SEQUENTIALLY P - CLOSED SPACES (*)

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SOMMARIO. - *Un $\mathcal{P}$ - spazio è sequenzialmente $\mathcal{P}$ - chiuso se e solo se esso è sequenzialmente chiuso in ogni $\mathcal{P}$ - spazio in cui esso sia immerso. Se $\mathcal{P}$ è una classe di spazi, rispettivamente, completamente regolari, normali, perfettamente normali, localmente compatti, paracompatti, metrici, gli spazi sequenzialmente $\mathcal{P}$ - chiusi sono esattamente i $\mathcal{P}$ - spazi numerabilmente compatti. Per varie categorie $\mathcal{P}$ consistenti di spazi di Hausdorff si danno caratterizzazioni interne degli spazi sequenzialmente $\mathcal{P}$ - chiusi che permettono di stabilirne molte altre proprietà.*

SUMMARY. - *A $\mathcal{P}$ - space is sequentially $\mathcal{P}$ - closed if it is sequentially closed in every $\mathcal{P}$ - space in which it is embedded. For $\mathcal{P}$ - completely regular, normal, perfectly normal, locally compact, paracompact and metric the sequentially $\mathcal{P}$ - closed spaces are precisely the countably compact $\mathcal{P}$ - spaces. Internal characterization of sequentially $\mathcal{P}$ - closed spaces are given for various categories $\mathcal{P}$ consisting of Hausdorff spaces which permits to establish a lot of other properties of the sequentially $\mathcal{P}$ - closed spaces.*

0. Introduction.

A space Y is called *sequentially determined extension* of its subspace X iff for every point $y \in Y$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = y$ [Go]. Let $\mathcal{P}$ be a class of topological spaces. A space $X \in \mathcal{P}$ is said to be *$\mathcal{P}$ - closed (sequentially $\mathcal{P}$ - closed)* iff X is closed (sequentially closed) in every $\mathcal{P}$ space in which it is embedded. In other words X is sequentially $\mathcal{P}$ - closed iff X has no sequentially determined extension $Y \in \mathcal{P}$ and $Y \neq X$ (this holds under very mild restrictions on $\mathcal{P}$ which are verified in all cases considered here).

Obviously, every $\mathcal{P}$ - closed space is sequentially $\mathcal{P}$ - closed. The

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$\mathcal{P}$ - closed spaces were extensively studied for categories $\mathcal{P}$ consisting of Hausdorff spaces (see [BPS]). For such $\mathcal{P}$ every compact (countably compact) $\mathcal{P}$ - space is $\mathcal{P}$ - closed (sequentially $\mathcal{P}$ - closed). The sequentially Hausdorff - closed spaces were introduced by P. Alexandroff and P. Urysohn [AU]. They were proved that the regular sequentially Hausdorff - closed spaces coincide with the regular countably compact spaces. In [DGo] sequentially $\mathcal{P}$ - closed spaces for some classes $\mathcal{P}$ of topological spaces between T_1 and T_2 are studied. It is proved in particular that for the category SUS of topological spaces in which every convergent sequence has a unique accumulation point the sequentially SUS - closed spaces are precisely the countably compact SUS - spaces. On the other hand it follows by results of A. Tozzi [To] that the sequentially SUS - closed spaces coincide with the absolutely SUS - closed spaces, introduced by D. Dikranjan and E. Giuli [DG3] in a more general situation. This was in fact the starting point of our study of sequentially $\mathcal{P}$ - closed spaces.

The aim of this paper is to study the sequentially $\mathcal{P}$ - closed spaces for $\mathcal{P} = S\ (n)$, regular, completely Hausdorff, completely regular, normal, perfectly normal, locally compact, paracompact and metric.

In section 1 we introduce open elementary filters which serve as the main tool in the study of sequentially $\mathcal{P}$ - closed spaces. We introduce also θ^n convergence and S^n - convergence (generalizations of the θ - convergence introduced by Veličko [Ve] and the usual convergence) and characterize the $S\ (n)$ - spaces by means of these convergences following Veličko [Ve], Dikranjan and Giuli [Di], [DG1], [DG3].

In section 2 we give internal characterization of the sequentially $\mathcal{P}$ - closed spaces in terms of special filters and covers for $\mathcal{P} = S\ (n)$, regular, completely Hausdorff and completely regular. For $\mathcal{P} = S\ (n)$ this characterization involves the θ^n - convergence and S^n - convergence. An example of a sequentially Hausdorff - closed completely Hausdorff space (and hence $S\ (n)$ - space for $n = 1, 2, ...$) is given which is neither Hausdorff - closed nor countably compact. We give also an example of a sequentially regular - closed space which is neither regular - closed nor countably compact. We discuss also the relations between sequentially $\mathcal{P}$ - closed spaces and $\mathcal{P}$ (1) - closed spaces (see [BPS]).

In section 3 various properties of the sequentially $\mathcal{P}$ - closed spaces are established. We show in particular that for $\mathcal{P}$ - completely regular, normal, perfectly normal, locally compact, paracompact and metric the sequentially $\mathcal{P}$ - closed spaces are precisely the countably compact $\mathcal{P}$ - spaces. Since for paracompact spaces countable compactness coincides with compactness in the last two cases we get the compact $\mathcal{P}$ - spaces.

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1. Preliminaries.

Troughout the paper the properties of regularity, completely regularity, etc. include the T_1 separation property, $\overline{U}$ denotes the closure of the set U in a given topological space, N denotes the set of positive integers and $\boldsymbol{R}$ denotes the real line with the usual topology. In general the terminology and notation follow [En].

