weakened by the following one:

there is $k > 0$ such that $\mathbf{M}'$ includes all mixtures of k_a and k_b;

- the hypothesis $\mathbf{M}' \supset \mathbf{M}_{oo}^{+}$ in Theorem 5.6 can be weakened by the following one:

there is $k > 0$ such that **M'** includes all mixtures of k_a and k_b for all a, b.

(ii) since $-g$ increases if g decreases, we may always suppose that the g involved in m_g is an *increasing* function whenever $\mathbf{M'} = \mathbf{M_k}[a, b]$ or $\mathbf{M'} \supset \mathbf{M}^+_{oo}$.

6. The Basic Lemma.

Now, we are going to prove a basic lemma which paves the way to some de Finetti-Kolmogoroff-Nagumo type integral representation theorems related with important classes of means.

6.1 LEMMA. (The Basic Lemma) *Let* m *be a mean on* $\mathbf{M_k}[a, b]$, *with* $k > 0$. *Then the following statements are equivalent:*

(i) m *is a continuous weakly associative mean;*

(ii) there is a bounded strictly monotone continuous real function g_k *on* $\boldsymbol{R}$ *such that*

$$m(\mu) = g_k^{-1}\left[\frac{S\int g_k\, d\mu}{\|\mu\|}\right]$$

for all μ.

Moreover, $g_k|_{[a,\, b]}$ *is unique up to non degenerate affine transformations.*

Proof. Since (ii) $\Rightarrow$ (i) immediately follows from Theorem 5.3, we prove here only the implication (i) $\Rightarrow$ (ii). Let f_k be the following function from $[0,1]$ to $\boldsymbol{R}$:

$$f_k(t) = m\,[(1 - t)\,k_a + tk_b].$$

Now, the proof is carried out in the following steps.

1°. Since, by Theorem 4.22, $m(\mu) \in [a, b]$ for all μ, we get $f_k(t) \in [a, b]$ for all t.

2°. We claim that:

$$m\left(\alpha_1 k_{f_k(t_1)} + \ldots + \alpha_n k_{f_k(t_n)}\right) = f_k(\alpha_1 t_1 + \ldots + \alpha_n t_n)$$

for all $t_i, \alpha_i \geq 0$ $(i = 1, \ldots, n)$ such that $\alpha_1 + \ldots + \alpha_n = 1$ and for all n.

By the weakly associative property we get:

$$m\left(\alpha_1 k_{f_k(t_1)} + \ldots + \alpha_n k_{f_k(t_n)}\right) =$$

$$= m\ \{\alpha_1\ [(1 - t_1)\ k_a + t_1 k_b] + \ldots + \alpha_n\ [(1 - t_n)\ k_a + t_n k_b]\} =$$

$$= m\ \{[\alpha_1\ (1 - t_1) + \ldots + \alpha_n(1 - t_n)]\ k_a + [\alpha_1 t_1 + \ldots + \alpha_n t_n]\ k_b\} =$$

$$= f_k\ (\alpha_1 t_1 + \ldots + \alpha_n t_n).$$

This proves the claim.

3°. We claim that f_k is a continuous function. See step 1° in the proof of Theorem 4.22 (put $a = x$ and $b = y$).

4°. We claim that f_k is an injective function. First we show that:

(*) if $f_k\ (t_1) = f_k\ (t_2)$ with $t_1 < t_2$, then f_k is a constant function on $[t_1, t_2]$.

Let $f_k\ (t_1) = f_k\ (t_2)$ with $t_1 < t_2$. Moreover, let $t \in [t_1, t_2]$. Then there is $\alpha \in [0,1]$ such that $t = (1 - \alpha)\ t_1 + \alpha t_2$. By 2°, we have

$$f_k(t) = m\left[(1-\alpha)\ k_{f_k(t_1)} + \alpha\ k_{f_k(t_2)}\right] = m\ (k_{f_k(t_1)}) = f_k(t_1)\ .$$

This proves the claim (*).

