SEQUENTIALLY $\mathcal{P}$ - CLOSED SPACES (*)

by IVAN GOTCHEV (in Sofia)(**) 

SOMMARIO. - Un $\mathcal{P}$ - spazio è sequenzialmente $\mathcal{P}$ - chiuso se e solo se esso è sequenzialmente chiuso in ogni $\mathcal{P}$ - spazio in cui esso sia immerso. Se $\mathcal{P}$ è una classe di spazi, rispettivamente, completamente regolari, normali, perfettamente normali, localmente compatti, paracompati, metrici, gli spazi sequenzialmente $\mathcal{P}$ - chiusi sono esattamente i $\mathcal{P}$ - spazi numerabilmente compatti. Per varie categorie $\mathcal{P}$ consistenti di spazi di Hausdorff si danno caratterizzazioni interne degli spazi sequenzialmente $\mathcal{P}$ - chiusi che permettono di stabilire molte altre proprietà.

SUMMARY. - A $\mathcal{P}$ - space is sequentially $\mathcal{P}$ - closed if it is sequentially closed in every $\mathcal{P}$ - space in which it is embedded. For $\mathcal{P}$ - completely regular, normal, perfectly normal, locally compact, paracompact and metric the sequentially $\mathcal{P}$ - closed spaces are precisely the countably compact $\mathcal{P}$ - spaces. Internal characterization of sequentially $\mathcal{P}$ - closed spaces are given for various categories $\mathcal{P}$ consisting of Hausdorff spaces which permits to establish a lot of other properties of the sequentially $\mathcal{P}$ - closed spaces.

0. Introduction.

A space $Y$ is called sequentially determined extension of its subspace $X$ iff for every point $y \in Y$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $X$ such that $\lim_{n \to \infty} x_n = y$ [Go]. Let $\mathcal{P}$ be a class of topological spaces. A space $X \in \mathcal{P}$ is said to be $\mathcal{P}$ - closed (sequentially $\mathcal{P}$ - closed) iff $X$ is closed (sequentially closed) in every $\mathcal{P}$ space in which it is embedded. In other words $X$ is sequentially $\mathcal{P}$ - closed iff $X$ has no sequentially determined extension $Y \in \mathcal{P}$ and $Y \neq X$ (this holds under very mild restrictions on $\mathcal{P}$ which are verified in all cases considered here).

Obviously, every $\mathcal{P}$ - closed space is sequentially $\mathcal{P}$ - closed. The

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(***) Indirizzo dell'Autore: Institute of Mathematics - Bulgarian Academy of Sciences - 1090 Sofia (Bulgaria).
\( \mathcal{P} \) - closed spaces were extensively studied for categories \( \mathcal{P} \) consisting of Hausdorff spaces (see [BPS]). For such \( \mathcal{P} \) every compact (countably compact) \( \mathcal{P} \) - space is \( \mathcal{P} \) - closed (sequentially \( \mathcal{P} \) - closed). The sequentially Hausdorff - closed spaces were introduced by P. Alexandroff and P. Urysohn [AU]. They were proved that the regular sequentially Hausdorff - closed spaces coincide with the regular countably compact spaces. In [DGo] sequentially \( \mathcal{P} \) - closed spaces for some classes \( \mathcal{P} \) of topological spaces between \( T_1 \) and \( T_2 \) are studied. It is proved in particular that for the category \( SUS \) of topological spaces in which every convergent sequence has a unique accumulation point the sequentially \( SUS \) - closed spaces are precisely the countably compact \( SUS \) - spaces. On the other hand it follows by results of A. Tozzi [To] that the sequentially \( SUS \) - closed spaces coincide with the absolutely \( SUS \) - closed spaces, introduced by D. Dikranjan and E. Giuli [DG3] in a more general situation. This was in fact the starting point of our study of sequentially \( \mathcal{P} \) - closed spaces.

The aim of this paper is to study the sequentially \( \mathcal{P} \) - closed spaces for \( \mathcal{P} = S (n) \), regular, completely Hausdorff, completely regular, normal, perfectly normal, locally compact, paracompact and metric.

In section 1 we introduce open elementary filters which serve as the main tool in the study of sequentially \( \mathcal{P} \) - closed spaces. We introduce also \( \theta^n \) convergence and \( S^n \) - convergence (generalizations of the \( \theta \) - convergence introduced by Veličko [Ve] and the usual convergence) and characterize the \( S (n) \) - spaces by means of these convergences following Veličko [Ve], Dikranjan and Giuli [Di], [DG1], [DG3].

In section 2 we give internal characterization of the sequentially \( \mathcal{P} \) - closed spaces in terms of special filters and covers for \( \mathcal{P} = S (n) \), regular, completely Hausdorff and completely regular. For \( \mathcal{P} = S (n) \) this characterization involves the \( \theta^n \) - convergence and \( S^n \) - convergence. An example of a sequentially Hausdorff - closed completely Hausdorff space (and hence \( S (n) \) - space for \( n = 1, 2, \ldots \)) is given which is neither Hausdorff - closed nor countably compact. We give also an example of a sequentially regular - closed space which is neither regular - closed nor countably compact. We discuss also the relations between sequentially \( \mathcal{P} \) - closed spaces and \( \mathcal{P} (1) \) - closed spaces (see [BPS]).

In section 3 various properties of the sequentially \( \mathcal{P} \) - closed spaces are established. We show in particular that for \( \mathcal{P} \) - completely regular, normal, perfectly normal, locally compact, paracompact and metric the sequentially \( \mathcal{P} \) - closed spaces are precisely the countably compact \( \mathcal{P} \) - spaces. Since for paracompact spaces countable compactness coincides with compactness in the last two cases we get the compact \( \mathcal{P} \) - spaces.

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1. Preliminaries.