Let X be a topological space and let A be a countable subset of X. A *maximal open elementary filter generated by* A is the open filter on X with base $\mathcal{B} = \{F \mid F \subset X, F$ is open in X and $A \setminus F$ is finite$\}$. An open filter $\mathcal{F}$ on X is a *maximal open elementary filter*, iff there exists a countable subset A of X, such that $\mathcal{F}$ is the maximal open elementary filter on X generated by A. If X is a *US* - space (every convergent sequence in X has a unique limit point [MN]) then A is uniquely determined by $\mathcal{F}$ up to a finite subset (see [DGo]). An open filter will be called *open elementary filter*, iff it is contained in some maximal open elementary filter. A filter $\mathcal{F}$ is *free* iff $\cap\{F \mid F \in \mathcal{F}\} = \phi$.

Let $\mathcal{F} = \{F_\alpha\}_{\alpha \in \mathcal{A}}$ be a free open elementary filter on X. On the set $X_{\mathcal{F}} = X \cup \{\mathcal{F}\}$ we introduce the following topology: The set X is open, the relative topology of X coincides with the original topology of X and for open base at the point $\{\mathcal{F}\}$ we take the family $\{\{\mathcal{F}\} \cup F_\alpha \mid F_\alpha \in \mathcal{F}, \alpha \in \mathcal{A}\}$. It is easy to verify that the topological space obtained in this way is a T_1 sequentially determined extension of X whenever X is T_1. In the sequel $X_{\mathcal{F}}$ will be called *standard sequentially determined extension of* X *by* $\mathcal{F}$.

Let X be a topological space, $M \subset X$ and $n \in N$. The point $x \in X$ is $S(n)$ *- separated* from M iff there exist open sets $U_i, i = 1, 2, \ldots, n$ such that $x \in U_1 \subset \ldots \subset U_n, \overline{U}_i \subset U_{i+1}$ and $\overline{U}_n \cap M = \phi$; x is $S(0)$ - separated from M iff $x \notin \overline{M}$. The space X is an $S(n)$ - space, iff every two different points in X are $S(n)$ separated [Vi]. It is obvious that $S(1) = T_2$, $S(2) = T_{2,5}$ (distinct points can be separated by disjoint closed neighbourhoods).

An open cover $\{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{A}}$ of the space X is an $S(n)$ *- cover*, iff every point $x \in X$ is $S(n)$ - separated from some $X \setminus U_\alpha$. Clearly the $S(0)$ - covers are exactly the open covers. A filter $\mathcal{F}$ on X is an $S(n)$ *- filter*, iff every point $x \in X$, which is not adherent point for $\mathcal{F}$ is $S(n)$ - separated from some $U \in \mathcal{F}$ [PV].

Let X be a topological space and $n \in N$. The point $x \in X$ will be called S^n - limit (θ^n - limit) of a sequence $\{x_n\}_{n=1}^{\infty}$ in X, iff for every chain $U_1 \subset U_2 \subset \ldots \subset U_n$ of open neighbourhoods of x such that $\overline{U}_i \subset U_{i+1}$ for

$i = 1, 2, ..., n - 1$, U_n ($\overline{U}_n$) contains all but a finite number of the members of the sequence. Every sequence which has an S^n - limit (θ^n - limit) will be called *S^n - convergent (θ^n - convergent)*. If for every chain $U_1 \subset U_2 \subset ... \subset U_n$ of open neighbourhoods of x such that $\overline{U}_i \subset U_{i+1}$ for $i = 1, 2, ..., n - 1$, U_n ($\overline{U}_n$) contains infinitely many members of the sequence then x will be called *S^n - adherent point (θ^n - adherent point*) of the sequence $\{x_n\}_{n=1}^{\infty}$. Clearly, the S^1 - convergence is the usual topological convergence and the θ^1 - convergence is the θ - convergence defined by Veličko [Ve]. Also every S^n - convergent sequence is θ^n - convergent and every θ^n - convergent sequence is S^{n+1} - convergent. So every convergent, in the usual sense, sequence is S^n - convergent and θ^n - convergent for every $n \in N$. If X is regular then the S^n - convergence and the θ^n - convergence coincides with the usual convergence. Similar facts are valid for S^n and θ^n - adherent points of a sequence. In the same way S^n (θ^n) convergence and S^n (θ^n) adherent points can be defined for nets. (For θ^n - adherent points see [DG3] and for $n = 1$ see also [DG2]).

The *$S(n)$* - spaces can be characterized by means of the θ^n - convergence, S^n - convergence and the usual convergence.

PROPOSITION 1.1. *Let X be a topological space and $n \in N$. The following conditions are equivalent:*

(a) *X is an $S(n)$ - space*
(b) *every convergent sequence in X has a unique θ^n - adherent point*
(c) *every convergent sequence in X has a unique θ^n - limit*
(d) *every convergent sequence in X has a unique S^{n+1} - adherent point*
(e) *every convergent sequence in X has a unique S^{n+1} - limit.*

Proof. Follows directly from the definition.

2. Characterization of the sequentially $\mathcal{P}$ - closed spaces.

The next theorem characterizes sequentially *$S(n)$* - closed spaces. Analogous results for *$S(n)$* - closed spaces were obtained in [PV] and for *$S(n)$ - θ* - closed spaces were obtained in [DG3].

THEOREM 2.1. *Let X be a T_1 space and $n \in N$. The following conditions are equivalent:*

(a) *every sequence in X has a θ^n - adherent point*

(b) *every sequence in X has an* S^{n+1} *- adherent point*
(c) *every countable S (n) - cover of X has a finite subcover*
(d) *every S (n) - filter with a countable base of closed sets has an adherent point*
(e) *every open elementary S (n) - filter has an adherent point*
(f) *every maximal open elementary S (n) - filter has an adherent point.*
If X is an S (n) - space then the above conditions are equivalent to:
(g) *X is sequentially S (n) - closed.*

