RECENT RESULTS ON PERIODIC SOLUTIONS OF THE FORCED PENDULUM EQUATION (*)

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SOMMARIO. - *Si descrivono i recenti progressi ottenuti nello studio delle soluzioni periodiche per l'equazione del pendolo forzato con attrito. Si dà particolare rilievo alla descrizione della struttura matematica dell'insieme delle funzioni forzanti per le quali esiste almeno una soluzione periodica.*

SUMMARY. - *The paper describes recent progress made in the study of periodic solutions for the forced pendulum equation with dissipation. Emphasis will be put on the description of the mathematical structure of the set of forcing functions for which at least one periodic solution exists.*

1. - Introduction.

In recent surveys [5, 6, 7], we have described some of the researches recently made about the determination of the structure of the set $\mathcal{R} = \mathcal{R}(c, A, T)$ of the $e \in L^1(0, T)$ for which the problem

$$(1) \qquad \begin{aligned} &x'' + cx' + A \sin x = e(t) \quad (A > 0) \\ &x(0) - x(T) = x'(0) - x'(T) = 0 \end{aligned}$$

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has at least one solution and about the corresponding number of solutions. Since the writing of those surveys, some progress has been made in the understanding of those questions and we want describe it here and to emphasize some interesting open questions. In the conservative case $(c = 0)$, little progress has been made by variational methods with respect to the results described in [6], except the fact that the existence of at least two solutions when $c = 0$ and $\int_0^T e(t)\,dt = 0$ can also be deduced from Morse theory (see the lecture of Willem at this conference). Consequently, we shall emphasize the situation where c is arbitrary. This paper will be divided into two main parts, namely the description of the mathematical structure of the set $\mathfrak{R}$ and of the corresponding multiplicity properties of the solutions of (1), and then the delicate question of the explicit location of $\mathfrak{R}$ in $L^1(0,T)$ in terms of c, A and T.

In this survey, which contains also some new results, the proofs will be sketched only when the results have been published elsewhere and we shall refer to the original papers for more details. Many of the results described here depend upon a simple transformation of (1), independently devised by Dancer (personal communication) and the author. For any $f \in L^1(0,T)$, let us denote by $\bar{f}$ its mean value

$$\bar{f} = (1/T)\int_0^T f(t)\,dt$$

and let $\tilde{f} = f - \bar{f}$. For $e = \bar{e} + \tilde{e} \in L^1(0,T)$, let $\tilde{E} = K\tilde{e}$ denote the unique solution of the linear problem

$$\begin{aligned} &u'' + cu' = \tilde{e}(t) \\ (2)\qquad &u(0) - u(T) = u'(0) - u'(T) = 0 \\ &\bar{u} = 0\,. \end{aligned}$$

If we set

$$x(t) = y(t) + \tilde{E}(t)$$

in (1), x will be a solution of (1) is and only if y is a solution of

$$(3)\qquad \begin{aligned} &y'' + cy' + A\sin(y + \tilde{E}(t)) = \bar{e} \\ &y(0) - y(T) = y'(0) - y'(T) = 0\,. \end{aligned}$$

It is useful to notice that if y is a solution of (3), then

$$A\,\overline{\sin(y(\cdot) + \tilde{E}(\cdot))} = \bar{e}$$

and y satisfies the (integral) equation

$$(4)\qquad y = \bar{y} + K[\tilde{e} - A\sin(y + \tilde{E})]\,.$$

Finally, if y is a solution to (3), the same is true for $y + 2k\pi$ $(k \in \mathbf{Z})$ so that, without loss of generality, we can always assume if necessary that $\bar{y} \in [0, 2\pi]$.

2. - The structure of the set $\mathfrak{R}$.

LEMMA 1 ([7]) - *$\mathfrak{R}$ is closed in $L^1(0,T)$.*

Proof. Considering the nontrivial case where $\mathfrak{R} \neq \phi$, let (e_k) be a sequence in $\mathfrak{R}$ converging to e and y_k a corresponding solution of (3) with $\tilde{E}$ (resp. $\bar{e}$) replaced by $\tilde{E}_k$ (resp. $\bar{e}_k$) such that $y_k \in [0,2\pi]$. Then, (4), and Ascoli's theorem imply the existence of a subsequence of (y_k) which converges in $C([0,T])$ to a solution y of (3).