Throughout the paper the properties of regularity, completely regularity, etc. include the $T_1$ separation property, $\overline{U}$ denotes the closure of the set $U$ in a given topological space, $N$ denotes the set of positive integers and $R$ denotes the real line with the usual topology. In general the terminology and notation follow [En].

Let $X$ be a topological space and let $A$ be a countable subset of $X$. A maximal open elementary filter generated by $A$ is the open filter on $X$ with base $\mathcal{B} = \{F \mid F \subseteq X, F \text{ is open in } X \text{ and } A \setminus F \text{ is finite}\}$. An open filter $\mathcal{F}$ on $X$ is a maximal open elementary filter, iff there exists a countable subset $A$ of $X$, such that $\mathcal{F}$ is the maximal open elementary filter on $X$ generated by $A$. If $X$ is a $US$ - space (every convergent sequence in $X$ has a unique limit point [MN]) then $A$ is uniquely determined by $\mathcal{F}$ up to a finite subset (see [DGo]). An open filter will be called open elementary filter, iff it is contained in some maximal open elementary filter. A filter $\mathcal{F}$ is free iff $\cap F \mid F \in \mathcal{F} = \phi$.

Let $\mathcal{F} = \{F_\alpha\}_{\alpha \in \mathcal{A}}$ be a free open elementary filter on $X$. On the set $X^\mathcal{F} = X \cup \{\mathcal{F}\}$ we introduce the following topology: The set $X$ is open, the relative topology of $X$ coincides with the original topology of $X$ and for open base at the point $\mathcal{F}$ we take the family $\{\mathcal{F} \cup F_\alpha \mid F_\alpha \in \mathcal{F}, \alpha \in \mathcal{A}\}$. It is easy to verify that the topological space obtained in this way is a $T_1$ sequentially determined extension of $X$ whenever $X$ is $T_1$. In the sequel $X^\mathcal{F}$ will be called standard sequentially determined extension of $X$ by $\mathcal{F}$.

Let $X$ be a topological space, $M \subseteq X$ and $n \in N$. The point $x \in X$ is $S(n)$ - separated from $M$ if there exist open sets $U_i, i = 1, 2, \ldots, n$ such that $x \in U_1 \subseteq \ldots \subseteq U_n, \overline{U}_i \subseteq U_{i+1}$ and $\overline{U}_n \cap M = \phi$; $x$ is $S(0)$ - separated from $M$ iff $x \notin \overline{M}$. The space $X$ is an $S(n)$ - space, iff every two different points in $X$ are $S(n)$ separated [Vi]. It is obvious that $S(1) = T_2, S(2) = T_{2,5}$ (distinct points can be separated by disjoint closed neighbourhoods).

An open cover $\{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{A}}$ of the space $X$ is an $S(n)$ - cover, iff every point $x \in X$ is $S(n)$ - separated from some $X \setminus U_\alpha$. Clearly the $S(0)$ - covers are exactly the open covers. A filter $\mathcal{F}$ on $X$ is an $S(n)$ - filter, iff every point $x \in X$, which is not adherent point for $\mathcal{F}$ is $S(n)$ - separated from some $U \in \mathcal{F}$ [PV].

Let $X$ be a topological space and $n \in N$. The point $x \in X$ will be called $S^n$ - limit ($\theta^n$ - limit) of a sequence $\{x_n\}_{n=1}^\infty$ in $X$, iff for every chain $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_n$ of open neighbourhoods of $x$ such that $\overline{U}_i \subseteq U_{i+1}$ for
$i = 1, 2, ..., n - 1, U_n$ ($\overline{U}_n$) contains all but a finite number of the members of the sequence. Every sequence which has an $S^n$ - limit ($\theta^n$ - limit) will be called $S^n$ - convergent ($\theta^n$ - convergent). If for every chain $U_1 \subset U_2 \subset ... \subset U_n$ of open neighbourhoods of $x$ such that $\overline{U}_i \subset U_i + 1$ for $i = 1, 2, ..., n - 1$, $U_n$ ($\overline{U}_n$) contains infinitely many members of the sequence then $x$ will be called $S^n$ - adherent point ($\theta^n$ - adherent point) of the sequence $\{x_n\}_{n=1}^{\infty}$. Clearly, the $S^1$ - convergence is the usual topological convergence and the $\theta^1$ - convergence is the $\theta$ - convergence defined by Veliičko [Ve]. Also every $S^n$ - convergent sequence is $\theta^n$ - convergent and every $\theta^n$ - convergent sequence is $S^n + 1$ - convergent. So every convergent, in the usual sense, sequence is $S^n$ - convergent and $\theta^n$ - convergent for every $n \in N$. If $X$ is regular then the $S^n$ - convergence and the $\theta^n$ - convergence coincides with the usual convergence. Similar facts are valid for $S^n$ and $\theta^n$ - adherent points of a sequence. In the same way $S^n$ ($\theta^n$) convergence and $S^n$ ($\theta^n$) adherent points can be defined for nets. (For $\theta^n$ - adherent points see [DG3] and for $n = 1$ see also [DG2]).

The $S(n)$ - spaces can be characterized by means of the $\theta^n$ - convergence, $S^n$ - convergence and the usual convergence.

**PROPOSITION 1.1.** Let $X$ be a topological space and $n \in N$. The following conditions are equivalent:

(a) $X$ is an $S(n)$ - space
(b) every convergent sequence in $X$ has a unique $\theta^n$ - adherent point
(c) every convergent sequence in $X$ has a unique $\theta^n$ - limit
(d) every convergent sequence in $X$ has a unique $S^n + 1$ - adherent point
(e) every convergent sequence in $X$ has a unique $S^n + 1$ - limit.

**Proof.** Follows directly from the definition.

2. Characterization of the sequentially $\mathcal{C}$ - closed spaces.

The next theorem characterizes sequentially $S(n)$ - closed spaces. Analogous results for $S(n)$ - closed spaces were obtained in [PV] and for $S(n)$ - $\theta$ - closed spaces were obtained in [DG3].