Proof. Obviously (a) implies (b) and (e) implies (f). To see that (b) implies (c) assume that $\{U_n\}_{n=1}^{\infty}$ is a countable $S\,(n)$ - cover which has no finite subcover. For every $k = 1, 2,...$ we choose $x_k \notin \bigcup_{i=1}^{k} U_i$. Let $x \in X$. Since $\{U_i\}_{i=1}^{\infty}$ is an $S\,(n)$ - cover then there exists an element U_i of the cover and chain $V_1 \subset V_2 ... \subset V_n$ of open neighbourhoods of x such that $\overline{V}_j \subset V_{j+1}$ for $j = 1, 2, ..., n - 1$ and $\overline{V}_n \subset U_i$. This means that x is not an S^{n+1} - adherent point for $\{x_k\}_{k=1}^{\infty}$ since for every $i \in N$ and $k \geq i$ we have $x_k \notin U_i$. Therefore $\{x_k\}_{k=1}^{\infty}$ has no S^{n+1} - adherent points in X : a contradiction. Let now $\mathcal{F}$ be an $S\,(n)$ - filter on X with countable closed base $\{F_i\}_{i=1}^{\infty}$. Assume that $\mathcal{F}$ has no adherent points. Then $\mathcal{U} = \{U_i \mid U_i = X \backslash F_i,\ i \in N\}$ is an $S\,(n)$ - cover of X. Let $U_{i_1}, U_{i_2}, ..., U_{i_k}$ be a finite subcover of $\mathcal{U}$. Since $\mathcal{F}$ is a filter we have $\bigcap_{j=1}^{k} \left(X \backslash U_{i_j}\right) = \bigcap_{j=1}^{k} F_{i_j} \neq \phi$, hence $U_{i_1}, U_{i_2}, ..., U_{i_k}$ is not a cover of X. This contradiction proves that (c) implies (d). Now we will prove that (d) implies (e). Let $\mathcal{F}$ be an open elementary $S\,(n)$ - filter without adherent points. There exists a maximal open elementary filter $\mathcal{F}'$ such that $\mathcal{F} \subset \mathcal{F}'$. Then $\mathcal{F}'$ has no adherent points and if $\{x_k\}_{k=1}^{\infty}$ determines $\mathcal{F}'$ then $\{x_k\}_{k=i}^{\infty}$ is a closed set for every $i \in N$. Thus the filter $\mathcal{F}''$ generated by $\left\{F_i \mid F_i = \{x_k\}_{k=i}^{\infty}, i \in N\right\}$ contains $\mathcal{F}'$. So $\mathcal{F}''$ is an $S\,(n)$ - filter with a countably base of closed sets without adherent points. To see that (f) implies (a) assume that $\{x_k\}_{k=1}^{\infty}$ is a sequence in X which has no θ^n - adherent points. Let $\mathcal{F}$ be the maximal open elementary filter generated by $\{x_k\}_{k=1}^{\infty}$ and x be an arbitrary point of X. There exists a chain $U_1 \subset U_2 \subset ... \subset U_n$ of open neighbourhoods of x such that $\overline{U_i} \subset U_{i+1}$, $i = 1, 2,..., n-1$ and $X \setminus \overline{U_n}$ contains all but a finite members of the sequence. Therefore $X \setminus \overline{U_n} \in \mathcal{F}$. On the other hand x and $X \setminus \overline{U_n}$ are $S(n)$ - separated. This means that $\mathcal{F}$ is a maximal open elementary $S(n)$ - filter with no adherent points. Contradiction. Now let we assume that X is not sequentially $S(n)$ - closed. Thus there exists an $S(n)$ - space $Y \supset X$, a point $y \in Y \setminus X$ and a sequence $\{x_k\}_{k=1}^{\infty}$ in X such that $\lim_{k \to \infty} x_k = y$. Clearly y is a θ^n - adherent point of $\{x_k\}_{k=1}^{\infty}$. From Proposition 1. 1 it follows that $\{x_k\}_{k=1}^{\infty}$ has no other θ^n - adherent points in Y. Thus $\{x_k\}_{k=1}^{\infty}$ has no θ^n - adherent points in X. This contradiction proves that (a) implies (g). To prove that (g) implies (f) assume that $\mathcal{F}$ is a maximal open elementary $S(n)$ - filter with no adherent points. Let $X_{\mathcal{F}}$ be the standard sequentially determined extension of X by $\mathcal{F}$. It is easy to verify that $X_{\mathcal{F}}$ is an $S(n)$ - space. Thus X is not sequentially $S(n)$ - closed. Contradiction.

The idea to characterize closed spaces with θ - convergence and elementary filters (but in a somewhat different sense, see [Bo], chap. 1. § 6) comes from Veličko [Ve].

Now we show that the class of sequentially $S(n)$ - closed spaces is not exhausted by the $S(n)$ - closed spaces and by the countably compact $S(n)$ - spaces.

EXAMPLE 2.2. Let N be the space of positive integers with the discrete topology and let βN be the Čech-Stone compactification of N. Let also $X = (\beta N \setminus N) \cup \{x_{ij}\}_{i,j=1}^{\infty} \cup \{y_i\}_{i=1}^{\infty}$. We provide X with a topology as follows: The points $\{x_{ij}\}_{i,j=1}^{\infty}$ are isolated for $i \in N$ and $j \in N$. For a neighbourhood base of y_i $(i \in N)$ we take the family $y_i \cup \left\{\{x_{ij}\}_{j=k}^{\infty}, k \in N\right\}$. Let $\{\mathcal{F}\} \in \beta N \setminus N$ and let $\{U_\alpha, \alpha \in \mathcal{A}\}$ be a neighbourhood base of $\{\mathcal{F}\}$ in βN. For a neighbourhood base of $\{\mathcal{F}\}$ in X we take the family $\left\{V_\alpha \,\middle|\, V_\alpha = \left(U_\alpha \setminus N\right) \cup \left\{\{x_{ij}\}_{i,j=1}^{\infty}, i \in U_\alpha \cap N\right\}, \alpha \in \mathcal{A}\right\}$. It is easy to verify that X is a Hausdorff-closed $S(n)$ - space for every $n \in N$ and X is not countably compact. Let now $\mathcal{F} \in \beta N \setminus N$ and $Y = X \setminus \{\mathcal{F}\}$. Clearly Y is $S(n)$ - closed for no n and Y is not countably compact. By (a) of the above theorem Y is sequentially $S(n)$ - closed.