LEMMA 2 ([2]) - *If y is a solution to (4), then y is a solution to*

$$(5) \qquad \begin{aligned} &y'' + cy' + A \sin(y + \tilde{E}(t)) = A\, \overline{\sin(y + \tilde{E})} \\ &y(0) - y(T) = y'(0) - y'(T) = 0. \end{aligned}$$

Proof. Integrate (3) over $(0,T)$ to get

$$\bar{e} = A\, \overline{\sin(y + \tilde{E})}.$$

LEMMA 3 ([2]) - *For each $\xi \in R$, problem (5) has at least one solution y such that $\bar{y} = \xi$.*

Proof. (5) with $\bar{y} = \xi$ is equivalent to the equation

$$y = \xi - K[A\, \widetilde{\sin(y + \tilde{E})}]$$

and the result follows from Schauder's fixed point theorem.

Let $\mathcal{S} = \mathcal{S}(c,A,e,T) = \{y \in C([0,T]) : y \text{ is a solution to } (5)\}$. By Lemma 3, $\mathcal{S} \neq \phi$ and $\mathcal{S} = \mathcal{S} + 2k\pi$, $k \in \mathbf{Z}$.
Let $\mathcal{S}' = \{y \in \mathcal{S} : \bar{y} \in [0,2\pi]$ so that $\mathcal{S}' \neq \phi$. For brevity, let us define $\gamma_{\bar{e}} : C([0,T]) \to [-1,1]$ by

$$\gamma_{\bar{e}}(y) = \overline{\sin(y + \tilde{E})}.$$

For $u \in L^1(0,T)$, let $\| u \| = [(1/T) \int_0^T u^2(t)\, dt]^{1/2}$ and let $\omega = 2\pi/T$.

LEMMA 4 ([2]) - *If $y \in \mathcal{S}$, then*

$$\| y'' \| \leq A, \| y' \| \leq (\omega^2 + c^2)^{-1/2} A, \| \tilde{y} \| \leq \omega^{-1}(\omega^2 + c^2)^{-1/2} A,$$
$$\| y \|_C \leq 3^{-1/2} \pi \omega^{-1} (\omega^2 + c^2)^{-1/2} A.$$

Proof. Multiply both members of (5) by $y'' + cy'$, integrate over $(0,T)$, use the Schwarz, Wirtinger and Sobolev inequalities as given in [9].

LEMMA 5 ([2]) - *$\gamma_{\bar{e}}$ achieves its infimum and its supremum on $\mathcal{S}$.*

Proof. It suffices to prove that $\gamma_{\bar{e}}$ achieves its infimum and its supremum on $\mathcal{S}'$. An argument similar to that used in the proof of Lemma 1 together with Lemma 4 shows that $\mathcal{S}'$ is compact and the result follows.

Let $\gamma_{\bar e}^1 = \gamma_{\bar e}^1(c, A, T) = \min_{\mathfrak{S}} \gamma_{\bar e}$ and $\gamma_{\bar e}^2 = \gamma_{\bar e}^2(c, A, T) = \max_{\mathfrak{S}} \gamma_{\bar e}$ and let y_i be such that $\gamma_{\bar e}(y_i) = \gamma_{\bar e}^i (i = 1, 2)$. We can assume, without loss of generality that

$$y_2(t) \leq y_1(t), \ t \in [0, T]$$

and that there is some $\tau \in [0, T]$ for which

$$y_1(\tau) - y_2(\tau) < 2\pi .$$

THEOREM 1 ([2]) - *Problem* (3) *has a solution if and only if* $\bar e \in [A\gamma_{\bar e}^1, A\gamma_{\bar e}^2]$. *If* $\gamma_{\bar e}^1 = \gamma_{\bar e}^2$, *then* (3) $\bar e = A\gamma_{\bar e}^1 = A\gamma_{\bar e}^2$ *has a solution* y *with* $\bar y = \xi$ *for each* $\xi \in R$. *If* $\gamma_{\bar e}^1 < \gamma_{\bar e}^2$ *and if* $\gamma_{\bar e}^1 < \bar e < \gamma_{\bar e}^2$, *then* (3) *has at least two solutions not differing by a multiple of* 2π.