**THEOREM 2.1.** Let $X$ be a $T_1$ space and $n \in N$. The following conditions are equivalent:

(a) every sequence in $X$ has a $\theta^n$ - adherent point
(b) every sequence in $X$ has an $S^{n+1}$-adherent point
(c) every countable $S$-cover of $X$ has a finite subcover
(d) every $S$-filter with a countable base of closed sets has an adherent point
(e) every open elementary $S$-filter has an adherent point
(f) every maximal open elementary $S$-filter has an adherent point.

If $X$ is an $S$-space then the above conditions are equivalent to:
(g) $X$ is sequentially $S$-closed.

Proof. Obviously (a) implies (b) and (e) implies (f). To see that (b) implies (c) assume that $\{U_n\}_{n=1}^\infty$ is a countable $S$-cover which has no finite subcover. For every $k = 1, 2, ..., \infty$ we choose $x_k \notin \bigcup_{i=1}^k U_i$. Let $x \in X$. Since $\{U_i\}_{i=1}^\infty$ is an $S$-cover then there exists an element $U_i$ of the cover and chain $V_1 \subset V_2 \subset \cdots \subset V_n$ of open neighbourhoods of $x$ such that $V_j \subset V_{j+1}$ for $j = 1, 2, ..., n-1$ and $V_n \subset U_i$. This means that $x$ is not an $S^{n+1}$-adherent point for $\{x_k\}_{k=1}^\infty$ since for every $i \in \mathbb{N}$ and $k \geq i$ we have $x_k \notin U_i$. Therefore $\{x_k\}_{k=1}^\infty$ has no $S^{n+1}$-adherent points in $X$: a contradiction. Let now $\mathcal{F}$ be an $S$-filter on $X$ with countable closed base $\{F_i\}_{i=1}^\infty$. Assume that $\mathcal{F}$ has no adherent points. Then $\mathcal{U} = \{U_i \mid U_i = X \setminus F_i, i \in \mathbb{N}\}$ is an $S$-cover of $X$. Let $U_{i_1}, U_{i_2}, ..., U_{i_k}$ be a finite subcover of $\mathcal{U}$. Since $\mathcal{F}$ is a filter we have $\bigcap_{j=1}^k (X \setminus U_{i_j}) = \bigcap_{j=1}^k F_j \neq \emptyset$, hence $U_{i_1}, U_{i_2}, ..., U_{i_k}$ is not a cover of $X$. This contradiction proves that (c) implies (d). Now we will prove that (d) implies (e). Let $\mathcal{F}$ be an open elementary $S$-filter without adherent points. There exists a maximal open elementary filter $\mathcal{F}'$ such that $\mathcal{F} \subset \mathcal{F}'$. Then $\mathcal{F}'$ has no adherent points and if $\{x_k\}_{k=1}^\infty$ determines $\mathcal{F}'$ then $\{x_k\}_{k=1}^\infty$ is a closed set for every $i \in \mathbb{N}$. Thus the filter $\mathcal{F}''$ generated by $\{F_i \mid F_i = \{x_k\}_{k=1}^\infty, i \in \mathbb{N}\}$ contains $\mathcal{F}'$. So $\mathcal{F}''$ is an $S$-filter with a countably base of closed sets without adherent points. To see that (f) implies (a) assume that $\{x_k\}_{k=1}^\infty$ is a sequence in $X$ which has no $\theta^n$-adherent points. Let $\mathcal{F}$ be the maximal open elementary filter generated by $\{x_k\}_{k=1}^\infty$ and $x$ be an arbitrary point.
of $X$. There exists a chain $U_1 \subset U_2 \subset \ldots \subset U_n$ of open neighbourhoods of $x$ such that $\bar{U}_i \subset U_i + 1$, $i = 1, 2, \ldots, n - 1$ and $X \setminus \bar{U}_n$ contains all but a finite members of the sequence. Therefore $X \setminus \bar{U}_n \in \mathcal{F}$. On the other hand $x$ and $X \setminus \bar{U}_n$ are $S(n)$-separated. This means that $\mathcal{F}$ is a maximal open elementary $S(n)$-filter with no adherent points. Contradiction. Now let we assume that $X$ is not sequentially $S(n)$-closed. Thus there exists an $S(n)$-space $X \supseteq Y$, a point $y \in Y \setminus X$ and a sequence $\{x_k\}_{k=1}^{\infty}$ in $X$ such that $\lim_{k \to \infty} x_k = y$. Clearly $y$ is a $\theta^n$-adherent point of $\{x_k\}_{k=1}^{\infty}$. From Proposition 1. it follows that $\{x_k\}_{k=1}^{\infty}$ has no other $\theta^n$-adherent points in $Y$. Thus $\{x_k\}_{k=1}^{\infty}$ has no $\theta^n$-adherent points in $X$. This contradiction proves that (a) implies (g). To prove that (g) implies (f) assume that $\mathcal{F}$ is a maximal open elementary $S(n)$-filter with no adherent points. Let $X\mathcal{F}$ be the standard sequentially determined extension of $X$ by $\mathcal{F}$. It is easy to verify that $X\mathcal{F}$ is an $S(n)$-space. Thus $X$ is not sequentially $S(n)$-closed. Contradiction.

The idea to characterize closed spaces with $\theta$-convergence and elementary filters (but in a somewhat different sense, see [Bo], chap. 1. § 6) comes from Veličko [Ve].

Now we show that the class of sequentially $S(n)$-closed spaces is not exhausted by the $S(n)$-closed spaces and by the countably compact $S(n)$-spaces.