Let X be a topological space. An open filter $\mathcal{F}$ on X is a *regular filter* iff for each $U \in \mathcal{F}$ there exists $V \in \mathcal{F}$ such that $\overline{V} \subset U$ [Ba]. Let $\mathcal{U}$ and

$\mathcal{V}$ be open covers of a space X. $\mathcal{V}$ is a *shrinkable refinement of* $\mathcal{U}$ iff for each $V \in \mathcal{V}$, there is $U \in \mathcal{U}$ such that $\overline{V} \subset U$. An open cover $\mathcal{U}$ *is regular* iff there exists an open cover $\mathcal{V}$ which refines $\mathcal{U}$ and $\mathcal{V}$ is a shrinkable refinement of itself [BPS].

THEOREM 2.3. *Let X be a T_1 space. The following conditions are equivalent*:

(a) *every open elementary regular filter on X has adherent points*
(b) *every countable regular cover of X has a finite subcover.*
If X is a regular space then the above conditions are equivalent to:
(c) *X is sequentially regular-closed.*

Proof. To see that (a) implies (b) assume that $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ is a countable regular cover without finite subcovers. For each $k = 1, 2, \dots$ we choose a point $x_k \in X$ such that $x_k \notin \bigcup_{i=1}^{k} U_i$. Then clearly $x_k \notin U_i$ whenever $k \geq i$. Let $\mathcal{V} = \{V_\alpha\}_{\alpha \in \mathcal{A}}$ be a cover of X which refines $\mathcal{U}$ and $\mathcal{V}$ is a shrinkable refinement of itself. Clearly the cover $\mathcal{V}$ has no finite subcovers. It is easy to verify that the filter $\mathcal{F}$ generated by the filter base $\left\{X \setminus \bigcup_{i=1}^{k} V_{\alpha_i}, \alpha_i \in \mathcal{A}, k \in N\right\}$ is an open elementary regular filter on X without adherent points. Contradiction. Let now $\mathcal{F}$ be an open elementary regular filter on X without adherent points. There exists a maximal open elementary filter such that $\mathcal{F} \subset \mathcal{F}'$. Let $\{x_k\}_{k=1}^{\infty}$ determines $\mathcal{F}'$. But $\mathcal{F}'$ has no adherent points. Then $\{x_k\}_{k=1}^{\infty}$ is a closed set for every $i \in N$. Let $\mathcal{U} = \left\{U_i \mid U_i = X \setminus \{x_k\}_{k=i}^{\infty}, i \in N\right\}$ and $\mathcal{V} = \{V \mid$ there exists an open set $W \in \mathcal{F}$ such that $V = X \setminus \overline{W}\}$. It is easy to verify that $\mathcal{V}$ is an open cover of X which refines $\mathcal{U}$ and $\mathcal{V}$ is a shrinkable refinement of itself. Thus $\mathcal{U}$ is a countable regular cover of X without finite subcovers and this proves that (b) implies (a). Now we prove that (a) implies (c). Assume that X is not sequentially regular - closed. Thus there exists a regular space $Y \supset X$, a point $y \in Y \setminus X$ and a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n \to \infty} x_n = y$. Let $\mathcal{V}_y$ be the filter of neighbourhoods of y on Y. Since Y is a regular space it follows that $\mathcal{V}_y$ is an open elementary regular filter on Y with no adherent points in X. Then $\mathcal{V} = \{V \mid$ there exists $W \in \mathcal{V}_y$ such that $V = X \cap W\}$ is an open elementary regular filter on X without adherent points. Contradiction. Assume that there exists an open elementary regular filter $\mathcal{F}$ on X without adherent points. Then the standard sequentially determined extension $X_{\mathcal{F}}$ of X by $\mathcal{F}$ will be a regular space. This contradicts the sequentially regular - closedness of X, so (c) implies (a).

Now we show that the class of sequentially regular - closed spaces is not exhausted by the regular - closed spaces and by the countably compact regular spaces.

EXAMPLE 2.4. The space X in Example 4.18 in [BPS] is a minimal regular space which is not countably compact [BS]. Let $x = (\omega_1, 1, 1) = (\omega_1, 1, 2)$ and $Y = X \setminus \{x\}$. Then by Lemma 3.10 Y is a sequentially regular - closed space which is neither regular - closed nor countably compact.

Let X be a topological space. X is *completely Hausdorff* iff for each pair x, y of distinct points, there exists a continuous real - valued function f such that $f(x) \neq f(y)$. An open filter $\mathcal{F}$ on X is *completely Hausdorff* iff for each $x \in X$ which is not an adherence point of $\mathcal{F}$ there exists an open set U containing x, $V \in \mathcal{F}$ and continuous real - valued function f on X such that $f(U) = \{1\}$ and $f(V) = \{0\}$. An open filter $\mathcal{F}$ on X is *completely regular* iff for each $U \in \mathcal{F}$, there exists $V \in \mathcal{F}$ and a continuous real - valued function f on X such that $f(V) = \{0\}$ and $f(X \setminus U) = \{1\}$. Let $\mathcal{V}$ and $\mathcal{U}$ be covers of a space X. $\mathcal{V}$ is a *continuous refinement* of $\mathcal{U}$ iff for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ and continuous real - valued function f on X such that $f(V) = \{0\}$ and $f(X \setminus U) = \{1\}$. An open cover is *completely Hausdorff* iff it has a continuous refinement [BPS]. An open cover $\mathcal{U}$ is *completely regular* iff there is an open cover $\mathcal{V}$ which refines $\mathcal{U}$ and $\mathcal{V}$ is a continuous refinement of itself.

