ON UNIFORMLY PARALINDELÖF SPACES (*)

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SOMMARIO. - *Nella presente nota vengono caratterizzate alcune classi di spazi uniformemente paralindelöf estendendo ad esse un importante teorema di Tamano* ([T], p. 1046) *concernente gli spazi paracompatti, teorema già esteso da Höthi agli spazi uniformemente paracompatti* ([H], p. 26).

SUMMARY. - *We characterize some classes of uniformly paralindelöf spaces in the spirit of the Tamano's characterization of paracompactness* ([T], p. 1046) *and of the Höthi's characterization of uniform paracompactness* ([H], p. 26).

INTRODUCTION - It is known ([H], Th. 5.1.1) that it is possible to characterize uniformly paracompact spaces as those uniform spaces X for which, if C is a closed set of a compactification K of X, disjointed from X, there is a uniform cover $\mathcal{U}$ of X such that, for each $U \in \mathcal{U}$, $cl_K U \cap K = \phi$. In this paper we study uniformly paralindelöf spaces, introduced in [H], which are a natural extension of uniformly paracompact ones, and we find some classes of such spaces for which a theorem analogous to the theorem of Höthi holds. All these results are also in the spirit of the Tamano's characterization of paracompactness.

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Definitions and basic properties of uniform spaces used in this paper may be found in [I], those of uniformly paracompact spaces in [H] and in [R]. Moreover, if $\mathcal{U}$ is a uniform cover of X and K is a space containing X as a subspace, we set, following [G-Z], $K(\mathcal{U}) = \bigcup_{U \in \mathcal{U}} cl_K U$ and $K'(\mathcal{U}) = \bigcup_{U \in \mathcal{U}} int_K cl_K U$.

A. Höthi ([H], p. 8) defines X to be uniformly paralindelöf if condition 1 in the following Proposition is satisfied. The other two conditions are analogous to characterizations of uniform paracompactness given in [R].

PROPOSITION 1 - *The following conditions are equivalent for a uniform space X:*

1. *Each open cover $\mathcal{A}$ of X admits a uniformly locally countable open refinement $\mathcal{U}$ (i.e. there exists a uniform cover each of whose elements meets countably many elements of $\mathcal{U}$).*
2. *If $\mathcal{A}$ is an open cover of X, the cover $\mathcal{A}_w$ consisting of the countable union of elements of $\mathcal{A}$ is a uniform cover.*
3. *If $\mathcal{A}$ is an open cover of X, there is a uniform cover $\mathcal{U}$ such that for each $U \in \mathcal{U}$, $\mathcal{A}/U$ has a countable subcover.*

Proof. $1 \to 2$. Let $\mathcal{A}$ be an open cover of X and $\mathcal{U}$ a uniformly locally countable open refinement of $\mathcal{A}$. Hence there is a uniform cover $\mathcal{V}$ such that for each $V \in \mathcal{V}$, there exists a countable subfamily $\{U_n\}_{n \in N}$ of $\mathcal{U}$ with $V \subset \bigcup_n U_n$. Therefore each $V \in \mathcal{V}$ is contained in a countable union of elements of $\mathcal{A}$ and consequently $\mathcal{A}_w$ is uniform.

$2 \to 3$. Obvious.

$3 \to 2$. If $\mathcal{A}$ is an open cover of X, there is a uniform cover $\mathcal{U}$ such that for each $U \in \mathcal{U}$, there exists a countable subfamily $\{A_n^{(U)}\}_{n \in N}$ of $\mathcal{A}$ with $U = \bigcup_{n \in N} (A_n^{(U)} \cap U)$. Hence $\mathcal{U}$ refines $\mathcal{A}_w$ and $\mathcal{A}_w$ is uniform.

$2 \to 1$. Let $\mathcal{A}$ be an open cover of X. Hence $\mathcal{A}_w$ is uniform.

From Th. 14, p. 7 of [I] there is a cover $\mathcal{B} = \{B_t\}_{t \in T}$ which is a locally finite open refinement of $\mathcal{A}_w$. For each B_t choose a countable family $\{A_n^t\}_{n \in N} \subset \mathcal{A}$ such that $B_t \subset \bigcup_{n \in N} A_n^t$. Then the open cover $\mathcal{W} = \{B_t \cap A_n^t\}_{t \in T, n \in N}$ refines $\mathcal{A}$ and for each $x \in X$, there is an open neighbourhood U_x of x which meets only a countable union of elements of $\mathcal{W}$.