Now, we suppose that f_k is not an injective function, i.e. $f_k\ (t_1) = f_k\ (t_2)$ for some $t_1 < t_2$. We are going to prove that f_k is a constant function on $]0,1[$, which is a contradiction, on noting that $f_k\ (0) = a < b = f_k\ (1)$ and, by 3°, f_k is a continuous function.

Obviously $t_1 > 0$ or $t_2 < 1$. Let, for instance, $t_2 < 1$. Then we consider the following real sequence (x_n):

$$x_1 = t_1\ ,$$

$$x_{2n} = 2x_{2n-1} - t_1\ , \quad \text{if } \ 2x_{2n-1} - t_1 < 1\ ,$$

$$= 1\ , \qquad \text{otherwise,}$$

$$x_{2n+1} = \frac{x_{2n-1} + x_{2n}}{2}\ , \quad \text{if } x_{2n} < 1\ ,$$

$$= \frac{x_{2n-1} + 1}{2}\ , \qquad \text{otherwise.}$$

Plainly, we can see that there is a natural number n_0 such that

$$x_{2n+1} = \frac{x_{2n-1} + 1}{2}$$

for all $n \geq n_0$. Hence

$$x_{2n+1} = x_{2n_0-1} + \left(\frac{1}{2} + \dots + \frac{1}{2^{n-n_0+1}}\right)\left(1 - x_{2n_0-1}\right)$$

for all $n \geq n_0$ and whence $x_{2n+1} \uparrow 1$. Therefore, by (*), to prove that f_k is a constant function on $[t_1, 1[$, it is sufficient to verify that $f_k(x_{2n+1}) = f_k(t_1)$ for all n; we prove it by induction. Since $f_k(t_1) = f_k(t_2)$ we immediately get $f_k(x_1) = f_k(t_1)$. Assume the induction hypothesis: $f_k(x_{2n+1}) = f_k(t_1)$.

Let $x_{2n+2} < 1$. Then $x_{2n+2} = 2x_{2n+1} - t_1$ and

$$x_{2n+3} = \frac{x_{2n+1} + x_{2n+2}}{2}.$$

Hence, by 2°, we have

$$f_k\left(x_{2n+3}\right) = m\left[\frac{1}{2}k_{f_k(x_{2n+1})} + \frac{1}{2}k_{f_k(x_{2n+2})}\right] = m\left[\frac{1}{2}k_{f_k(t_1)} + \frac{1}{2}k_{f_k(x_{2n+2})}\right] =$$

$$= f_k\left(\frac{t_1 + x_{2n+2}}{2}\right) = f_k(x_{2n+1}) = f_k(t_1).$$

Now, let $x_{2n+2} \geq 1$. Then

$$x_{2n+3} = \frac{x_{2n+1}+1}{2}$$

and hence

$$f_k(x_{2n+3}) = f_k\left(\frac{t_1+1}{2}\right).$$

Since $2x_{2n+1} - t_1 \geq 1$, we have $\frac{t_1+1}{2} \in [t_1, x_{2n+1}]$ and hence, by (*),

$f_k\left(\frac{t_1+1}{2}\right) = f_k(t_1)$. This completes the proof by induction.

By similar arguments we can prove that f_k is a constant function on $]0, t_2]$. Hence f_k is a constant function on $]0, 1[$. This contradiction establishes the claim.

5°. Since f_k is an injective continuous function and $f_k(0) = a < b = f_k(1)$ there is a bounded increasing continuous function g_k on $\mathbf{R}$ such that $g_k|_{[a,b]} = f_k^{-1}$ and $g_k([a, b]) = [0,1]$.

6°. We claim that $g_k \circ \boldsymbol{m}$ is a linear functional on $\mathbf{M_k}[a, b]$. Let $\mu_1, \mu_2 \in \mathbf{M_k}[a, b]$ and let $\alpha \in [0,1]$. Then, by 1° and 5°, there are $t_i \in [0,1]$ such that $f(t_i) = \boldsymbol{m}(\mu_i)$ $(i = 1,2)$. Now, by the weak associative property and 2°, we have

$$(g_k \circ \boldsymbol{m})[(1-\alpha)\mu_1+\alpha\mu_2] = g_k\left\{\boldsymbol{m}[(1-\alpha)\,\boldsymbol{k}_{\boldsymbol{m}(\mu_1)}+\alpha\,\boldsymbol{k}_{\boldsymbol{m}(\mu_2)}]\right\} =$$

