ON THE FREDHOLM ALTERNATIVE FOR a-HOMOGENEOUS OPERATORS (*)

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SOMMARIO. - *Si considera la relazione tra la risolubilità dell'equazione*

(1) $$J(x) - \mu S(x^+) + \nu S(x^-) + G(x) = f$$

e l'insieme dei parametri reali μ *e* ν *per i quali esiste una soluzione non banale di*

$$J(x) - \mu S(x^+) + \nu S(x^-) = 0$$

(x^+*, risp.* x^-*, è la parte «positiva», risp. «negativa», di* x*). Usando un procedimento topologico basato sulla teoria del grado di Leray-Schauder si fa vedere, in alcuni casi speciali di problemi di valori al contorno per equazioni differenziali ordinarie, che esiste una stretta relazione tra le asserzioni riguardanti la risolubilità di* (1) *e l'alternativa classica di Fredholm (nei casi speciali degli operatori* J, S *e* $\mu = \nu$*).*

SUMMARY. - *We are interested in the relation between the solvability of the equation*

(1) $$J(x) - \mu S(x^+) + \nu S(x^-) + G(x) = f$$

and the set of real parameters μ *and* ν *for which there exists a nontrivial solution of*

$$J(x) - \mu S(x^+) + \nu S(x^-) = 0$$

(x^+*, resp.* x^-*, is the «positive», resp. «negative», part of* x*). Using a topological argument based on the Leray-Schauder degree theory it is shown, in some special cases of boundary value problems for ordinary differential equations, that there is a close relation between the assertions concerning the solvability of* (1) *and the classical Fredholm alternative (in the special cases of operators* J, S *and* $\mu = \nu$*).*

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1. - Introduction.

Let us consider the operator equation

$$(1.1) \qquad J(x) - \mu S(x^+) + \nu S(x^-) = f,$$

where μ and ν are real parameters, J and S are a-homogeneous operators, the properties of which will be specified in Section 2, x^+, resp. x^-, denote the «positive», resp. «negative», part of an element x. Our aim is to show some connections between the solvability of the equation (1.1) and the solvability of the equation

$$(1.2) \qquad y - \lambda T y = f,$$

where T is a linear completely continuous operator.

It is well known that the Fredholm alternative gives an exhaustive answer to the solvability of the equation (1.2). In Section 2 we formulate some problems, the solutions of which are more or less known in some special cases of operators J and S. Moreover the solutions of these problems could be understood as a generalization of the Fredholm alternative for the operator equations of the type (1.1). We are also interested in the investigation of the perturbed equation

$$(1.3) \qquad J(x) - \mu S(x^+) + \nu S(x^-) + G(x) = f$$

with some suitable completely continuous operator G and we formulate Landesman-Lazer-type problem concerning the solvability of the equation (1.3), i.e. under the assumption that (1.1) has a nontrivial solution for $f = 0$ to formulate by means of the properties of G the sufficient conditions on f in order for (1.3) to be solvable.

In addition we are also interested in the solvability of the operator equations which are asymptotically close to the equation (1.1) and in Theorems 2.1, 2.2 we obtain the generalization of some results contained in Fučík, Nečas, Souček, Souček [9], Fučík [8], Kučera [15], Nečas [19], Pochozajev [22].

In Section 3 we give some partial answers to the questions formulated in Section 2 when (1.1)-(1.3) are the operator representations of some boundary value problems for ordinary differential equations. The proofs of all these results use the topological argument based on the Leray-Schauder degree theory. Moreover, it is shown that using the specific form of ordinary differential operators one may obtain existence results even for the equations containing more general completely continuous operators than that considered in Section 2. The proofs of these specific results are based on the properties of the Leray-Schauder degree combined with the «shooting method» for ordinary differential equations.

Section 4 contains some final remarks concerning the multiplicity results and there is shown the connection of the problems investigated above with the case when (1.1)-(1.3) are operator representations of boundary value problems for partial differential equations.

2. - Abstract setting.

Let X, Y, Z be Banach spaces with zero elements 0_X, 0_Y, 0_Z and with norms $\|x\|_X$, $\|y\|_Y$, $\|z\|_Z$, respectively. A subset C of Z is called a cone if it is closed, convex, invariant under multiplication by non-negative real numbers and if $C \cap (-C) = \{0_Z\}$. In the sequel we shall suppose that a given fixed cone C in Z has the following properties:

(Z1) C induces the semi-ordering $x \leqslant y$ (i.e. $y - x \in C$) such that $z^+ = \max\{z, 0_Z\} \in C$, $z^- = \max\{-z, 0_Z\} \in C$ exists for every $z \in Z$.

(Z2) The mapping $z \mapsto z^+$ is continuous.