**Example 2.2.** Let $\mathcal{N}$ be the space of positive integers with the discrete topology and let $\beta \mathcal{N}$ be the Čech-Stone compactification of $\mathcal{N}$. Let also $X = (\beta \mathcal{N} \setminus \mathcal{N}) \cup \{x_{ij}\}_{i,j=1}^{\infty} \cup \{y_i\}_{i=1}^{\infty}$. We provide $X$ with a topology as follows: The points $\{x_{ij}\}_{i,j=1}^{\infty}$ are isolated for $i \in \mathcal{N}$ and $j \in \mathcal{N}$. For a neighbourhood base of $y_i$ ($i \in \mathcal{N}$) we take the family $y_i \cup \{x_{ij}\}_{j=k}^{\infty}$, $k \in \mathcal{N}$. Let $\{\mathcal{F}\} \in \beta \mathcal{N} \setminus \mathcal{N}$ and let $\{U_\alpha, \alpha \in \mathcal{A}\}$ be a neighbourhood base of $\{\mathcal{F}\}$ in $\beta \mathcal{N}$. For a neighbourhood base of $\{\mathcal{F}\}$ in $X$ we take the family $\{V_\alpha \mid V_\alpha = (U_\alpha \setminus \mathcal{N}) \cup \{x_{ij}\}_{i,j=1}^{\infty}, i \in U_\alpha \cap \mathcal{N}\}, \alpha \in \mathcal{A}\}$. It is easy to verify that $X$ is a Hausdorff-closed $S(n)$-space for every $n \in \mathcal{N}$ and $X$ is not countably compact. Let now $\mathcal{F} \in \beta \mathcal{N} \setminus \mathcal{N}$ and $Y = X \setminus \{\mathcal{F}\}$. Clearly $Y$ is $S(n)$-closed for no $n$ and $Y$ is not countably compact. By (a) of the above theorem $Y$ is sequentially $S(n)$-closed.

Let $X$ be a topological space. An open filter $\mathcal{F}$ on $X$ is a regular filter iff for each $U \in \mathcal{F}$ there exists $V \in \mathcal{F}$ such that $\bar{V} \subset U$ [Ba]. Let $\mathcal{U}$ and
$\mathcal{V}$ be open covers of a space $X$. $\mathcal{V}$ is a shrinkable refinement of $\mathcal{U}$ iff for each $V \in \mathcal{V}$, there is $U \in \mathcal{U}$ such that $\overline{V} \subseteq U$. An open cover $\mathcal{U}$ is regular iff there exists an open cover $\mathcal{V}$ which refines $\mathcal{U}$ and $\mathcal{V}$ is a shrinkable refinement of itself [BPS].

**Theorem 2.3.** Let $X$ be a $T_1$ space. The following conditions are equivalent:

(a) every open elementary regular filter on $X$ has adherent points
(b) every countable regular cover of $X$ has a finite subcover.

If $X$ is a regular space then the above conditions are equivalent to:

(c) $X$ is sequentially regular-closed.

**Proof.** To see that (a) implies (b) assume that $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ is a countable regular cover without finite subcovers. For each $k = 1, 2, \ldots$ we choose a point $x_k \in X$ such that $x_k \notin \bigcup_{i=1}^{k} U_i$. Then clearly $x_k \notin U_i$ whenever $k \geq i$. Let $\mathcal{V} = \{V_\sigma\} \sigma \in \mathcal{A}$ be a cover of $X$ which refines $\mathcal{U}$ and $\mathcal{V}$ is a shrinkable refinement of itself. Clearly the cover $\mathcal{V}$ has no finite subcovers. It is easy to verify that the filter $\mathcal{F}$ generated by the filter base $\{X \setminus \bigcup_{i=1}^{k} V_{\alpha_i}, \alpha_i \in \mathcal{A}, k \in \mathbb{N}\}$ is an open elementary regular filter on $X$ without adherent points. Contradiction. Let now $\mathcal{F}$ be an open elementary regular filter on $X$ without adherent points. There exists a maximal open elementary filter such that $\mathcal{F} \subseteq \mathcal{F}'$. Let $\{x_k\}_{k=1}^{\infty}$ determines $\mathcal{F}'$. But $\mathcal{F}'$ has no adherent points. Then $\{x_k\}_{k=1}^{\infty}$ is a closed set for every $i \in \mathbb{N}$. Let $\mathcal{U} = \{U_i \mid U_i = X \setminus \overline{\{x_k\}_{k=i}^{\infty}}, i \in \mathbb{N}\}$ and $\mathcal{V} = \{V\}$ there exists an open set $W \in \mathcal{F}$ such that $V = X \setminus \overline{W}$. It is easy to verify that $\mathcal{V}$ is an open cover of $X$ which refines $\mathcal{U}$ and $\mathcal{V}$ is a shrinkable refinement of itself. Thus $\mathcal{U}$ is a countable regular cover of $X$ without finite subcovers and this proves that (b) implies (a). Now we prove that (a) implies (c). Assume that $X$ is not sequentially regular-closed. Thus there exists a regular space $Y \supset X$, a point $y \in Y \setminus X$ and a sequence $\{x_n\}_{n=1}^{\infty}$ in $X$ such that $\lim_{n \to \infty} x_n = y$. Let $\mathcal{V}_y$ be the filter of neighbourhoods of $y$ on $Y$. Since $Y$ is a regular space it follows that $\mathcal{V}_y$ is an open elementary regular
filter on $Y$ with no adherent points in $X$. Then $\mathcal{U} = \{V \mid \text{there exists } W \in \mathcal{U}_Y \text{ such that } V = X \cap W\}$ is an open elementary regular filter on $X$ without adherent points. Contradiction. Assume that there exists an open elementary regular filter $\mathcal{F}$ on $X$ without adherent points. Then the standard sequentially determined extension $X\mathcal{F}$ of $X$ by $\mathcal{F}$ will be a regular space. This contradicts the sequentially regular - closedness of $X$, so (c) implies (a).

Now we show that the class of sequentially regular - closed spaces is not exhausted by the regular - closed spaces and by the countably compact regular spaces.