$$= g_k\left\{\boldsymbol{m}[(1-\alpha)\,\boldsymbol{k}_{f_k(t_1)}+\alpha\,\boldsymbol{k}_{f_k(t_2)}]\right\} =$$

$$= g_k\left\{f_k[(1-\alpha)t_1 + \alpha t_2]\right\} = (1-\alpha)\,t_1 + \alpha t_2 =$$

$$= (1-\alpha)g_k(\boldsymbol{m}(\mu_1)) + \alpha\, g_k(\boldsymbol{m}(\mu_2)) =$$

$$= (1-\alpha)(g_k \circ \boldsymbol{m})(\mu_1)+\alpha\,(g_k \circ \boldsymbol{m})(\mu_2)$$

This proves the claim.

7°. We claim that every mass of $\mathbf{M_k}[a, b]$ is a weak limit of a sequence of mixtures of degenerate masses in $\mathbf{M_k}[a, b]$. See Step 3° in the proof of Theorem 4.22 (put $a = x$ and $b = y$).

8°. We claim that

$$\boldsymbol{m}(\mu) = g_k^{-1}\left[\frac{S\int g_k\, d\mu}{k}\right]$$

for all $\mu \in \mathbf{M_k}[a, b]$.

Given $\mu \in \mathbf{M_k}[a, b]$, let (μ_n) be the sequence considered in 7°. Since, by Remark 3.25, $\mu_n \xrightarrow{d} \mu$ we have, by the continuity of $\boldsymbol{m}$ and g_k, $(g_k \circ \boldsymbol{m})(\mu_n) \to (g_k \circ \boldsymbol{m})(\mu)$. Now, from 6° we get

$$(g_k \circ \boldsymbol{m})(\mu_n) = \frac{\mu(F_1^{(n)})}{k}(g_k \circ \boldsymbol{m})(\boldsymbol{k}_{x_1^{(n)}}) + \ldots + \frac{\mu(F_{h_n}^{(n)})}{k}(g_k \circ \boldsymbol{m})(\boldsymbol{k}_{x_{h_n}^{(n)}}) =$$

$$= \frac{1}{k} [\mu (F_1^{(n)}) g_k (x_1^{(n)}) + \ldots + \mu(F_{h_n}^{(n)}) g_k (x_{h_n}^{(n)})] = \frac{1}{k} S \int g_k \, d\mu_n$$

and hence $\frac{1}{k} S \int g_k \, d\mu_n \to (g_k \circ \boldsymbol{m}) (\mu)$.

Since g_k is a bounded continuous function, by Corollary 3.17, g_k is S-integrable with respect to μ, μ_n for all n. Then, from $\mu_n \overset{w}{\to} \mu$, we get $S \int g_k \, d\mu_n \to S \int g_k \, d\mu$ and hence

$$\boldsymbol{m} (\mu) = g_k^{-1} \left[\frac{S \int g_k \, d\mu}{k} \right].$$

This proves the claim.

This proves the desired implication.

The last part of the theorem immediately follows from Theorem 5.5. This completes the proof. ♦

6.2 REMARK. (i) Keeping in mind Corollary 3.17 and Proposition 3.20, instead of S-integral and g_k given in (ii) of the Basic Lemma, we can consider improper S-integral of any strictly monotone continuous function g_k^o, *bounded or not*, such that $g_k^o |_{[a,\, b]} = g_k |_{[a,\, b]}$. Thus we can see $\boldsymbol{m}$ as a g_k^o-mean on $\mathbf{M_k}$ $[a, b]$.

(ii) Keeping in mind the proof of the Basic Lemma we may replace in it the set $\mathbf{M_k}$ $[a, b]$ with $\mathbf{M'} \subset \mathbf{M_k}$ $[a, b]$ such that $\mathbf{M'}$ includes all the mixtures of degenerate masses of $\mathbf{M_k}$ $[a, b]$.