THEOREM 2.5. *Let X be a T_1 space. The following conditions are equivalent*:

(a) *every countable completely Hausdorff cover of X has a finite subcover*
(b) *every open elementary completely Hausdorff filter on X has adherent points*
(c) *every maximal open elementary completely Hausdorff filter on X has adherent points.*

If X is a completely Hausdorff space then the above conditions are equivalent to:

(d) *X is sequentially completely Hausdorff - closed.*

Proof. To see that (a) implies (b) let $\mathcal{F}$ be an open elementary completely Hausdorff filter on X without adherent points. There exists a maximal open elementary filter $\mathcal{F}'$ on X such that $\mathcal{F} \subset \mathcal{F}'$. If $\{x_k\}_{k=1}^{\infty}$ is a sequence which generate $\mathcal{F}'$ then $\{x_k\}_{k=1}^{\infty}$ is a closed set for every $i \in N$. Let $U = \left\{U_i \mid U_i = X \setminus \{x_k\}_{k=i}^{\infty}, i \in N\right\}$ and $\mathcal{V} = \{V_x \mid V_x = f^{-1}\left[0, \tfrac{1}{2}\right)$, $x \in X, W \in \mathcal{F}$ and $f: X \to \boldsymbol{R}$ is such that $f(x) = 0$ and $f(W) = \{1\}\}$. Then $\mathcal{U}$ and $\mathcal{V}$ are open covers of X and $\mathcal{V}$ is a continuous refinement of $\mathcal{U}$. Thus $\mathcal{U}$ is a completely Hausdorff cover of X without a finite subcover. Obviously (b) implies (c). We prove that (c) implies (a). Let $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ be a countable completely Hausdorff cover of X without finite subcovers. Let $\mathcal{V} = \{V_\alpha\}_{\alpha \in \mathcal{A}}$ be a cover of X which is a continuous refinement of $\mathcal{U}$. For every $k \in N$ we choose $x_k \notin \bigcup_{i=1}^{k} U_i$ and let $\mathcal{F}$ be the maximal open elementary filter generated by $\{x_k\}_{k=1}^{\infty}$. If x is an arbitrary point of X then there exists $\alpha \in \mathcal{A}$ such that $x \in V_\alpha$ and there exists $i \in N$ and $f: X \to \boldsymbol{R}$ such that $f(V_\alpha) = \{0\}$ and $f(X \setminus U_i) = \{1\}$. Let $W = f^{-1}\left(\tfrac{1}{2}, 1\right]$ and $g(x) = 2.\min(f(x), \tfrac{1}{2})$. Then $W \in \mathcal{F}, g(V_\alpha) = \{0\}$ and $g(W) = \{1\}$. Thus $\mathcal{F}$ is a maximal open elementary filter which is completely Hausdorff and it has no adherent points. To prove that (b) implies (d) assume that X is not sequentially completely Hausdorff - closed. Then there exists a completely Hausdorff space $Y \supset X$, a point $y \in Y \setminus X$ and a sequence $\{x_k\}_{k=1}^{\infty}$ of points of X such that $\lim_{k \to \infty} x_k = y$. The filter $\mathcal{V}_y$ of neighbourhoods of y is a completely Hausdorff filter on Y. Let $\mathcal{V} = \{V \mid$ there exists $W \in \mathcal{V}_y$ such that $V = X \cap W\}$. Then $\mathcal{V}$ is an open elementary completely Hausdorff filter on X without adherent points. Contradiction. If there exists an open elementary completely Hausdorff filter $\mathcal{F}$ on X without adherent points, then $X_{\mathcal{F}}$ will be a completely Hausdorff, sequentially determined extension of X. This proves that (d) implies (b).

The space Y in Example 2.2 is also completely Hausdorff, consequently sequentially completely Hausdorff - closed. On the other hand it is neither completely Hausdorff - closed nor countably compact.

LEMMA 2.6. *In a completely regular space X every open cover of X is a completely regular cover.*

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of X. For every $\alpha \in \mathcal{A}$ and every $x \in U_\alpha$ let $f_{\alpha,x}$ be a continuous real - valued function such that $f_{\alpha,x}(x) = 1$ and $f_{\alpha,x}(X\backslash U_\alpha) = \{0\}$. If $\mathcal{V} = \left\{V \mid V = f_{\alpha,x}^{-1}(1/n,1], \alpha \in \mathcal{A}, x \in U_\alpha, n \geq 2\right\}$ then $\mathcal{V}$ is an open cover of X, $\mathcal{V}$ refines $\mathcal{U}$ and $\mathcal{V}$ is a continuous refinement of itself.

THEOREM 2.7. *Let X be a T_1 space. The following conditions are equivalent*:

(a) *every countable completely regular cover of X has a finite subcover*
(b) *every open elementary completely regular filter on X has adherent points.*
If X is a completely regular space then the above conditions are equivalent to:
(c) *X is sequentially completely regular - closed*
(d) *X is countably compact.*