Proof. By Lemma 2, if (3) has a solution y, then

$$\bar e = A\gamma_{\bar e}(y) \in [A\gamma_{\bar e}^1, A\gamma_{\bar e}^2] .$$

To prove the converse, it suffices to notice that if $\bar e \in [A\gamma_{\bar e}^1, A\gamma_{\bar e}^2]$, then$y_1$ (resp. y_2) defined above is an upper (resp. lower) solution for (3) and hence (3) has a solution y with $y_2(t) \leq y(t) \leq y_1(t)$ for $t \in [0, T]$. When $\gamma_{\bar e}^1 = \gamma_{\bar e}^2$ and $\xi \in R$, the existence of a solution y of (3) with $\bar e = \gamma_{\bar e}^1 = \gamma_{\bar e}^2$ such that $\bar y = \xi$ follows from Lemma 3. Finally, if $\gamma_{\bar e}^1 < \gamma_{\bar e}^2$ and $A\gamma_{\bar e}^1 < \bar e < A\gamma_{\bar e}^2$, one can show that $y_2(t) < y_1(t)$, $t \in [0, T]$. Now $(y_2, y_1 + 2\pi)$ and $(y_2 + 2\pi, y_1 + 2\pi)$ are also couples of upper and lower solutions, and using degree arguments in the line of [8] or [7], one obtains a second solution which does not differ from the first one by a multiple of 2π (see [2] for the details).

To complete the description of $\mathfrak{R}$ we need another characterization of $\gamma_{\bar e}^i (i = 1, 2)$.

LEMMA 6 ([7]) - *One has*

$$A\gamma_{\bar e}^2 = \min_{\beta} \max_{t \in [0,T]} [z''(t) + cz'(t) + A \sin(z(t) + \tilde E(t))] \tag{6}$$

$$A\gamma_{\bar e}^2 = \max_{\beta} \min_{t \in [0,T]} [z''(t) + cz'(t) + A \sin(z(t) + \tilde E(t))] \tag{7}$$

for all $z \in C^2([0, T])$ *such that* $z(0) - z(T) = z'(0) - z'(T) = 0$.

Proof. Let us consider, say, the first case. As

$$y_1'' + cy_1' + A \sin(y_1 + \tilde E) = A\gamma_{\bar e}^1,$$

we have

$$m' = \inf_{\beta} \max_{t \in [0,T]} [z''(t) + cz'(t) + A \sin(z(t) + \tilde E(t))] \leq A\gamma_{\bar e}^1 .$$

If $m' < A\gamma_{\bar e}^1$, there will be some $m' < m'' < A\gamma_{\bar e}^1$ and some $v \in C^2([0, T])$ such that $v(0) - v(T) = v'(0) - v'(T) = 0$ and

$$\max_{t\in[0,T]} [v''(t) + cv'(t) + A\sin(v(t) + \tilde{E}(t))] \leq m''$$

and, without loss of generality, we can assume that $v(t) \geq y_1(t)$, $t \in [0,T]$. This (y_1, v) is then a couple of lower and upper solutions for the problem

$$(8) \qquad \begin{aligned} &y'' + cy' + A\sin(y + \tilde{E}) = m'' \\ &y(0) - y(T) = y'(0) - y'(T) = 0 . \end{aligned}$$

Consequently, (8) has a solution, which contradicts Theorem 1 and completes the proof.

We deduce from Lemma 6 some results on the continuous dependence of $\gamma^i_{\tilde{e}}$ with respect to $\tilde{e}$.

LEMMA 7 - *If $(\tilde{e}_k) \in \tilde{L}^1(0,T)$ and $\tilde{e} \in \tilde{L}^1(0,T)$ are such that*

$$K\tilde{e}_k \to K\tilde{e}$$

in $C([0,T])$ (in particular, if $\tilde{e}_k \to \tilde{e}$ in $L^1(0,T)$), then