(iii) Let $\mathbf{M'}$ be the set of all σ-additive probabilities on $[a, b]$. It is well known that there is a one-to-one correspondence between $\mathbf{M'}$ and the set Δ $[a, b]$ of all probability distribution functions with support $[a,b]$, preserving the weak convergence. Therefore each mean on $\mathbf{M'}$ defines a corresponding de Finetti type mean on Δ $[a, b]$. Since this correspondence preserves the associativity, the de Finetti-Kolmogoroff-Nagumo Representation Theorem, the Basic Lemma (with $k = 1$) and (ii) assure that the continuity and the strictly monotonicity are equivalent notions in the context of associative means on Δ $[a, b]$.

Since $\mathbf{M'} \subset \mathbf{M_1}$ $[a, b]$ our Basic Lemma may be seen as a generalization of the de Finetti-Kolmogoroff-Nagumo Representation Theorem to the means on the set of all distribution functions F such that $F(b + 0) - F(a - 0) = k$.

(iv) Of course we may always suppose that g_k involved in the Basic

Lemma is an increasing function.

7. Axiomatic Treatment of G-Means.

Now we have developed enough machinery to give our de Finetti-Kolmogoroff-Nagumo type integral representation theorems related with some important classes of means. In this way we show how the g-means may be defined axiomatically, i.e. by means of its characteristic properties. The first characterization theorem is an easy consequence of the Basic Lemma.

7.1 THEOREM. *Let* $\boldsymbol{m}$ *be a mean on* $\mathrm{M}^+[a, b]$. *Then the following statements are equivalent:*

(i) $\boldsymbol{m}$ *is a weakly associative mean. Moreover* $\boldsymbol{m}$ *is continuous on* $\mathrm{M_k}\,[a, b]$ *for all* $k > 0$ *(i.e.* $\boldsymbol{m}(\mu_n) \to \boldsymbol{m}(\mu)$ *whenever* $\mu_n \xrightarrow{d} \mu$, $\|\mu_n\| = \|\mu\|$ *for all* n*);*

(ii) there is a real function g on $\boldsymbol{R} \times \boldsymbol{R}^+$ *such that the function* $g_k = g(\cdot, k)$ *is a strictly monotone continuous function for all* $k > 0$ *and*

$$\boldsymbol{m}(\mu) = g_{\|\mu\|}^{-1}\left[\frac{\int g_{\|\mu\|}\,d\mu}{\|\mu\|}\right] = m_{g_{\|\mu\|}}(\mu)$$

for all μ.

7.2 THEOREM. *Let* $\boldsymbol{m}$ *be a mean on* $\mathrm{M}^+[a, b]$. *Then the following statements are equivalent:*

(i) $\boldsymbol{m}$ *is a continuous weakly associative mean;*

(ii) there is a real continuous function g on $\boldsymbol{R} \times \boldsymbol{R}^+$ *such that the function* $g_k = g(\cdot, k)$ *is strictly increasing for all* $k > 0$ *and*

$$\boldsymbol{m}(\mu) = m_{g_{\|\mu\|}}(\mu)$$

for all μ.

Proof. (i) $\Rightarrow$ (ii). Given $k > 0$ let f_k be the function considered in the proof of the Basic Lemma. Moreover let h be a bounded increasing continuous function on $\boldsymbol{R}$ such that $h(a) = 0$ and $h(b) = 1$. Finally, let

$$g_k(x) = f_k^{-1}(x), \quad \text{if } x \in [a, b],$$

$$g_k(x) = h(x), \qquad \text{if } x \notin [a, b]$$

for all $k > 0$; hence g_k is a bounded strictly increasing continuous function.

To prove the desired implication it is sufficient to verify that the continuity of the function $g(t, k) = g_k(t)$ holds for all (t, k). Let $(t_n, k_n) \to (t, k)$, with $k_n, k > 0$ for all n. From Lemma 4.9 and Remark 3.25 we get

$$(1 - t)(k_n)_a + t(k_n)_b \overset{d}{\to} (1 - t)k_a + tk_b$$

for all $t \in [0,1]$ and hence, by the continuity of $\boldsymbol{m}$, we have $f_{k_n} \to f_k$ in $[0,1]$. Therefore, by Corollary 8.3, (g_{k_n}) is uniformly convergent to g_k in $[a, b]$ and hence, by the definition of g_k, (g_{k_n}) is uniformly convergent to g_k in $\boldsymbol{R}$.