Proof. Let $\mathcal{F}$ be an open elementary completely regular filter on X without adherent points and let $\mathcal{F}'$ be a maximal open elementary filter such that $\mathcal{F} \subset \mathcal{F}'$. If $\{x_k\}_{k=1}^{\infty}$ is a sequence which generates $\mathcal{F}'$, then $\{x_k\}_{k=i}^{\infty}$ is a closed set for every $i \in N$. Let $\mathcal{U} = \left\{U_i \mid U_i = X\backslash\{x_k\}_{k=i}^{\infty}, i \in N\right\}$ and $\mathcal{V} = \{V \mid$ there exists an open set $W \in \mathcal{F}$ such that $V = X\backslash\overline{W}\}$. Then $\mathcal{U}$ and $\mathcal{V}$ are open covers of X, $\mathcal{V}$ refines $\mathcal{U}$ and $\mathcal{V}$ is continuous refinement of itself. Thus $\mathcal{U}$ is a completely regular cover of X without finite subcovers. This proves that (a) implies (b). To see that (b) implies (a) let $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ be a countable completely regular cover of X without finite subcovers and let $\mathcal{V} = \{V_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of X which refines $\mathcal{U}$ and $\mathcal{V}$ is a continuous refinement of itself. For every $k \in N$ we choose $x_k \notin \bigcup_{i=1}^{k} U_i$ and let $\mathcal{B} = \{W \mid$ there exist $V_{\alpha_i} \in \mathcal{V}$ such that $W = X\backslash \bigcup_{i=1}^{k} \overline{V}_{\alpha_i}$, $k \in N\}$. Let $\mathcal{F}'$ be the maximal open elementary filter on X determined by $\{x_k\}_{k=1}^{\infty}$. If $\mathcal{F}$ is the open filter with base $\mathcal{B}$ then $\mathcal{F} \subset \mathcal{F}'$ and hence $\mathcal{F}$ is an open elementary completely regular filter without adherent points. Now we prove that (b) implies (c). Assume that X is not sequentially completely regular - closed. Then there exists a completely regular space $Y \supset X$, a point $y \in Y \setminus X$ and a sequence $\{x_k\}_{k=1}^{\infty}$ of points of X such that $\lim_{k \to \infty} x_k = y$. The filter $\mathcal{V}_y$ of neighbourhoods of y is a completely regular filter on Y. Let $\mathcal{V} = \{W \mid$ there exists an open set $V \in \mathcal{V}_y$ such that $W = X \cap V\}$, then $\mathcal{V}$ is an open elementary completely regular filter on X without adherent points. Contradiction. If there exists an open elementary completely regular filter $\mathcal{F}$ on X without adherent points, then $X_{\mathcal{F}}$ will be a completely regular, sequentially determined extension of X. This proves that (c) implies (b). The equivalence of conditions (a) and (d) follows directly by Lemma 2.6.

For a class $\mathcal{P}$, the class of all first countable $\mathcal{P}$ - spaces will be denoted by $\mathcal{P}$ (1) [BPS]. Evidently every sequentially $\mathcal{P}$ - closed $\mathcal{P}$ (1) space is $\mathcal{P}$ (1) - closed. Hence the sequentially $\mathcal{P}$ (1) - closed spaces coincide with the $\mathcal{P}$ (1) - closed spaces. For various classes $\mathcal{P}$ the $\mathcal{P}$ (1) - closed spaces were studied in [Ste2].

A family of open sets $\mathcal{U}$ in a space X is a *proximate cover* of X iff $\cup \{\overline{U} \mid U \in \mathcal{U}\} = X$ [Ka].

THEOREM 2.8. *Let X be a T_1 space and $n \in N$. The following conditions are equivalent*:

(a) *every countable S $(n - 1)$ - cover of X contains a finite proximate subcover*
(b) *every countable open S (n) - filter has adherent points.*
If X is an S (n) (1) - space then the above conditions are equivalent to:
(c) *X is S (n) (1) - closed.*

Proof. To see that (a) implies (b) suppose that $\mathcal{F}$ is a countable open S (n) - filter on X without adherent points. Then $\mathcal{U} = \{U \mid U = X \setminus \overline{V}, V \in \mathcal{F}\}$ is a countable S $(n - 1)$ - cover of X and $\mathcal{U}$ has a proximate subcover. So that if $X = \bigcup_{i=1}^{k} \overline{U}_i$ then $\bigcap_{i=1}^{k} (X \setminus \overline{U}_i) = \bigcap_{i=1}^{k} V_i = \phi$. But $V_i \in \mathcal{F}$ for $i = 1, 2, \ldots k$. Contradiction. Now let us assume that $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ is a countable S $(n - 1)$ - cover of X which has no finite proximate subcovers. For every

$i \in N$ we consider $V_i = X \setminus (\bigcup_{j=1}^{i} \overline{U_j})$. Obviously $\mathcal{B} = \{V_i\}_{i=1}^{\infty}$ is a countable open base of a filter $\mathcal{F}$. One can easily verify that $\mathcal{F}$ is a countable open $S(n)$ - filter without adherent points. This contradiction proves that (b) implies (a). If X is not $S(n)$ (1) - closed then there exists an $S(n)$ (1) extension Y of X and a point $y \in Y \setminus X$. But the trace on X of the neighbourhood filter of the point y is a countable open $S(n)$ - filter on X without adherent points in X and this proves that (b) implies (c). To see that (c) implies (b) we suppose that $\mathcal{F}$ is a countable open $S(n)$ - filter without adherent points. Then the standard extension $X_{\mathcal{F}}$ of X by $\mathcal{F}$ is an $S(n)$ (1) space. Contradiction.
The above theorem for n = 1,2 is proved by R. Stephenson [Ste2].

3. Properties of the sequentially $\mathcal{P}$ - closed spaces.

It was proved by P. Alexandroff and P. Urysohn [AU] that the regular Hausdorff - closed spaces (regular sequentially Hausdorff - closed spaces) are precisely the compact (regular countably compact) spaces. In fact every regular $S(n)$ - closed space is compact as shown by Herlich [He] for $n = 2$ and by Porter and Votaw [PV] for $n > 2$. On the other hand every completely regular, regular - closed space is compact ([He], [BS]). We show next that similar results are valid for sequentially $\mathcal{P}$ - closed spaces.

COROLLARY 3.1. (a) *Let X be a regular space and $n \in N$. Then X is sequentially $S(n)$ - closed iff X is countably compact.*

(b) *Let X be a completely regular space. Then X is sequentially regular - closed iff X is countably compact.*

Proof. (a). Follows by the fact that every open cover of a regular space is an $S(n)$ - cover and by Theorem 2.1. (b). Follows by Theorem 2.3. and Theorem 2.7.

COROLLARY 3.2. *Let X be a Lindelöf, regular space and $n \in N$. The following conditions are equivalent*:

(a) *X is compact*
(b) *X is regular - closed*
(c) *X is sequentially regular - closed*
(d) *X is sequentially $S(n)$ - closed.*

Proof. For the equivalence of (a) and (b) see [He]. The equivalence of the other conditions follows by the fact that every Lindelöf, regular space is normal [En] and by Corollary 3.1.

THEOREM 3.3. *Let X be a normal space. Then X is sequentially normal - closed iff X is countably compact.*

Proof. The proof follows immediately from Corollary 3.1 and Lemma 3.4.