(Z3) $X \subset Z$ and the identity mapping $X \hookrightarrow Z$ is continuous.

Let us suppose that $a > 0$ is a fixed real number and $J : X \to Y$ is the mapping which satisfies the following properties:

(J1) J is positively a-homogeneous.

(J2) J is homeomorphism X onto Y.

(J3) J is odd, i.e. $x \in X \Rightarrow J(-x) = -J(x)$.

Let $S : Z \to Y$ be the operator defined on Z and satisfying:

(S1) S is positively a-homogeneous.

(S2) S is continuous.

(S3) $x \mapsto S(x^+)$, $x \mapsto S(x^-)$ are completely continuous mappings from X into Y.

Suppose that $G : X \to Y$ is a completely continuous operator. We shall denote

$$R(G) = \{f \in Y ;\ \exists x_0 \in X : J(x_0) - \mu S(x_0^+) + \nu S(x_0^-) + G(x_0) = f\},$$

where $R(0)$ is written in the case $G(x) = 0$, for all $x \in X$.

$$A_{-1} = \{(\mu, \nu) \in \mathbf{R}^2 ;\ \exists x_0 \neq O_X : J(x_0) - \mu S(x_0^+) + \nu S(x_0^-) = O_Y\},$$

$$A_0 = \mathbf{R}^2 \setminus A_{-1},$$

$$A_1 = \{(\mu, \nu) \in A_0 ;\ \deg[\tilde{F} ; B_Y(1), O_Y] \neq 0\},$$

where $\tilde{F}: y \mapsto y - \mu S(J^{-1}(y))^+ + \nu S(J^{-1}(y))^-$, $y \in Y$, and «deg» denotes the Leray-Schauder degree of $\tilde{F}$ with respect to O_Y and open ball $B_Y(1) \subset Y$ with radius 1.

$A_2 = \{ (\mu, \nu) \in A_0 ; R(0) \neq Y \}$,

$A_3 = \{ (\mu, \nu) \in \mathbf{R}^2 ; R(0) = Y \}$.

The proofs of the following assertions (i)-(x) are based on the topological properties of the Leray-Schauder degree and the reader may find them in Fučík [12].

(i) The sets $A_i \subset \mathbf{R}^2$, $i = -1, 0, 1, 2, 3$, are symmetric with respect to the diagonal $\mu = \nu$.

(ii) A_0 is open in $\mathbf{R}^2$ and moreover if

$$(\alpha, \beta) \in \mathbf{R}^2, |\alpha| + |\beta| < \frac{C_1(\mu, \nu)}{s}, (\mu, \nu) \in A_0,$$

then $(\mu + \alpha, \nu + \beta) \in A_0$, where $C_1(\mu, \nu)$ and s are defined as follows:

$$C_1(\mu, \nu) = \inf_{\|x\|_X = 1} \| J(x) - \mu S(x^+) + \nu S(x^-) \|_Y > 0$$

and

$$s = \max \{ \sup_{\|x\|_X = 1} \| S(x^+) \|_Y, \sup_{\|x\|_X = 1} \| S(x^-) \|_Y \} < \infty .$$

(iii) For $(\mu, \nu) \in A_0$ the set $R(0)$ is closed in Y.

(iv) $A_1 \subset A_3$.

(v) A_1 is an open subset of $\mathbf{R}^2$.

(vi) A_1 is a union of some components of A_0.

(vii) If $T_\lambda \subset A_0$ is a component containing the point (λ, λ) for some real λ then $T_\lambda \subset A_1$.

(viii) Let $(\mu, \nu) \in A_1$ and suppose that

$$\limsup_{\|x\|_X \to \infty} \frac{\| G(x) \|_Y}{\| x \|_X^a} < C_1(\mu, \nu).$$

Then $R(G) = Y$.

(ix) For given $(\mu, \nu) \in A_2$ there exists $C_2(\mu, \nu) > 0$ such that if

$$\limsup_{\|x\|_X \to \infty} \frac{\| G(x) \|_Y}{\| x \|_X^a} < C_2(\mu, \nu)$$

then $R(G) \neq Y$.

(x) A_2 is an open set in $\mathbf{R}^2$.

In the following part of Section 2 we shall suppose that $S(x) = S(x^+) - S(x^-)$, $x \in X$. Let us denote by σ the set of all real numbers λ for which there exists a nontrivial solution $x \in X$ (i.e. $x \neq O_X$) of the equation

(2.1) $J(x) - \lambda S(x) = O_Y$.