Take $\mathcal{U} = \{U_x\}_{x \in X}$. Hence $\mathcal{U}_w$ is uniform and each element of $\mathcal{U}_w$ meets countably many elements of $\mathcal{W}$.

DEFINITION 1 - A uniform space satisfying one of the conditions of Proposition 1 is called uniformly paralindelöf.

The following result corresponds in spirit to [R, Th. 1].

PROPOSITION 2 - *A uniform space $\langle X, u\rangle$ is uniformly paralindelöf if and only if the induced topology is paracompact and $\alpha = e\,\alpha/u$, where α is the finest uniformity inducing the same topology* (1).

Proof. To prove that X is uniformly paralindelöf, we take an open cover $\mathcal{U}$. Since X is paracompact we have from the definition of α that $\mathcal{U} \in \alpha$ i.e. there exists a uniform cover $\mathcal{V} = \{V_j\}_{j\in J}$ such that the cover

$$\{V_j \cap W_j^n\}_{j\in J,\, n\in N},\ \text{where for each } j \in J, \{W_j^n\}_{n\in N} \in e\alpha,$$

refines $\mathcal{U}$. Then for each $V_j \in \mathcal{V}$, $\mathcal{U}/V_j$ has a countable subcover and Condition 3 of Proposition 1 follows.

Now we remark that if X is uniformly paralindelöf then X is paracompact. In fact each countably additive open cover of X is uniform and then it has a σ-discrete open refinement. By Proposition 1 of [F] it follows that X is paracompact.

Finally to show that $e\alpha/u = \alpha$, let $\mathcal{U}$ be an open cover and $\{F_t\}_{t\in T}$ a closed uniform cover refining $\mathcal{U}_w$. For each F_t there is a countable family $\{U_n^t\}_{n\in N} \subset \mathcal{U}$ that covers F_t. Then

$$\mathcal{V}_t = \{U_n^t\}_{n\in N} \cup \{X - F_t\}$$

is a countable open cover, element of $e\alpha$ since X is paracompact. Therefore $\{F_t \cap V\}_{V\in\mathcal{V}_t,\, t\in T}$ is an element of $e\alpha/u$ refining $\mathcal{U}$.

In the attempt to characterize uniformly paralindelöf spaces analogously to Höthi's we introduce the following definition

DEFINITION 2 - A uniform space X is strongly uniformly paralindelöf if there is a Lindelöf space L, extension of X, such that for each closed set C, $C \subset L - X$, there is a uniform cover $\mathcal{U}$ of X such that $L(\mathcal{U}) \cap C = \phi$.

PROPOSITION 3 - *A strongly uniformly paralindelöf space is uniformly paralindelöf.*

Proof. Let X be a strongly paralindelöf space, L a Lindelöf space satisfying the condition of Definition 2, $\mathcal{A} = \{A_i\}_{i\in I}$ be an open cover of X and, for each $i \in I$, W_i be an open set of L such that

(1) We recall that, if u is a uniformity, eu is the uniformity having as a base the countable elements of u and that, if u and v are uniformities, u/v has for its base all covers of the form $\{V_s \cap U_t^s\}$ where $\{V_s\} \in v$ and, for each $s, \{U_t^s\} \in u$.

$A_i = W_i \cap X$. Since $F = L - \bigcup_{i \in I} W_i$ is a closed set contained in $L - X$, there is a uniform cover $\mathcal{U} = \{U_j\}_{j \in J}$ such that for each $j \in J$, $\mathrm{cl}_L U_j \subset \bigcup_{i \in I} W_i$. But, since $\mathrm{cl}_L U_j$ is Lindelöf, there is a countable family $\{W_{i_n}\}_{n \in N}$ such that $\mathrm{cl}_L U_j \subset \bigcup_{n \in N} W_{i_n}$. Hence $\mathrm{cl}_L U_j \cap X \subset \bigcup_{n \in N} A_{i_n}$. It follows that for each $j \in J$, $U_j \subset \bigcup_{n \in N} A_{i_n}$. This means that $\mathcal{A}_w$ is uniform i.e. X is uniformly paralindelöf.

Is the converse of Proposition 3 true? We do not know the answer, but in the following propositions we find some classes of uniformly paralindelöf spaces which are also strongly uniformly paralindelöf. First of all we prove the following result which is useful for the other ones.