LEMMA 3.4. *Let X be a regular space,* $x \in X$ *and let* $X \setminus \{x\}$ *be a normal space. Then X is a normal space.*

THEOREM 3.5. *Let X be a perfectly normal space. The following conditions are equivalent*:

(a) *X is perfectly normal - closed*
(b) *X is sequentially perfectly normal - closed*
(c) *X is countably compact.*

Proof. For the equivalence of conditions (a) and (b) see [Ste2]. Obviously (c) implies (b). It is known that in a normal space the countable compactness coincides with the feeble compactenes (see [Ste1] and [Hew]) and that a regular space X is feebly compact iff every countable open regular filter on X has adherent points [Ste1]. Then the proof that (b) implies (c) follows by Lemma 3.4 and by the fact that if X is a normal space, x is a point in X and $X \setminus \{x\}$ is a perfectly normal space, then X is a perfectly normal space whenever x is a G_δ set in X.

THEOREM 3.6. *Let X be a locally compact space. Then X is sequentially locally compact - closed iff X is countably compact.*

Proof. It is obvious that if X is countably compact then X is sequentially locally compact - closed. Let X be a sequentially locally compact - closed space and let we assume that X is not countably compact. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points of X without adherent points. Let ωX be the Alexandroff compactification of X (see [En]) and $y = \omega X \setminus X$. It is easy to verify that $\lim_{n\to\infty} x_n = y$. Thus X is not sequentially locally compact - closed. Contradiction.

THEOREM 3.7. *For $\mathcal{P}$ = paracompact or metric if X is a $\mathcal{P}$ - space then the following conditions are equivalent:*

(a) *X is $\mathcal{P}$ - closed*
(b) *X is compact*
(c) *X is sequentially $\mathcal{P}$ - closed*
(d) *X is countably compact.*

Proof. For the equivalence of (a) and (b) see [SSe] and for the equivalence of (b) and (d) see [En]. Obviously (d) implies (c). The proof that (c) implies (d) for $\mathcal{P}$ = paracompact follows by Corollary 3.1 and by the fact that if X is a regular space, $x \in X$ and $X \setminus \{x\}$ is a paracompact space then X is a paracompact space. For $\mathcal{P}$ = metric it follows by Corollary 3.1 and by the fact that if X is a regular first countable space, $x \in X$ and $X \setminus \{x\}$ is a metric space then X is a metric space.

The spaces satisfying the equivalent conditions (a) - (f) of Theorem 2.1 and the equivalent conditions (a), (b) of Theorem 2.3 and Theorem 2.7 and the equivalent conditions (a) - (c) of Theorem 2.5 are in fact natural generalizations of the countable compactness. Moreover for $\mathcal{P}$ = US (SUS) the sequentially $\mathcal{P}$ - closed spaces are precisely the sequentially compact (countably compact) spaces [DGo]. The next theorem shows that some properties of the countably compact spaces are valid also for the sequentially $\mathcal{P}$ - closed spaces.

THEOREM 3.8. *Let $n \in N$ and $\mathcal{P}$ be one of the following classes of topological spaces: US, SUS, $S(n)$, regular, completely Hausdorff, completely regular, normal, perfectly normal, locally compact, paracompact or metric. Then the following conditions are satisfies:*

(a) *Sequentially $\mathcal{P}$ - closedness is preserved by continuous functions onto a $\mathcal{P}$ space.*
(b) *If a product of nonvoid spaces is sequentially $\mathcal{P}$ - closed then each coordinate is sequentially $\mathcal{P}$ - closed.*
(c) *Every sequentially $\mathcal{P}$ - closed space is pseudocompact.*

Proof. Obviously (a) implies (b). Clearly (a) and (c) are true when sequentially $\mathcal{P}$ - closedness coinsides with countable compactness or sequential compactness, i.e. for $\mathcal{P}$ = US, SUS, completely regular, normal, perfectly normal, locally compact, paracompact or metric. For the others $\mathcal{P}$ (a) follows from Theorem 2.1, Theorem 2.3 and Theorem 2.5. To see that (c) is true let $f : X \rightarrow \mathbf{R}$ be a continuous function. Then $f(X)$

is a sequentially $\mathcal{P}$ - closed metric space by (a). Thus $f(X)$ is a compact space by Theorem 2.7 and Theorem 3.7. This shows that f is bounded.

THEOREM 3.9. *Let $n \in N$. The following conditions are valid:*

(a) *$S(n)$ (1) - closedness is preserved by continuous functions onto an $S(n)$ - space.*
(b) *If a product of nonvoid spaces is an $S(n)$ (1) - closed space then each coordinate is $S(n)$ (1) - closed.*
(c) *Every $S(n)$ (1) - closed space is pseudocompact.*

Proof. (a) follows by Theorem 2.8 and (a) implies (b). We shall proof (c). Let $f : X \to R$ be a continuous function. Then $\mathcal{U} = \{f^{-1}(-k, k)\}_{k=1}^{\infty}$ is a countable regular cover of X. Since X is $S(n)$ (1) - closed then by Theorem 2.8 we can choose a finite proximate subcover of X. This implies that f is bounded.

The above theorem for $n = 1,2$ is proved by R. Stephenson [Ste2].

Let $\mathcal{P}$ be a class of topological spaces. $X \in \mathcal{P}$ is called *$\mathcal{P}$ - minimal* iff X has no strictly coarser $\mathcal{P}$ topologies. (For $\mathcal{P}$ - minimal spaces see [BPS]).