**Example 2.4.** The space $X$ in Example 4.18 in [BPS] is a minimal regular space which is not countably compact [BS]. Let $x = (\omega_1, 1, 1) = (\omega_1, 1, 2)$ and $Y = X \setminus \{x\}$. Then by Lemma 3.10 $Y$ is a sequentially regular - closed space which is neither regular - closed nor countably compact.

Let $X$ be a topological space. $X$ is completely Hausdorff iff for each pair $x, y$ of distinct points, there exists a continuous real - valued function $f$ such that $f(x) \neq f(y)$. An open filter $\mathcal{F}$ on $X$ is completely Hausdorff iff for each $x \in X$ which is not an adherence point of $\mathcal{F}$ there exists an open set $U$ containing $x$, $V \in \mathcal{F}$ and continuous real - valued function $f$ on $X$ such that $f(U) = \{1\}$ and $f(V) = \{0\}$. An open filter $\mathcal{F}$ on $X$ is completely regular iff for each $U \in \mathcal{F}$, there exists $V \in \mathcal{F}$ and a continuous real - valued function $f$ on $X$ such that $f(V) = \{0\}$ and $f(X \setminus U) = \{1\}$. Let $\mathcal{U}$ and $\mathcal{V}$ be covers of a space $X$. $\mathcal{V}$ is a continuous refinement of $\mathcal{U}$ iff for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ and continuous real - valued function $f$ on $X$ such that $f(V) = \{0\}$ and $f(X \setminus U) = \{1\}$. An open cover is completely Hausdorff iff it has a continuous refinement [BPS]. An open cover $\mathcal{U}$ is completely regular iff there is an open cover $\mathcal{V}$ which refines $\mathcal{U}$ and $\mathcal{V}$ is a continuous refinement of itself.

**Theorem 2.5.** Let $X$ be a $T_1$ space. The following conditions are equivalent:

(a) every countable completely Hausdorff cover of $X$ has a finite subcover

(b) every open elementary completely Hausdorff filter on $X$ has adherent points

(c) every maximal open elementary completely Hausdorff filter on $X$ has adherent points.

If $X$ is a completely Hausdorff space then the above conditions are equivalent to:
(d) \( X \) is sequentially completely Hausdorff - closed.

\textbf{Proof.} To see that (a) implies (b) let \( \mathcal{F} \) be an open elementary completely Hausdorff filter on \( X \) without adherent points. There exists a maximal open elementary filter \( \mathcal{F}' \) on \( X \) such that \( \mathcal{F} \subset \mathcal{F}' \). If \( \{x_k\}_{k=1}^{\infty} \) is a sequence which generate \( \mathcal{F}' \) then \( \{x_k\}_{k=1}^{\infty} \) is a closed set for every \( i \in N \). Let \( U = \{ U_i \mid U_i = X \setminus \{x_k\}_{k=i}^{\infty}, i \in N \} \) and \( \mathcal{U} = \{ V_x \mid V_x = f^{-1}[0, v_2) \} \). \( x \in X, W \in \mathcal{F} \) and \( f : X \to \mathbb{R} \) is such that \( f(x) = 0 \) and \( f(W) = \{1\} \). Then \( \mathcal{U} \) and \( \mathcal{V} \) are open covers of \( X \) and \( \mathcal{V} \) is a continuous refinement of \( \mathcal{U} \). Thus \( \mathcal{U} \) is a completely Hausdorff cover of \( X \) without a finite subcover. Obviously (b) implies (c). We prove that (c) implies (a). Let \( \mathcal{U} = \{ U_i \}_{i=1}^{k} \) be a countable completely Hausdorff cover of \( X \) without finite subcovers. Let \( \mathcal{V} = \{ V_a \}_{a \in A} \) be a cover of \( X \) which is a continuous refinement of \( \mathcal{U} \). For every \( k \in N \) we choose \( x_k \notin \bigcup_{i=1}^{k} U_i \) and let \( \mathcal{F} \) be the maximal open elementary filter generated by \( \{x_k\}_{k=1}^{\infty} \). If \( x \) is an arbitrary point of \( X \) then there exists \( a \in A \) such that \( x \in V_a \) and there exists \( i \in N \) and \( f : X \to \mathbb{R} \) such that \( f(V_a) = \{0\} \) and \( f(X \setminus U_i) = \{1\} \). Let \( W = f^{-1}(v_2, 1] \) and \( g(x) = 2 \cdot \min(f(x), v_2) \). Then \( W \in \mathcal{F}, g(V_a) = \{0\} \) and \( g(W) = \{1\} \). Thus \( \mathcal{F} \) is a maximal open elementary filter which is completely Hausdorff and it has no adherent points. To prove that (b) implies (d) assume that \( X \) is not sequentially completely Hausdorff - closed. Then there exists a completely Hausdorff space \( Y \supset X \), a point \( y \in Y \setminus X \) and a sequence \( \{x_k\}_{k=1}^{\infty} \) of points of \( X \) such that \( \lim_{k \to \infty} x_k = y \). The filter \( \mathcal{V}_y \) of neighbourhoods of \( y \) is a completely Hausdorff filter on \( Y \). Let \( \mathcal{V} = \{ V \mid \text{there exists } W \in \mathcal{V}_y \text{ such that } V = X \cap W \} \). Then \( \mathcal{V} \) is an open elementary completely Hausdorff filter on \( X \) without adherent points. Contradiction. If there exists an open elementary completely Hausdorff filter \( \mathcal{F} \) on \( X \) without adherent points, then \( X_{\mathcal{F}} \) will be a completely Hausdorff, sequentially determined extension of \( X \). This proves that (d) implies (b).

The space \( Y \) in Example 2.2 is also completely Hausdorff, consequently sequentially completely Hausdorff - closed. On the other hand it is neither completely Hausdorff - closed nor countably compact.
LEMMA 2.6. In a completely regular space $X$ every open cover of $X$ is a completely regular cover.