PROPOSITION 4 - *A space X uniformly locally Lindelöf (i.e. there is a uniform cover whose elements are Lindelöf) is strongly uniformly paralindelöf.*

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a uniform cover of X whose elements are closed and Lindelöf. Then, for a compactification K of X, we set $L = X \cup [K - K'(\mathcal{U})]$. Firstly we note that $K - K'(\mathcal{U}) = \bigcap_{i \in I} (K - \mathrm{int}_K \mathrm{cl}_K U_i)$ is compact.

Now we have to prove that L is a Lindelöf space. For this let $\mathcal{W} = \{W_j\}_{j \in J}$ be an open cover of L. Since $K - K'(\mathcal{U})$ is compact, there is a finite family $\{W_h\}_{h=1,2,\dots,n}$ contained in $\mathcal{W}$ and covering it. Let $W'_h (h = 1, 2, \dots, n)$ be open sets of K such that $W_h = W'_h \cap L$. Then the family $\{\mathrm{int}_K \mathrm{cl}_K U_i\}_{i \in I} \cup \{W'_h\}_{h=1,2,\dots,n}$ is an open cover of K which of course admits a finite subcover.

Hence $F = X - (W_1 \cup W_2 \cup \dots \cup W_n)$ is covered by a finite number of elements of the form $(\mathrm{int}_K \mathrm{cl}_K U_1) \cap X, \dots, (\mathrm{int}_K \mathrm{cl}_K U_m) \cap X$.

But, for $p = 1, 2, \dots, m$ we have $y \in (\mathrm{int}_K \mathrm{cl}_K U_p) \cap X$. It follows that there is a neighbourhood of y in K, U^K_y, such that $U^K_y \subset \mathrm{cl}_K U_p$.

Then $U^K_y \cap X \subset \mathrm{cl}_K U_p \cap X$ and $y \in U_p$. Thus $(\mathrm{int}_K \mathrm{cl}_K U_p) \cap X \subset U_p$ and F is covered by finitely many U_i which are Lindelöf. Then it is easy to obtain from $\mathcal{W}$ countably many W_j covering F. Hence L is a Lindelöf space.

Let now C be a closed set such that $C \subset L - X$. Then $C \cap K'(\mathcal{U}) = \phi$.

But from Proposition 1 of [G-Z], if $\mathcal{V}$ is a uniform cover star-refining $\mathcal{U}$, $K(\mathcal{V}) \subset K'(\mathcal{U})$. Thus $C \cap L(\mathcal{V}) \subset C \cap K(\mathcal{V}) = \phi$ and this completes the proof.

PROPOSITION 5 - *Each locally Lindelöf and uniformly paralindelöf space X is strongly uniformly paralindelöf.*

Proof. It is enough to prove that X is uniformly locally Lindelöf. For each x, let A_x and U_x be neighbourhoods of x such that $A_x \subset U_x$, A_x open and U_x Lindelöf. Take the cover $\mathcal{U} = \{U_x\}_{x \in X}$ and the open cover $\mathcal{A} = \{A_x\}_{x \in X}$. Then $\mathcal{A}_w$ and consequently also $\mathcal{U}_w$ are uniform. Besides it is obvious that the elements of $\mathcal{U}_w$ are Lindelöf.

PROPOSITION 6 - *If X is a uniformly paralindelöf space such that the set F of the points of X which have no neighbourhood which is Lindelöf, is compact, then X is strongly uniformly paralindelöf.*

Proof. We fix an open base $\{V_j\}_{j \in J}$ of F and set $L_j = X - V_j$. Since L_j is uniformly locally Lindelöf, there exists a uniform cover $\mathcal{U}_j = \{U^j_i\}_{i \in I}$ of L_j whose elements are closed and Lindelöf.

Let now K be a compactification of X and, for each $j \in J$,

$$K'(\mathcal{U}_j) = \bigcup_{U^j_i \in \mathcal{U}_j} \operatorname{int}_K \operatorname{cl}_K U^j_i .$$

Then we consider $L = X \cup [\bigcap_{j \in J} (K - K'(\mathcal{U}_j))]$ and in a way analogous to the one used in Proposition 4 we have that L is a Lindelöf space. After let $C \subset L - X$ be closed in L. Being C a subspace of the compact space $\bigcap_{j \in J} (K - K'(\mathcal{U}_j))$, it is closed in K. Since K is normal and $C \cap F = \phi$, there are open sets A'_1 and A'_2 such that $A'_1 \cap A'_2 = \phi$, $F \subset A'_1$, $C \subset A'_2$. If A_1 and A_2 are open sets of L such that $A_1 = A'_1 \cap L$ and $A_2 = A'_2 \cap L$, then $A_1 \cap X \supset F$, $A_2 \supset C$, $A_1 \cap A_2 = \phi$. Then there is a V_j such that $V_j \subset A_1 \cap X$.