Then we obtain $(\lambda, \lambda) \in A_{-1}$ whenever $\lambda \in \sigma$. Let us suppose for a moment that $a = 1$, J, S are linear operators satisfying (J1)-(J3), (S1)-(S3) and Y is a Hilbert space. Then (2.1) may be written in an equivalent form

(2.2) $y - \lambda S(J^{-1}(y)) = O_Y$

and the set σ becomes the spectrum of completely continuous linear operator $S \circ J^{-1}$. This spectrum is countable, isolated, and each eigenvalue has a finite multiplicity. The Fredholm alternative then implies that the equation

(2.3) $J(x) - \lambda S(x) = f$

has a solution for arbitrary right-hand side $f \in Y$ provided $\lambda \notin \sigma$. If $\lambda \in \sigma$ then (2.3) has a solution if and only if f belongs to the orthogonal complement of the eigenspace corresponding to λ.

If we suppose, now, that $a > 0$ and generally not equal to 1 then the following question may be formulated.

QUESTION 1 - What is there to say about the structure of the set σ and about the multiplicity of $\lambda \in \sigma$? Moreover, what is there to say about the solvability of the equation (2.3) if $\lambda \notin \sigma$ and $\lambda \in \sigma$?

REMARK 2.1 - Let X and Y be two Banach spaces, $T : X \to Y$. The mapping T is said to be *regularly surjective from X onto Y* if $T(X) = Y$ and for any $R > 0$ there exists $r > 0$ such that $\|x\|_X \leqslant r$ for all $x \in X$ with $\|T(x)\|_Y \leqslant R$. Then the partial answer to the question above is contained in [9, Th. 3.2, Chp. II]. The following assertion is proved:

«Let X and Y be two Banach spaces. Then $J - \lambda S$ is regularly surjective from X onto Y if and only if $\lambda \notin \sigma$».

Coming back to the assertion (ii) and taking into account that $C_1(\mu, \nu) > 0$ for $(\mu, \nu) \in A_0$ we obtain that also the following generalization of [9, Th. 3.2] holds:

«Let X and Y be two Banach spaces. Then

$$x \mapsto J(x) - \mu S(x^+) + \nu S(x^-)$$

is regularly surjective from X onto Y if and only if $(\mu, \nu) \in A_0 \cap A_3$».

We arrive, now, from the equation (2.3) to

(2.4) $J(x) - \mu S(x^+) + \nu S(x^-) = f$,

where μ, ν are real numbers. The results concerning the solvability of weakly nonlinear problems with jumping nonlinearities (see Dancer [3,4], Drábek [6], Fučík [10,12], Ruf [23] for ordinary differential equations and Dancer [5], Galouet, Kavian [27,28] for partial differential equations) show that the set A_{-1} will play the same role in the investigation of the equation (2.4) as the set σ in the investigation of the equation (2.3). More precisely, the understanding of the solvability of the equation (2.4) is determined by the understanding of the structure of the set A_{-1}. But the problem of precise description of the set A_{-1} seems to be very difficult and up to now it was solved in some particular cases of ordinary differential equations of second and fourth order (see Dancer [4], Drábek [6], Fučík [10], Krejčí [14]). The case of partial differential equations seems to be much more complicated because the essential role in the structure of the set A_{-1} will play not only the type of the corresponding partial differential operator but also the domain and boundary conditions (see open problems formulated in Dancer [5], Fučík [12]).

Nevertheless concerning the solvability of the equation (2.4) the following problems may be formulated.

QUESTION 2 - Is it possible to characterize the set of right-hand sides $f \in Y$ for which the equation (2.4) is solvable if $(\mu, \nu) \in A_{-1}$?

QUESTION 3 - Does the set A_3 contain also the components of A_0 which have an empty intersection with the diagonal $\mu = \nu$?

QUESTION 4 - Let $\lambda \notin \sigma$ and let us denote

$$\sigma_- = \{\eta \in \sigma ; \eta < \lambda\} , \quad \sigma_+ = \{\eta \in \sigma ; \eta > \lambda\},$$

$$\lambda_s = \sup \sigma_- , \quad \lambda_i = \inf \sigma_+ ,$$

$$K = \{ (\mu, \nu) \in \mathbf{R}^2 ; \lambda_s < \mu < \lambda_i , \lambda_s < \nu < \lambda_i\}.$$

Then the question is: $K \subset A_3$?

QUESTION 5 - Is it possible to characterize the right-hand sides $f \in Y$ for which (2.4) is solvable in the case $(\mu, \nu) \in A_2$?

Let us consider, now, the perturbed equation

(2.5) $J(x) - \mu S(x^+) + \nu S(x^-) + G(x) = f$.

The assertions (viii) and (ix) of this section give an answer concerning the solvability of (2.5) under the conditions $(\mu, \nu) \in A_1$, resp. $(\mu, \nu) \in A_2$. Let us suppose that G is the Nemitskii operator generated by some real continuous functions $s \mapsto g(s)$. Then one may be interested in the following Landesman-Lazer-type problem.