**Proof.** Let $\mathcal{U} = \{U_\alpha\} \alpha \in \mathcal{A}$ be an open cover of $X$. For every $\alpha \in \mathcal{A}$ and every $x \in U_\alpha$ let $f_{\alpha, x}$ be a continuous real-valued function such that $f_{\alpha, x}(x) = 1$ and $f_{\alpha, x}(X \setminus U_\alpha) = \{0\}$. If $\mathcal{V} = \{V | V = f_{\alpha, x}^{-1}(\mathbb{V}_n, 1), \alpha \in \mathcal{A}, x \in U_\alpha, n \geq 2\}$ then $\mathcal{V}$ is an open cover of $X$, $\mathcal{V}$ refines $\mathcal{U}$ and $\mathcal{V}$ is a continuous refinement of itself.

THEOREM 2.7. Let $X$ be a $T_1$ space. The following conditions are equivalent:

(a) every countable completely regular cover of $X$ has a finite subcover
(b) every open elementary completely regular filter on $X$ has adherent points.

If $X$ is a completely regular space then the above conditions are equivalent to:

(c) $X$ is sequentially completely regular - closed
(d) $X$ is countably compact.

**Proof.** Let $\mathcal{F}$ be an open elementary completely regular filter on $X$ without adherent points and let $\mathcal{F}'$ be a maximal open elementary filter such that $\mathcal{F} \subset \mathcal{F}'$. If $\{x_k\}_{k=1}^\infty$ is a sequence which generates $\mathcal{F}'$, then $\{x_k\}_{k=i}^\infty$ is a closed set for every $i \in \mathbb{N}$. Let $\mathcal{U} = \{U_i | U_i = X \setminus \{x_k\}_{k=i}^\infty, i \in \mathbb{N}\}$ and $\mathcal{V} = \{V | \text{there exists an open set } W \in \mathcal{F} \text{ such that } V = X \setminus \overline{W}\}$. Then $\mathcal{U}$ and $\mathcal{V}$ are open covers of $X$, $\mathcal{V}$ refines $\mathcal{U}$ and $\mathcal{V}$ is continuous refinement of itself. Thus $\mathcal{U}$ is a completely regular cover of $X$ without finite subcovers. This proves that (a) implies (b). To see that (b) implies (a) let $\mathcal{U} = \{U_i\}_{i=1}^\infty$ be a countable completely regular cover of $X$ without finite subcovers and let $\mathcal{V} = \{V_\alpha\} \alpha \in \mathcal{A}$ be an open cover of $X$ which refines $\mathcal{U}$ and $\mathcal{V}$ is a continuous refinement of itself. For every $k \in \mathbb{N}$ we choose $x_k \notin \bigcup_{i=1}^k U_i$ and let $\mathcal{B} = \{W | \text{there exist } V_\alpha \in \mathcal{V} \text{ such that } W = X \setminus \bigcup_{i=1}^k \overline{V_\alpha}, k \in \mathbb{N}\}$. Let $\mathcal{F}'$ be the maximal open elementary filter on $X$ determined by $\{x_k\}_{k=1}^\infty$. If $\mathcal{F}$ is the open filter
with base $\mathcal{B}$ then $\mathcal{F} \subseteq \mathcal{F}^*$ and hence $\mathcal{F}$ is an open elementary completely regular filter without adherent points. Now we prove that (b) implies (c). Assume that $X$ is not sequentially completely regular - closed. Then there exists a completely regular space $Y \supseteq X$, a point $y \in Y \backslash X$ and a sequence $\{x_k\}_{k=1}^{\infty}$ of points of $X$ such that $\lim_{k \to \infty} x_k = y$. The filter $\mathcal{V}_y$ of neighbourhoods of $y$ is a completely regular filter on $Y$. Let $\mathcal{V} = \{W\}$ there exists an open set $V \in \mathcal{V}_y$ such that $W = X \cap V$, then $\mathcal{V}$ is an open elementary completely regular filter on $X$ without adherent points. Contradiction. If there exists an open elementary completely regular filter $\mathcal{F}$ on $X$ without adherent points, then $X_{\mathcal{F}}$ will be a completely regular, sequentially determined extension of $X$. This proves that (c) implies (b). The equivalence of conditions (a) and (d) follows directly by Lemma 2.6.

For a class $\mathcal{P}$, the class of all first countable $\mathcal{P}$ - spaces will be denoted by $\mathcal{P} (1)$ [BPS]. Evidently every sequentially $\mathcal{P}$ - closed $\mathcal{P} (1)$ space is $\mathcal{P} (1)$ - closed. Hence the sequentially $\mathcal{P} (1)$ - closed spaces coincide with the $\mathcal{P} (1)$ - closed spaces. For various classes $\mathcal{P}$ the $\mathcal{P} (1)$ - closed spaces were studied in [Ste2].

A family of open sets $\mathcal{U}$ in a space $X$ is a proximate cover of $X$ iff $\bigcup \{\overline{U} | U \in \mathcal{U}\} = X$ [Ka].

**Theorem 2.8.** Let $X$ be a $T_1$ space and $n \in \mathbb{N}$. The following conditions are equivalent:

(a) every countable $S (n - 1)$ - cover of $X$ contains a finite proximate subcover

(b) every countable open $S (n)$ - filter has adherent points.

If $X$ is an $S (n) (1)$ - space then the above conditions are equivalent to:

(c) $X$ is $S (n) (1)$ - closed.

**Proof.** To see that (a) implies (b) suppose that $\mathcal{F}$ is a countable open $S (n)$ - filter on $X$ without adherent points. Then $\mathcal{U} = \{U | U = X \backslash \overline{V} | V \in \mathcal{F}\}$ is a countable $S (n - 1)$ - cover of $X$ and $\mathcal{U}$ has a proximate subcover.