$$\gamma^i_{\tilde{e}_k} \to \gamma^i_{\tilde{e}} \qquad (i = 1,2).$$

Proof. Let us prove it, say, for $\gamma^1_{\tilde{e}}$. Let

$$\Gamma(z)(t) = z''(t) + cz'(t) + A\sin(z(t) + \tilde{E}(t))$$

and

$$\Gamma_k(z)(t) = z''(t) + cz'(t) + A\sin(z(t) + \tilde{E}_k(t)).$$

By Lemma 6, if z_k is such that

$$\max_t \Gamma_k(z_k)(t) = \min_z \max_t \Gamma_k(z)(t),$$

and if t_k is such that

$$\Gamma(z_k)(t_k) = \max_t \Gamma(z_k)(t),$$

then

$$A\gamma^1_{\tilde{e}_k} - A\gamma^1_{\tilde{e}} \geqslant \max_{t\in[0,T]} \Gamma_k(z_k)(t) - \max_{t\in[0,T]} \Gamma(z_k)(t) \geqslant \Gamma_k(z_k)(t_k) - \Gamma(z_k)(t_k) =$$

$$= A\sin[z_k(t_k) + \tilde{E}_k(t_k) - A\sin(z_k(t_k) + \tilde{E}(t_k))].$$

Similarly, if v is such that

$$\max_t \Gamma(v)(t) = \min_z \max_t \Gamma(z)(t),$$

and if τ_k is such that $\Gamma_k(v)(\tau_k) = \max_t \Gamma_k(v)(t)$,

then

$$A\gamma^1_{\tilde{e}_k} - A\gamma^1_{\tilde{e}} \leq \max_{t\in[0,T]} \Gamma_k(v)(t) - \max_{t\in[0,T]} \Gamma(v)(t) \leq \Gamma_k(v)(\tau_k) - \Gamma(v)(\tau_k) =$$

$= A \sin [v(\tau_k) + \tilde{E}_k(\tau_k)] - A \sin [v(\tau_k) + \tilde{E}(\tau_k)]$.

Consequently,

$$|\gamma^1_{\tilde{e}_k} - \gamma^1_{\tilde{e}}| \leq \| \tilde{E}_k - \tilde{E} \|_C$$

and the proof is complete.

We summarize the above results in the following

THEOREM 2 - *$\mathfrak{R}$ is a closed subset of $L^1(0,T)$ such that, for each $\tilde{e} \in \tilde{L}^1(0,T)$, the set $\mathfrak{R}_{\tilde{e}}$ of $\bar{e} \in \mathbf{R}$ such that $\bar{e} + \tilde{e} \in \mathfrak{R}$ is a closed interval $[\gamma^1_{\tilde{e}}, \gamma^2_{\tilde{e}}]$ and the $\gamma^i_{\tilde{e}}$ depend continuously on $\tilde{e}$.*

COROLLARY 1 - *The set of $\tilde{e} \in \tilde{L}^1(0,T)$ for which $\tilde{\gamma}^1_{\tilde{e}} < \tilde{\gamma}^2_{\tilde{e}}$ is open*

OPEN PROBLEM 1 - It is not known if there exists some $\tilde{e} \in \tilde{L}^1(0,T)$ for which $\gamma^1_{\tilde{e}} = \gamma^2_{\tilde{e}}$. Some special results described in the next section make likely the existence of such an e and the following density result, which generalizes the ones of [8] and [7], is of interest.

THEOREM 3 - *The set of $\tilde{e}$ such that $\gamma^1_{\tilde{e}} < \gamma^2_{\tilde{e}}$ is dense in $L^1(0,T)$.*

Proof. If it is not the case, there will exist $\tilde{h} \in \tilde{L}^1(0,T)$ and $r > 0$ such that $\gamma^1_{\tilde{e}} = \gamma^2_{\tilde{e}}$ whenever $\tilde{e} \in \tilde{L}^1(0,T)$ and $\| \tilde{e} - \tilde{h} \|_{L^1} < r$. By Theorem 1, there is a solution $y \in C^2([0,T])$ to

$$y'' + cy' + A \sin(y + \tilde{H}(t)) = \gamma^1_{\tilde{h}} \qquad (9)$$
$$y(0) - y(T) = y'(0) - y'(T) = 0$$

and, by continuity, there is $R > 0$ such that

$$\| v'' + cv' + A \sin(v + \tilde{H}(\cdot)) - \gamma^1_{\tilde{h}} \|_C < r/2T \qquad (10)$$

whenever $\| v - y \|_{C^2} \leq R$. We show that

$$\int_0^T [v'' + cv' + A \sin(v + \tilde{H}(\cdot)) - \gamma^1_{\tilde{h}}]\, dt \geqslant 0 \qquad (11)$$

whenever $\| v - y \|_{C^2} \leqslant R$ and $v(0) - v(T) = v'(0) - v'(T) = 0$. If it is not the case, there is such a v with