Let $\varepsilon > 0$. Then there is n' such that

$$|g_{k_n}(x) - g_k(x)| < \varepsilon, \; |g_k(t_n) - g_k(t)| < \varepsilon$$

for all $n > n'$ and for all x. On noting that

$$|g_{k_n}(t_n) - g_k(t)| \leq |g_{k_n}(t_n) - g_k(t_n)| + |g_k(t_n) - g_k(t)|$$

holds for all n, we have $|g_{k_n}(t_n) - g_k(t)| < 2\varepsilon$ for all $n > n'$. This proves the desired implication.

(ii) $\Rightarrow$ (i). Since the weakly associative property is an obvious consequence of the Basic Lemma, we prove here only the continuity of $\boldsymbol{m}$. Let $\mu_n \overset{d}{\to} \mu$. Hence $\|\mu_n\| \to \|\mu\|$ and whence, by the continuity of g, $g_{\|\mu_n\|} \to g_{\|\mu\|}$. Therefore, by Theorem 8.1, the sequence $(g_{\|\mu_n\|})$ is uniformly convergent to $g_{\|\mu\|}$ in $[a', b']$ with $a' < a$ and $b' > b$.

We claim that $\int g_{|\mu_n|}\, d\mu_n \to \int g_{|\mu|}\, d\mu$. Keeping in mind Theorem 3.13 we have

$$\left| \int g_{\|\mu_n\|}\, d\mu_n - \int g_{\|\mu\|}\, d\mu \right| \leq \left| S \int_{[a', b']} g_{\|\mu_n\|}\, d\mu_n - \right.$$

$$- S \int_{[a',b']} g_{|\mu|}\, d\mu_n \,| + |\, \int g_{|\mu|}\, d\mu_n - \int g_{|\mu|}\, d\mu \,|$$

Let $\varepsilon > 0$. Then there exists n' such that

$$|g_{|\mu_n|}(x) - g_{|\mu|}(x)| < \varepsilon$$

for all $n > n'$ and for all $x \in [a', b']$. Let $m > n'$. Then, by Corollary 3.21 and Remark 3.25, there is $n'' > n'$ such that

$$|\int g_{|\mu|}\, d\mu_n - \int g_{|\mu|}\, d\mu \,| < \varepsilon$$

for all $n > n''$. Finally, given $n > n''$ we have

$$|\int g_{\|\mu_n\|}\, d\mu_n - \int g_{\|\mu\|}\, d\mu \,| < \varepsilon(\|\mu_n\| + 1) \leq \varepsilon\alpha,$$

for some $\alpha > 0$, where the last inequality holds, on noting that $\|\mu_n\| \to \|\mu\|$. This proves the claim.

Plainly we have

$$y_n = \frac{\int g_{|\mu_n|}\, d\mu_n}{\|\mu_n\|} \to y = \frac{\int g_{|\mu|}\, d\mu}{\|\mu\|}.$$

Now we claim that $y \in [\, g_{\|\mu\|}(a), g_{|\mu|}(b)]$. From Theorem 5.3 we get

$$y_n \in \left[g_{|\mu_n|}(a), g_{|\mu_n|}(b)\right]$$

for all n and hence we have the claim, on noting that $g_{|\mu_n|} \to g_{|\mu|}$.

Finally, we claim that $g^{-1}_{\|\mu_n\|}(y_n) \to g^{-1}_{\|\mu\|}(y)$. By Theorem 8.2,$(g^{-1}_{\|\mu_n\|})$ is uniformly convergent to $g^{-1}_{\|\mu\|}$ in $\left[g_{\|\mu\|}(a'), g_{\|\mu\|}(b')\right]$. Since $y_n \to y$ there is n' such that $y_n \in \left[g_{\|\mu\|}(a'), g_{\|\mu\|}(b')\right]$ for all $n > n'$. Now, on nothing that

$$|g^{-1}_{|\mu_n|}(y_n) - g^{-1}_{|\mu|}(y)| \leq |\, g^{-1}_{|\mu_n|}(y_n) - g^{-1}_{|\mu|}(y_n)| + |g^{-1}_{|\mu|}(y_n) - g^{-1}_{|\mu|}(y)|\,,$$

the claim may be proved by arguments similar to those used to verify $g_{k_n}(t_n) \to g_k(t)$ in the proof of (i) $\Rightarrow$ (ii). This proves the desired implication.