So that if $X = \bigcup_{i=1}^{k} \overline{U}_i$ then $\bigcap_{i=1}^{k} (X \backslash \overline{U}_i) = \bigcap_{i=1}^{k} V_i = \phi$. But $V_i \in \mathcal{F}$ for $i = 1, 2, \ldots, k$. Contradiction. Now let us assume that $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ is a countable $S (n - 1)$ - cover of $X$ which has no finite proximate subcovers. For every
$i \in N$ we consider $V_i = X \setminus (\bigcup_{j=1}^{i} U_j)$. Obviously $\mathcal{B} = \{V_i\}_{i=1}^{\infty}$ is a countable open base of a filter $\mathcal{F}$. One can easily verify that $\mathcal{F}$ is a countable open $S(n)$ - filter without adherent points. This contradiction proves that (b) implies (a). If $X$ is not $S(n)$ (1) - closed then there exists an $S(n)$ (1) extension $Y$ of $X$ and a point $y \in Y \setminus X$. But the trace on $X$ of the neighbourhood filter of the point $y$ is a countable open $S(n)$ - filter on $X$ without adherent points in $X$ and this proves that (b) implies (c). To see that (c) implies (b) we suppose that $\mathcal{F}$ is a countable open $S(n)$ - filter without adherent points. Then the standard extension $X_{\mathcal{F}}$ of $X$ by $\mathcal{F}$ is an $S(n)$ (1) space. Contradiction.

The above theorem for $n = 1, 2$ is proved by R. Stephenson [Ste2].

3. Properties of the sequentially $\mathcal{P}$ - closed spaces.

It was proved by P. Alexandroff and P. Urysohn [AU] that the regular Hausdorff - closed spaces (regular sequentially Hausdorff - closed spaces) are precisely the compact (regular countably compact) spaces. In fact every regular $S(n)$ - closed space is compact as shown by Herlich [He] for $n = 2$ and by Porter and Votaw [PV] for $n > 2$. On the other hand every completely regular, regular - closed space is compact ([He], [BS]). We show next that similar results are valid for sequentially $\mathcal{P}$ - closed spaces.

**Corollary 3.1.** (a) Let $X$ be a regular space and $n \in N$. Then $X$ is sequentially $S(n)$ - closed iff $X$ is countably compact.

(b) Let $X$ be a completely regular space. Then $X$ is sequentially regular - closed iff $X$ is countably compact.

**Proof:** (a). Follows by the fact that every open cover of a regular space is an $S(n)$ - cover and by Theorem 2.1. (b). Follows by Theorem 2.3. and Theorem 2.7.

**Corollary 3.2.** Let $X$ be a Lindelöf, regular space and $n \in N$. The following conditions are equivalent:

(a) $X$ is compact
(b) $X$ is regular - closed
(c) $X$ is sequentially regular - closed
(d) $X$ is sequentially $S(n)$ - closed.
Proof. For the equivalence of (a) and (b) see [He]. The equivalence of the other conditions follows by the fact that every Lindelöf, regular space is normal [En] and by Corollary 3.1.

**THEOREM 3.3.** Let $X$ be a normal space. Then $X$ is sequentially normal -closed iff $X$ is countably compact.

Proof. The proof follows immediately from Corollary 3.1 and Lemma 3.4.

**LEMMA 3.4.** Let $X$ be a regular space, $x \in X$ and let $X \setminus \{x\}$ be a normal space. Then $X$ is a normal space.

**THEOREM 3.5.** Let $X$ be a perfectly normal space. The following conditions are equivalent:

(a) $X$ is perfectly normal -closed
(b) $X$ is sequentially perfectly normal -closed
(c) $X$ is countably compact.

Proof. For the equivalence of conditions (a) and (b) see [Ste2]. Obviously (c) implies (b). It is known that in a normal space the countable compactness coincides with the feeble compactness (see [Ste1] and [Hew]) and that a regular space $X$ is feeably compact iff every countable open regular filter on $X$ has adherent points [Ste1]. Then the proof that (b) implies (c) follows by Lemma 3.4 and by the fact that if $X$ is a normal space, $x$ is a point in $X$ and $X \setminus \{x\}$ is a perfectly normal space, then $X$ is a perfectly normal space whenever $x$ is a $G_{\delta}$ set in $X$.

**THEOREM 3.6.** Let $X$ be a locally compact space. Then $X$ is sequentially locally compact -closed iff $X$ is countably compact.

Proof. It is obvious that if $X$ is countably compact then $X$ is sequentially locally compact -closed. Let $X$ be a sequentially locally compact -closed space and let we assume that $X$ is not countably compact. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points of $X$ without adherent points. Let $\omega X$ be the Alexandroff compactification of $X$ (see [En]) and $y = \omega X \setminus X$. It is easy to verify that $\lim_{n \to \infty} x_n = y$. Thus $X$ is not sequentially locally compact - closed. Contradiction.
THEOREM 3.7. For $\mathcal{P} = \text{paracompact or metric}$ if $X$ is a $\mathcal{P}$ - space then the following conditions are equivalent:

(a) $X$ is $\mathcal{P}$ - closed
(b) $X$ is compact
(c) $X$ is sequentially $\mathcal{P}$ - closed
(d) $X$ is countably compact.

Proof. For the equivalence of (a) and (b) see [SSe] and for the equivalence of (b) and (d) see [En]. Obviously (d) implies (c). The proof that (c) implies (d) for $\mathcal{P} = \text{paracompact}$ follows by Corollary 3.1 and by the fact that if $X$ is a regular space, $x \in X$ and $X \setminus \{x\}$ is a paracompact space then $X$ is a paracompact space. For $\mathcal{P} = \text{metric}$ it follows by Corollary 3.1 and by the fact that if $X$ is a regular first countable space, $x \in X$ and $X \setminus \{x\}$ is a metric space then $X$ is a metric space.