$$\int_0^T [v'' + A \sin(v + \tilde{H}(\cdot)) - \gamma^1_{\tilde{h}}]\, dt = -\varepsilon < 0$$

so that

$$v'' + cv' + A \sin(v + \tilde{H}(\cdot)) - \gamma^1_{\tilde{h}} = \tilde{e}(t) - \varepsilon$$

with $\tilde{e} \in C([0,T])$, $\int_0^T \tilde{e}(t)\, dt = 0$. Then, by (10),

$$\| \tilde{e} - \varepsilon \|_C < r/2T$$

and, as $\tilde{e}(\tau) = 0$ for some τ,

$$\varepsilon = |\tilde{e}(\tau) - \varepsilon| \leqslant \|\tilde{e} - \varepsilon\|_C < r/2T,$$

so that

$$\|\tilde{e}\|_C < r/T.$$

Setting $v = w + \tilde{E}$, we see that w is solution of

$$w'' + cw' + A\sin(w + \tilde{E} + \tilde{H}) = \gamma^1_{\tilde{h}} - \varepsilon$$
$$w(0) - w(T) = w'(0) - w'(T) = 0,$$

with

$$\|\tilde{e} + \tilde{h} - \tilde{h}\|_{L^1} = \|\tilde{e}\|_{L^1} < T\|\tilde{e}\|_C < r,$$

a contradiction. Thus (11) holds and, with (9), it implies that

$$0 \leq \int_0^T [z'' + cz' + A\cos(y + \tilde{H})\, z]\, dt = A\int_0^T \cos(y + \tilde{H})\, z dt$$

for all $z \in C^2([0,T])$ with $z(0) - z(T) = z'(0) - z'(T) = 0$.

Consequently,

$$\cos(y(\cdot) + \tilde{H}(\cdot)) = 0,$$

which, by continuity, is only possible if

$$y(t) + \tilde{H}(t) = \frac{\pi}{2} + k\pi$$

for some $k \in \mathbf{Z}$ and all $t \in [0,T]$. But then, by (9),

$$-\tilde{h}(t) + (-1)^k A = \gamma^1_{\tilde{h}},$$

i.e.

$$\gamma^2_{\tilde{h}} = \gamma^1_{\tilde{h}} = (-1)^k A\ ,\ \tilde{h} = 0,$$

a contradiction with $\gamma^1_0 = -A = -\gamma^2_0$.

3. - Explicit determination of $[\gamma^1_{\tilde{e}}, \gamma^2_{\tilde{e}}]$ in some special cases.

In this section, we shall describe some recent results about the determination of explicit subintervals of $[\gamma^1_{\tilde{e}}, \gamma^2_{\tilde{e}}]$. We first recall a result of Dancer [1] (see also [7]).

THEOREM 4 ([1]) - *If* $\operatorname{osc}_{[0,T]} \tilde{E} \leq \pi$, *then*

$$[-\cos(\frac{1}{2}\operatorname{osc}_{[0,T]} \tilde{E}),\ \cos(\frac{1}{2}\operatorname{osc}_{[0,T]} \tilde{E})] \subset [\gamma^1_{\tilde{e}}, \gamma^2_{\tilde{e}}].$$

Proof. It suffices to check that when

$$|\bar{e}| \leq A\cos(\frac{1}{2}\operatorname{osc}_{[0,T]} \tilde{E}),$$

then $\frac{\pi}{2} - \frac{1}{2}(\max_{[0,T]} \tilde{E} + \min_{[0,T]} \tilde{E})$ is a lower solution and

$$\frac{3\pi}{2} - \frac{1}{2}(\max_{[0,T]} \tilde{E} + \min_{[0,T]} \tilde{E})$$

is an upper solution for (3).

An interesting consequence of Theorem 4 given in [7] is that, for each nonzero integer n, $\tilde{e}(t) = n\omega \sin n\omega t$ is such that $\tilde{E}(t) = -\frac{1}{n\omega} \sin n\omega t$ and hence

$$\| \tilde{e} \|_{L^1} = 4n$$

and

$$\operatorname*{osc}_{[0,T]} \tilde{E} = \frac{2}{n\omega} \leq \pi$$

if $n \geqslant 2/\pi\omega$. In this case,

$$[\gamma^1_{\tilde{e}}, \gamma^2_{\tilde{e}}] \supset [-\cos\frac{1}{n\omega}, \cos\frac{1}{n\omega}],$$

which shows that there exist $\tilde{e}$ with arbitrary large norm for which $[A\gamma^1_{\tilde{e}}, A\gamma^2_{\tilde{e}}]$ is arbitrarily close to its maximal possible value $[-A, A]$.