LEMMA 3.10. *Let $n \in N$, $\mathcal{P} = S(n)$, regular, completely Hausdorff or completely regular and X be a $\mathcal{P}$ - minimal space. If $x \in X$ and x is not a limit point for a non trivial sequence in X then $Y = X \setminus \{x\}$ is a sequentially $\mathcal{P}$ - closed space.*

Proof. Let us assume that Y is not sequentially $\mathcal{P}$ - closed space. Then there exists an open elementary $\mathcal{P}$ - filter $\mathcal{F}_1$ on Y without adherent points. Let $\mathcal{F}_x$ be the filter of neighbourhoods of the point x on X. We consider the filter $\mathcal{F} = \{U \mid U = V \cup W, V \in \mathcal{F}_1, W \in \mathcal{F}_x\}$. Obviously $\mathcal{F}$ is an open elementary $\mathcal{P}$ - filter on X and x is the unique adherent point for $\mathcal{F}$. Let $\mathcal{F}'_1$ be a maximal open elementary filter on Y containing $\mathcal{F}_1$. Suppose that $\{x_k\}_{k=1}^{\infty}$ determines $\mathcal{F}'_1$. But $\lim_{k \to \infty} x_k \neq x$. Thus $\mathcal{F} \underset{\neq}{\subset} \mathcal{F}_x$. But this contradicts to the $\mathcal{P}$ - minimality of X.

COROLLARY 3.11. *Let X be a compact Hausdorff space and $x \in X$. The point x is not the limit of a some (non trivial) sequence of X iff $X \setminus \{x\}$ is a countably compact space.*

Proof. Every compact Hausdorff space is a minimal completely regular space (see [Ba]). Now the corollary holds by Lemma 3.10 and Theorem 2.7.

Let $C(X)$ be the set of all real - valued continuous functions on a space $(X, \mathcal{T})$. The weak-topology $\mathcal{T}_\omega$ on X is the smallest topology on X such that all functions in $C(X)$ are continuous. Clearly $\mathcal{T}_\omega$ is coarser than $\mathcal{T}$ and the space $(X, \mathcal{T}_\omega)$ is completely regular iff $(X, \mathcal{T})$ is completely Hausdorff.

THEOREM 3.12. *Let $(X, \mathcal{T})$ be a completely Hausdorff space. $(X, \mathcal{T})$ is sequentially completely Hausdorff - closed iff $(X, \mathcal{T}_\omega)$ is countably compact.*

Proof. It follows by Theorem 2.7 and by the fact that $(X, \mathcal{T})$ is sequentially completely Hausdorff - closed iff $(X, \mathcal{T}_\omega)$ is sequentially completely regular - closed.

REFERENCES

[AU] P. ALEXANDROFF and P. URYSOHN, *Zur Teorie der topologischen Räume*, Math. Ann., *92* (1924), 258-266.

[Ba] B. BANASCHEWSKI, *Über zwei Extremaleigenschaften topologischen Räume*, Math. Nachr., *13* (1955), 141-150.

[Bo] N. BOURBAKI, *General Topology*, Part 1, Addison-Wesley, 1966.

[BPS] M. BERRI, J. PORTER and R. STEPHENSON, JR., *A survey of minimal topological spaces*, Proc. Kanpur Top. Conf. 1968, Acad. Press. New York, 1970, 93-114.

[BS] M. BERRI and R. SORGENFREY, *Minimal regular spaces*, Proc. Amer. Math. Soc., *14* (1963), 454-458.

[DG1] D. DIKRANJAN and E. GIULI, *Closure operators I*, Topology Appl., *27* (1987), 129-143.

[DG2] D. DIKRANJAN and E. GIULI, *Ordinal invariants and epimorphisms in some categories of weak Hausdorff spaces*, Comm. Math. Univ. Carolinae, *27* (2) (1986), 395-417.

[DG3] D. DIKRANJAN and E. GIULI, *$S(n)$ - θ - closed spaces*, Topology Appl., *28* (1988), 59-74.

[DGo] D. DIKRANJAN and I. GOTCHEV, *Sequentially closed and absolutely closed spaces*, Bollettino U.M.I., (7) 1-B (1987), 849-860.

[Di] D. DIKRANJAN, *Epimorphic order in $S(\alpha)$*, (preprint).

[En] R. ENGELKING, *General Topology*, Warszawa 1977.

[Go] I. GOTCHEV, *Superneighbourhood spaces and extensions of topological spaces*, to appear in Ann. Univ. de Sofia, Fac. Math. et Mec., *80* (1986).

[He] H. HERRLICH, *T_ν - Abgeschlossenheit and T_ν* - Minimalitat, Math. Z., *88* (1965), 285-294.

[Hew] E. HEWITT, *Rings of real-valued continuous functions*, Trans. Amer. Math. Soc., *64* (1948), 45-99.

[Ho] R. HOFFMAN, *On weak Hausdorff spaces*, Archiv. der Math., *32* (1979), 487-504.

[Ka] M. KATETOV, *Über H-abgeschlossene und bikompakt Räume*, Casopis pest. Mat., *69* (1940), 36-49.

[MN] M. MURDESHWAR and S. NAIMPALLY, *Semi-Hausdorff spaces*, Canad. Math. Bull., *9* (1966), 353-356.

[PV] J. PORTER and C. VOTAW, *$S(\alpha)$ spaces and regular Hausdorff extensions*, Pac. J. Math. *45*, 1 (1973), 327-345.

[SSe] C. SCARBOROUGH and R. STEPHENSON, JR., *Minimal topologies*, Colloq. Math., *19* (1968), 215-219.

[Ste1] R. STEPHENSON, JR., *Pseudocompact spaces*, Trans. Amer. Math. Soc., *134* (1968), 437-448.

[Ste2] R. STEPHENSON, JR., *Minimal first countable topologies*, Trans. Amer. Math. Soc., *138* (1969), 115-127.

[To] A.TOZZI, *US-spaces and closure operators,* Rend. Circolo Mat. Palermo Suppl., 12 (1986), 291-300.

[Ve] H.VELIČKO, *H-closed topological spaces*, Math. Sb. (N.S.) *70* (112) (1966), 98-112; Amer. Math. Soc., Transl. *78* (2) (1969), 103-118.

[Vi] G. VIGLINO, *T_n - spaces*, Notices Amer. Math. Soc., *16* (1969), 846.