This completes the proof. ♦

7.3 REMARK. Of course we may choose the function g involved in (ii) of the previous theorem so that g_k is a strictly decreasing function for all k.

7.4 THEOREM. *Let* m *be a mean on* $\mathbf{M}^+[a, b]$. *Then the following statements are equivalent:*

(i) m *is a continuous associative mean;*

(ii) there is a strictly monotone continuous real function g *on* $\boldsymbol{R}$ *such that*

$$m(\mu) = m_g(\mu)$$

for all μ.

Moreover, $g|_{[a,b]}$ *is unique up to non degenerate affine transformations.*

Proof. (i) ⇒ (ii). By Remark 4.14 and Theorem 4.22, we have $m(\mu) = m\left(\frac{\mu}{\|\mu\|}\right)$ and hence, by the Basic Lemma, we get

$$m(\mu) = g_1^{-1}\left[\frac{\int g_1\, d\mu}{\|\mu\|}\right] = m_{g_1}(\mu).$$

This proves the desired implication.

(ii) ⇒ (i). It immediately follows from Theorem 5.3.

Finally, the last statement of the theorem easily follows from (i) of Remark 5.7. This completes the proof. ♦

7.5 PROPOSITION. *Let* m *be a continuous mean on* $\mathbf{M}' \supset \mathbf{M}^+[a, b]$. *Then* m *is not a strongly associative mean.*

Proof. Assume this is not true. Then $m' = m|_{\mathbf{M}^+[a,b]}$ is a continuous associative mean. Hence, by the previous theorem, m' is a g-mean. Keeping in mind Theorem 5.3, we get a contradiction. This completes the proof. ♦

7.6 THEOREM. *Let* m *be a mean on* $\mathbf{M}^+_{oo}$. *Then the following statements are equivalent:*

(i) m *is a conditionally continuous associative mean;*

(ii) there is a strictly monotone continuous real function g on $\boldsymbol{R}$ *such that*

$$m(\mu) = m_g(\mu)$$

for all μ.

Moreover, g is unique up to non degenerate affine transformations.

Proof. (i) ⇒ (ii). Let (a_n), (b_n) be two sequences such that $a_n \downarrow -\infty$, $b_n \uparrow +\infty$ and $a_n < b_n$ for all n.

Then, by Theorem 7.4, there is a strictly monotone continuous function $g^{(1)}$ such that $m(\mu) = m_{g^{(1)}}(\mu)$ for all $\mu \in \mathbf{M}^+[a_1, b_1]$.

Now, always by Theorem 7.4, there is a strictly monotone continuous function $g^{(2)}$ such that $m(\mu) = m_{g^{(2)}}(\mu)$ for all $\mu \in \mathbf{M}^+[a_2, b_2]$. Since $\mathbf{M}^+[a_1, b_1] \subset \mathbf{M}^+[a_2, b_2]$ and $m_{g^{(2)}}(\mu) = m_{g^{(1)}}(\mu)$ for all $\mu \in \mathbf{M}^+[a_1, b_1]$, we can always assume that

$$g^{(2)}\Big|_{[a_1, b_1]} = g^{(1)}\Big|_{[a_1, b_1]},$$

on noting that $g^{(2)}$ is unique up to non degenerate affine transformations.

In this way, by induction, we get a strictly monotone continuous function g such that

$$g\Big|_{[a_n, b_n]} = g^{(n)}\Big|_{[a_n, b_n]}$$

for all n.

Therefore, by Proposition 3.20, we have constructed a strictly monotone continuous function g such that $m(\mu) = m_{g^{(n)}}(\mu) = m_g(\mu)$ whenever $\mu \in \mathbf{M}^+[a_n, b_n]$ for some n. This proves the desired implication.