Another explicit subinterval of $[\gamma^1_{\tilde{e}}, \gamma^2_{\tilde{e}}]$ can be obtained by comparing $\min_{\mathcal{S}} \gamma_{\tilde{e}}$ and $\max_{\mathcal{S}} \gamma_{\tilde{e}}$ to $\min_{\mathbf{R}} \gamma_{\tilde{e}}$ and $\max_{\mathbf{R}} \gamma_{\tilde{e}}$. Clearly, for each $a \in \mathbf{R}$,

$$\gamma_{\tilde{e}}(a) = \sin a \, \overline{\cos \tilde{E}} + \cos a \, \overline{\sin \tilde{E}},$$

so that

$$-\min_{\mathbf{R}} \gamma_{\tilde{e}} = \max_{\mathbf{R}} \gamma_{\tilde{e}} = \delta(\tilde{e}) \geqslant 0$$

where

$$\delta(\tilde{e}) = (\overline{\cos \tilde{E}}^2 + \overline{\sin \tilde{E}}^2)^{1/2}.$$

By Schwarz inequality, we have

$$\delta(\tilde{e}) \leq 1$$

for each $\tilde{e} \in \tilde{L}^1(0, T)$. Let

$$\beta = \beta(A, c, \omega) = \omega^{-1}(\omega^2 + c^2)^{-1/2} A.$$

THEOREM 5 ([2]) - $[\gamma^1_{\tilde{e}}, \gamma^2_{\tilde{e}}] \subset [-(\delta(\tilde{e}) + \beta), \delta(\tilde{e}) + \beta]$, *and if*

$$\beta \leq \delta(\tilde{e}).$$

then

$$[-(\delta(\tilde{e})-\beta),\ \delta(\tilde{e})-\beta] \subset [\gamma^1_{\tilde{e}}, \gamma^2_{\tilde{e}}].$$

Proof. By Lemma 4 and Schwarz inequality, we have, for each

$$y \in \mathcal{S},\ |\gamma_{\tilde{e}}(y)-\gamma_{\tilde{e}}(\bar{y})| \leq \|\tilde{y}\| \leq \omega^{-1}(\omega^2+c^2)^{-1/2}A = \beta,$$

and the result follows immediately.

One can find in [2] an example, devised by Dalmaso and Invernizzi (personal communication), of functions $\tilde{e}$ for which $\delta(\tilde{e})=0$. For such a function, $[\gamma^1_{\tilde{e}}, \gamma^2_{\tilde{e}}] \subset [-\beta, \beta]$ can be made arbitrarily close to $\{0\}$ by taking c or ω sufficiently large.

Let us also mention that, as it was noticed in [4] in another context, the condition

$$\operatorname*{osc}_{[0,T]} \tilde{E} < \pi$$

implies that 0 does not belong to the closed convex hull of the set $\{\exp.\ i\tilde{E}(t) : t \in [0,T]\}$, i.e.

$$\frac{1}{T}\int_0^T \exp i\tilde{E}(t)\, dt \neq 0,$$

i.e.

$$\delta(\tilde{e}) = \left|\frac{1}{T}\int_0^T \exp i\tilde{E}(t)\, dt\right| > 0.$$

Open Problem 2 - Theorem 4 and 5 furnish sufficient conditions on $\tilde{e}$ under which $0 \in [\gamma^1_{\tilde{e}}, \gamma^2_{\tilde{e}}]$. It is still an open question to know if this property holds for each $\tilde{e} \in \tilde{L}^1(0,T)$, $c \in \mathbf{R}$, $\omega > 0$ and $A > 0$. The answer is yes when $c = 0$ as shown independently by Dancer [1] and Willem [10] using a variational argument which essentially comes back to Hamel [3]. This is also true for $|c|$ «sufficiently large» as shown in [8] when $\tilde{e} \in \tilde{L}^2(0,T)$, and as it follows from Theorem 5 when $\delta(\tilde{e}) > 0$. The answer is positive also when the Fourier series of $\tilde{e}$ only contains harmonics of odd order (the so-called odd-harmonic functions). In this case, the same is true for $\tilde{E} = K\tilde{e}$ and hence $\tilde{E}(t) = -\tilde{E}(t+\frac{T}{2})$, $t \in [0, T/2]$. Consequently, if $u \in C([0,T])$ and $u(t) = -u(t+\frac{T}{2})$, $t \in [0,T/2]$,