The spaces satisfying the equivalent conditions (a) - (f) of Theorem 2.1 and the equivalent conditions (a), (b) of Theorem 2.3 and Theorem 2.7 and the equivalent conditions (a) - (c) of Theorem 2.5 are in fact natural generalizations of the countable compactness. Moreover for $\mathcal{P} = \text{US (SUS)}$ the sequentially $\mathcal{P}$ - closed spaces are precisely the sequentially compact (countably compact) spaces [DGo]. The next theorem shows that some properties of the countably compact spaces are valid also for the sequentially $\mathcal{P}$ - closed spaces.

THEOREM 3.8. Let $n \in N$ and $\mathcal{P}$ be one of the following classes of topological spaces: $\text{US, SUS, S (n), regular, completely Hausdorff, completely regular, normal, perfectly normal, locally compact, paracompact or metric}$. Then the following conditions are satisfies:

(a) Sequentially $\mathcal{P}$ - closedness is preserved by continuous functions onto a $\mathcal{P}$ space.

(b) If a product of nonvoid spaces is sequentially $\mathcal{P}$ - closed then each coordinate is sequentially $\mathcal{P}$ - closed.

(c) Every sequentially $\mathcal{P}$ - closed space is pseudocompact.

Proof. Obviously (a) implies (b). Clearly (a) and (c) are true when sequentially $\mathcal{P}$ - closedness coincides with countable compactness or sequential compactness, i.e. for $\mathcal{P} = \text{US, SUS}$, completely regular, normal, perfectly normal, locally compact, paracompact or metric. For the others $\mathcal{P}$ (a) follows from Theorem 2.1, Theorem 2.3 and Theorem 2.5. To see that (c) is true let $f : X \to \mathbb{R}$ be a continuous function. Then $f (X)$
is a sequentially $\mathcal{P}$-closed metric space by (a). Thus $f(X)$ is a compact space by Theorem 2.7 and Theorem 3.7. This shows that $f$ is bounded.

**Theorem 3.9.** Let $n \in \mathbb{N}$. The following conditions are valid:

(a) $S(n)(1)$ - closedness is preserved by continuous functions onto an $S(n)$-space.

(b) If a product of nonvoid spaces is an $S(n)(1)$-closed space then each coordinate is $S(n)(1)$-closed.

(c) Every $S(n)(1)$-closed space is pseudocompact.

**Proof.** (a) follows by Theorem 2.8 and (a) implies (b). We shall proof (c). Let $f : X \to \mathbb{R}$ be a continuous function. Then $\mathcal{U} = \{f^{-1}(-k, k)\}_{k=1}^{\infty}$ is a countable regular cover of $X$. Since $X$ is $S(n)(1)$-closed then by Theorem 2.8 we can choose a finite proximate subcover of $X$. This implies that $f$ is bounded.

The above theorem for $n = 1, 2$ is proved by R. Stephenson [Ste2].

Let $\mathcal{P}$ be a class of topological spaces. $X \in \mathcal{P}$ is called $\mathcal{P}$-minimal iff $X$ has no strictly coarser $\mathcal{P}$ topologies. (For $\mathcal{P}$-minimal spaces see [BPS]).

**Lemma 3.10.** Let $n \in \mathbb{N}$, $\mathcal{P} = S(n)$, regular, completely Hausdorff or completely regular and $X$ be a $\mathcal{P}$-minimal space. If $x \in X$ and $x$ is not a limit point for a non trivial sequence in $X$ then $Y = X \setminus \{x\}$ is a sequentially $\mathcal{P}$-closed space.

**Proof.** Let us assume that $Y$ is not sequentially $\mathcal{P}$-closed space. Then there exists an open elementary $\mathcal{P}$-filter $\mathcal{F}_1$ on $Y$ without adherent points. Let $\mathcal{F}_x$ be the filter of neighbourhoods of the point $x$ on $X$. We consider the filter $\mathcal{F} = \{U \mid U = V \cup W, V \in \mathcal{F}_1, W \in \mathcal{F}_x\}$. Obviously $\mathcal{F}$ is an open elementary $\mathcal{P}$-filter on $X$ and $x$ is the unique adherent point for $\mathcal{F}$. Let $\mathcal{F}'_1$ be a maximal open elementary filter on $Y$ containing $\mathcal{F}_1$. Suppose that $\left\{x_k\right\}_{k=1}^{\infty}$ determines $\mathcal{F}'_1$. But $\lim_{k \to \infty} x_k \neq x$. Thus $\mathcal{F} \subseteq \mathcal{F}_x$. But this contradicts to the $\mathcal{P}$-minimality of $X$.

**Corollary 3.11.** Let $X$ be a compact Hausdorff space and $x \in X$. The point $x$ is not the limit of a some (non trivial) sequence of $X$ iff $X \setminus \{x\}$ is a countably compact space.

**Proof.** Every compact Hausdorff space is a minimal completely regular space (see [Bal]). Now the corollary holds by Lemma 3.10 and Theo-
rem 2.7.

Let $C(X)$ be the set of all real-valued continuous functions on a space $(X, \mathcal{T})$. The weak-topology $\mathcal{T}_\omega$ on $X$ is the smallest topology on $X$ such that all functions in $C(X)$ are continuous. Clearly $\mathcal{T}_\omega$ is coarser than $\mathcal{T}$ and the space $(X, \mathcal{T}_\omega)$ is completely regular iff $(X, \mathcal{T})$ is completely Hausdorff.

**Theorem 3.12.** Let $(X, \mathcal{T})$ be a completely Hausdorff space. $(X, \mathcal{T})$ is sequentially completely Hausdorff-closed iff $(X, \mathcal{T}_\omega)$ is countably compact.

**Proof.** It follows by Theorem 2.7 and by the fact that $(X, \mathcal{T})$ is sequentially completely Hausdorff-closed iff $(X, \mathcal{T}_\omega)$ is sequentially completely regular-closed.
REFERENCES