(ii) ⇒ (i). It immediately follows from Theorem 5.3.

Finally, the last statement of the theorem follows from Theorem 5.6. This completes the proof. ♦

7.7 THEOREM. *Let* $\boldsymbol{m}$ *be a mean on* $\mathbf{M}_0^+$. *Then the following statements are equivalent:*

(i) $\boldsymbol{m}$ *is a continuous associative mean;*

(ii) there is a bounded strictly monotone continuous real function g on $\boldsymbol{R}$ *such that*

$$m(\mu) = m_g(\mu)$$

for all μ.

Moreover, g is unique up to non degenerate affine transformations.

Proof. (i) $\Rightarrow$ (ii) Since $\mathbf{M}_{oo}^+ \subset \mathbf{M}_o^+$ by Theorem 7.6, there is a strictly monotone continuous function g such that $m(\mu) = m_g(\mu)$ for all $\mu \in \mathbf{M}_{oo}^+$. Moreover, by Proposition 3.24, g is a bounded function. Hence, by (ii) of Remark 5.2, m_g may be defined on $\mathbf{M}_o^+$. Of course, we must still prove that $m = m_g$.

Let $\mu \in \mathbf{M}_o^+$. Now we can choose two sequences (a_n), (b_n) such that $a_n \downarrow -\infty$, $b_n \uparrow +\infty$ and $a_n < b_n$ for all n. Let

$$\mu_n(F) = \mu(F \cap [a_n, b_n])$$

for all n and for all F; hence $\mu_n \in \mathbf{M}^+[a_n, b_n]$.

We claim that $\mu_n \xrightarrow{d} \mu$. Let $\varepsilon > 0$. Given a, there is n' such that $b_n > a$, $a_n < a$ and $\mu(]-\infty, a_n]) < \varepsilon$ for all $n > n'$. Then we have

$$|F_\mu(a) - F_{\mu_n}(a)| = |\mu(]-\infty, a_n]) +$$

$$+ \mu(]a_n, a]) - \mu_n(]-\infty, a_n]) - \mu_n(]a_n, a])| =$$

$$= |\mu(]-\infty, a_n]) - \mu(\{a_n\})| = \mu(]-\infty, a_n[) < \varepsilon$$

for all $n > n'$. This proves the claim on noting that $\|\mu_n\| = \mu([a_n, b_n]) \to \mu(R) = \|\mu\|$, because μ is a tight mass.

Since g is a bounded function, by Theorem 5.3, we have $m(\mu_n) = m_g(\mu_n) \to m_g(\mu)$. Hence $m(\mu) = m_g(\mu)$ on noting that m is a continuous mean, as well. This proves the desired implication.

(ii) $\Rightarrow$ (i). This immediately follows from Theorem 5.3.

The last statement of the theorem immediately follows from Theorem 5.6. This completes the proof. ♦

7.8 REMARK. (i) It is interesting to note that, by Theorem 5.3 and the previous one, any associative continuous mean on $\mathbf{M}_o^+$, is also an internal mean.

(ii) Let **F** be the Borel σ-field on $\boldsymbol{R}$ and let **M'** be the set of all bounded positive measures on **F**. Hence $\mathbf{M'} \subset \mathbf{M}_0^+$. Going through the proofs of the representations theorems, we can see that every continuous associative mean on **M'** is a g- mean.

7.9 REMARK. It is interesting to observe that we cannot assure that the function g involved in (ii) of Theorem 7.6 is a bounded function. Therefore the improper S-integral is a basic tool to study integral representations of means on $\mathbf{M}_{0o}^+$. On the contrary, in the other integral representation theorems we can consider S-integral instead of improper S-integral, since we may always suppose, if we please, that the functions involved in (ii) of these theorems are bounded functions.

8. Appendix.

We are going to give here some useful properties of converging sequences of monotone functions.