Now we consider $\mathcal{U}_j$ and after a uniform cover $\mathcal{W}_j$ of L_j star-refining $\mathcal{U}_j$. It follows that $K(\mathcal{W}_j) \subset K'(\mathcal{U}_j)$, $C \cap K(\mathcal{W}_j) = C \cap L(\mathcal{W}_j) = \phi$.

Let $\mathcal{R}_j$ be a uniform cover of X with $\mathcal{R}_j / L_j = \mathcal{W}_j$. Now

— if $R_{j_i} \in \mathcal{R}_j$ is contained in L_j, then $R_{j_i} \in \mathcal{W}_j$ and $\operatorname{cl}_K R_{j_i} \cap C = \operatorname{cl}_L R_{j_i} \cap C = \phi$,

— if $R_{j_i} \subset X - L_j = V_j \subset A_1$ then $\operatorname{cl}_L R_{j_i} \cap C = \phi$,

— if $R_{j_i} = R^1_{j_i} \cup R^2_{j_i}$ where $R^1_{j_i} \subset X - L_j$ and $R^2_{j_i} \in \mathcal{W}_j$, then $\operatorname{cl}_L R_{j_i} \cap C = \phi$ since this happens also for the closures in L of $R^1_{j_i}$ and $R^2_{j_i}$.

This completes the proof.

COROLLARY - *Each metrizable uniformly paralindelöf space is strongly uniformly paralindelöf.*

Proof. By Lemma 3.1.1 of [H], $X = F \cup (X - F)$ with F compact and $X - F$ locally Lindelöf.

PROPOSITION 7 - *Let X be a uniformly paralindelöf space containing a subspace F which is closed, locally Lindelöf and such that $X-F$ is uniformly paracompact. Then X is strongly uniformly paralindelöf.*

Proof. It is routine to prove that F is uniformly paralindelöf and locally Lindelöf, hence uniformly locally Lindelöf. Then there is a uniform cover $\mathcal{U}$ of F whose elements are closed and Lindelöf.

If K is a compactification of X, we set $L = X \cup [K - K'(\mathcal{U})]$. Like in Proposition 4, L is a Lindelöf space.

Let now C be a closed set of $L - X$. Then $C \cap K'(\mathcal{U}) = \phi$. If $\mathcal{V}$ is a uniform cover of F star-refining $\mathcal{U}$, we have that $C \cap K(\mathcal{V}) = C \cap L(\mathcal{V}) = \phi$.

Being $C \subset K - (X - F)$ and $X - F$ uniformly paracompact, there exists ([H] Th. 5.1.1) a uniform cover $\mathcal{W}$ of $X - F$ such that $K(\mathcal{W}) \cap C = \phi$.

Afterwards we take two uniform covers $\mathcal{R}$ and $\mathcal{S}$ of X such that $\mathcal{R}/F = \mathcal{V}$, $\mathcal{S}/X - F = \mathcal{W}$ and note that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ where $R_1 \in \mathcal{R}_1$ if and only if $R_1 \subset F$, $R_3 \in \mathcal{R}_3$ if and only if $R_3 \subset X - F$, $\mathcal{R}_2 = \mathcal{R} - (\mathcal{R}_1 \cup \mathcal{R}_3)$. Analogously $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$.

Then $\mathcal{R} \wedge \mathcal{S}$ is a uniform cover of X such that each of its elements either is contained in an element of $\mathcal{V}$ or in an element of $\mathcal{W}$ or in the union of an element of $\mathcal{V}$ and of an element of $\mathcal{W}$.

Now we compute $K(\mathcal{R} \wedge \mathcal{S})$. Let $R \cap S \in \mathcal{R} \wedge \mathcal{S}$.

If $R \cap S \subset W \in \mathcal{W}$ or $R \cap S \subset V \in \mathcal{V}$, obviously $\mathrm{cl}_K(R \cap S) \cap C = \phi$.

If $R \cap S \subset V \cup W (V \in \mathcal{V}, W \in \mathcal{W})$, we note that $\mathrm{cl}_K(R \cap S) \subset \mathrm{cl}_K V \cup \mathrm{cl}_K W$ and both $\mathrm{cl}_K V$ and $\mathrm{cl}_K W$ are disjointed from C.

All this means that $K(\mathcal{R} \wedge \mathcal{S}) \cap C = L(\mathcal{R} \wedge \mathcal{S}) \cap C = \phi$.

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