$$\sin(u(t)+\tilde{E}(t)) = -\sin(u(t+\frac{T}{2})+\tilde{E}(t+\frac{T}{2}))$$

and the determination of the solutions of (3) which are odd-harmonic is equivalent to the fixed point problem

$$(12) \qquad y = -K[A\sin(y(\cdot)+\tilde{E}(\cdot))]$$

in the space of continuous odd-harmonic functions. The Schauder's fixed point theorem immediately implies the existence of a solution for (12), and hence $0 \in [\gamma^1_{\tilde{e}}, \gamma^2_{\tilde{e}}]$ for each odd-harmonic $\tilde{e} \in \tilde{L}^1(0,T)$.

4. - Uniqueness of solutions with a fixed mean value.

The discussion in this section will be based upon the following result.

THEOREM 6 ([2]) - *If*

$$A \leq \omega^2, \tag{13}$$

if v_i *is solution of* (3) *with* $\bar{e} = \bar{e}_i \ (i = 1,2)$ *and if* $\bar{v}_1 = \bar{v}_2$, *then* $v_1 = v_2$.

In particular, when (13) *holds,* (3) *has at most one solution with given mean value.*

Proof. If $u = v_1 - v_2$, so that $\bar{u} = 0$, we have

$$u'' + cu' + A[\sin(v_1 + \tilde{E}(t)) - \sin(v_2 + \tilde{E}(t))] = \bar{e}_1 - \bar{e}_2$$
$$u(0) - u(T) = u'(0) - u'(T) = 0,$$

so that

$$\int_0^T u'^2(t)\,dt = A\int_0^T [\sin(v_1(t) + E(t)) - \sin(v_2(t) + E(t))]\,u(t)\,dt. \tag{14}$$

Using the inequality

$$\frac{\sin r - \sin s}{r - s} < 1$$

for $r \neq s$, the assumption that $u(t) \neq 0$ on a set of positive messure implies by (14) that

$$\int_0^T u'^2(t)\,dt < A\int_0^T u^2(t)\,dt \leq \omega^2 \int_0^T u^2(t)\,dt,$$

a contradiction with Wirtinger's inequality.

In particular, if $\tilde{e}$ is odd-harmonic, problem (3) has an odd-harmonic solution y which has therefore mean value zero. When $A \leq \omega^2$, y is the unique solution with mean value zero which, together with Theorem 1, immediately implies the following

THEOREM 7 - *If* $A \leq \omega^2$, $e \in L^1(0,T)$ *is odd-harmonic and if*

$$\gamma^1_{\tilde{e}} < 0 < \gamma^2_{\tilde{e}} \ \text{or} \ \gamma^1_{\tilde{e}} = \gamma^2_{\tilde{e}} = 0,$$

then problem (3) *has exactly one odd-harmonic solution and at least a second solution which is not odd-harmonic.*

REMARK 1 - Theorem 7 is rather sharp because if we take $\tilde{e} = 0$, so that $\gamma^1_{\tilde{e}} = -A = -\gamma^2_{\tilde{e}}$, problem (3) has always the odd-harmonic solution $y = 0$ and the second solution $y = \pi$. It has also families of periodic solutions $2k\pi + u(t + \varphi, a)$ $(k \in \mathbf{Z}, \varphi \in \mathbf{R}, a > 0)$ with periods $S = S(a)$ fulfilling the interval $]2\pi / \sqrt{A}, +\infty[$ and where $u(\cdot\,; a)$ is choosen to be odd-harmonic. Thus, if $2\pi / \sqrt{A} < T$, i.e. if $A > \omega^2$, problem (3) with an odd-harmonic $\tilde{e}$ may have more than one odd-harmonic solution.

On can find in [4] some conditions upon A, e, ω and c which insure the existence of exactly two solutions (mod $2k\pi$).

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