8.1 THEOREM. *Let (g_n) be a sequence of increasing functions on $[a, b]$ converging to a continuous function g. Then the sequence is uniformly convergent in $[a, b]$.*

Proof. It is easy to verify that g is an increasing function. Let $\varepsilon > 0$. Then there are $a_0, a_1, ..., a_m$ such that

$$a = a_0 < a_1 < ... < a_m = b,$$

$$|g(x_1) - g(x_2)| < \varepsilon \text{ for all } x_1, x_2 \in [a_i, a_{i+1}] \ (i = 0, ..., m-1).$$

Moreover there exists n' such that $|g_n(a_i) - g(a_i)| < \varepsilon$ ($i = 0, ... m$), for every $n \geq n'$.

Let $x \in [a_i, a_{i+1}]$ for some i. Then for all $n > n'$ we have:

$$g_n(x) \in [g_n(a_i), g_n(a_{i+1})] \subset [g(a_i) - \varepsilon, g(a_{i+1}) + \varepsilon]$$

$$g(x) \in [g(a_i) - \varepsilon, \ g(a_{i+1}) + \varepsilon].$$

Therefore $|g_n(x) - g(x)| < 3\varepsilon$. Since x is arbitrarily choosen, we get the thesis.

This completes the proof. ♦

8.2 THEOREM. *Let g_n, g be a sequence of strictly increasing continuous* *functions on* $\boldsymbol{R}$, *such that* $g_n \to g$. *Then the sequence* (g_n^{-1}) *is uniformly convergent to* g^{-1} *in* $[g(a), g(b)]$ *for all* a, b.

Proof. Let $\mathrm{I} = [g(a), g(b)]$. We claim that there is n' such that $\mathrm{I} \subset \mathrm{dom}\,(g_n^{-1})$ for all $n > n'$. Let a', b' such that $a' < a < b < b'$. Since $g_n \to g$ and g is strictly increasing, there is n' such that $g_n(a') < g(a)$, $g_n(b') > g(b)$ for all $n > n'$. Therefore, by the continuity of g_n, we get $g(a), g(b) \in \mathrm{dom}\,(g_n^{-1})$ for all $n > n'$. This proves the claim.

Keeping in mind the previous theorem, it is sufficient to prove $g_n^{-1} \to g^{-1}$ in I. Let $y \in\,]g(a), g(b)[$ and let $x = g^{-1}(y)$. Let $\varepsilon > 0$. Since g^{-1} is a continuous function there are $x_1, x_2 \in [a, b]$ such that

$$0 < x_2 - x < \varepsilon,$$

$$0 < x - x_1 < \varepsilon,$$

$$g(x_2) - y = y - g(x_1).$$

Now let $\delta = y - g(x_1)$. Since, by Theorem 8.1, g_n is uniformly convergent to g in $[a, b]$, there is $n'' > n'$ such that $|g_n(x) - g(x)| < \delta$ for all $n > n''$ and for all $x \in [a, b]$. Therefore, given $n > n''$, we have

$$g_n(x_1) < g(x_1) + \delta = y,$$

$$g_n(x_2) > g(x_2) - \delta = y,$$

and hence $g_n^{-1}(y) \in\,]x_1, x_2[$, on noting that g_n is a strictly increasing continuous function. It follows that $|g_n^{-1}(y) - g^{-1}(y)| < \varepsilon$ for all $n > n''$. This completes the proof. ♦

8.3 COROLLARY. *Let* g_n, g *be a strictly increasing continuous functions such that* $g_n(a) = g(a)$ *and* $g_n(b) = g(b)$ *for all* n. *Moreover let* $g_n \to g$ *in* $[a, b]$. *Then the sequence* (g_n^{-1}) *is uniformly convergent to* g^{-1} *in* $[g(a), g(b)]$.

Proof. Let g^o, g_n^o be strictly increasing continuous functions on $\boldsymbol{R}$ such that $g_n^o \to g^o$ and

$$g_n^o\big|_{[a,b]} = g_n\big|_{[a,b]} \text{ for all } n,$$

$$g^o \big|_{[a,b]} = g \big|_{[a,b]} .$$

Now, the thesis easily follows from Theorem 8.2. This completes the proof. ♦

8.4 REMARK. Of course, all statements proved in this appendix hold for sequences of decreasing functions as well.

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