QUESTION 6 - Is it possible to formulate by means of

$$g_+ = \lim_{s \to +\infty} g(s) \text{ and } g_- = \lim_{s \to -\infty} g(s)$$

any conditions on $f \in Y$ in order for (2.5) to be solvable in the case $(\mu, \nu) \in A_{-1}$?

In Section 3 we shall give some partial answers to the questions formulated above. In the following part of this section we shall deal with the equations of the type (2.4) containing the operators which are asymptotically close to J and S.

DEFINITION 2.1 - The mapping $T: X \to Y$ is said to be a *(K, L, a)-homeomorphism of X onto Y* if

(1) T is a homeomorphism of X onto Y;

(2) there exist real numbers $K > 0$, $L > 0$ such that

$$L \| x \|_X^a \leqslant \| T(x) \|_Y \leqslant K \| x \|_X^a ,$$

for each $x \in X$.

DEFINITION 2.2 - Let $T_0: X \to Y$ be an a-homogeneous operator.

(i) T is said to be *a-quasihomogeneous with respect to T_0* if

$$t_n \searrow 0 \ , \ x_n \rightharpoonup x_0 \ , \ t_n^a T \left(\frac{x_n}{t_n}\right) \to y_0 \in Y$$

imply

$$T_0(x_0) = y_0 .$$

(ii) T is said to be *a-strongly quasihomogeneous with respect to T_0* if

$$t_n \searrow 0 \ , \ x_n \rightharpoonup x_0 \ , \text{ imply } t_n^a T \left(\frac{x_n}{t_n}\right) \to T_0(x_0).$$

REMARK 2.2 - The symbols « $\rightharpoonup$ » and « $\to$ » denote the weak and the strong convergence, respectively.

Using the homotopy invariance property of the Leray-Schauder degree we shall prove the following assertion.

THEOREM 2.1 - *Let X be a reflexive Banach space and A an odd (K, L, a)-homeomorphism of X onto Y which is a-quasihomogeneous with respect to J. Then if $(\mu, \nu) \in T_\lambda \subset A_0$, where T_λ is some component containing the point (λ, λ) for some $\lambda \in \mathbf{R}$ then the equation*

(2.6) $\quad A(x) - \mu S(x^+) + \nu S(x^-) = f$

has at least one solution for arbitrary right-hand side $f \in Y$.

Proof. Let us denote by $\eta(\tau) = (\eta_1(\tau), \eta_2(\tau))$, $\tau \in [0, 1]$ the smooth curve which lies in T_λ and such that $\eta(0) = (\lambda, \lambda), \eta(1) = = (\mu, \nu)$. We shall prove the existence of such a ball $B_X(r) \subset X$ that the mapping defined by

$$H(x, \tau) = A(x) - \eta_1(\tau)\, S(x^+) + \eta_2(\tau)\, S(x^-) - \tau f \tag{2.7}$$

satisfies

$$H(x, \tau) \neq O_Y, \tag{2.8}$$

for all $x \in \partial B_X(r)$ (the boundary of $B_X(r)$) and $\tau \in [0, 1]$. If (2.8) is proved then the assertion will follow from the fact that A is (K, L, a)-homeomorphism, the Borsuk theorem and from the basic property of the Leray-Schauder degree.

Let us suppose via contradiction that there are

$$\tau_n \in [0, 1] \ , \ \| x_n \|_X \to \infty$$

such that

$$H(x_n, \tau_n) = O_Y. \tag{2.9}$$

Passing to the suitable subsequences we may suppose that

$$\tau_n \to \tau_0 \in [0, 1], \frac{x_n}{\| x_n \|_X} = v_n \rightharpoonup v_0 \in X.$$

Hence dividing (2.9) by $\| x_n \|_X^a$ we obtain using (2.7)

$$\frac{A(\| x_n \|_X v_n)}{\| x_n \|_X^a} \to \eta_1(\tau_0)\, S(v_0^+) - \eta_2(\tau_0)\, S(v_0^-).$$

This implies that

$$J(v_0) - \eta_1(\tau_0)\, S(v_0^+) + \eta_2(\tau_0)\, S(v_0^-) = O_Y. \tag{2.10}$$

Since A is (K, L, a)-homeomorphism, we have

$$\frac{A(x_n)}{\| x_n \|_X^a} \geqslant L, \tag{2.11}$$

for all $n \in \mathbf{N}$, and hence $v_0 \neq O_X$. But it is $(\eta_1(\tau), \eta_2(\tau)) \in A_0$, for all $\tau \in [0, 1]$, which contradicts (2.10). We have just proved that (2.8) is true and the proof of Theorem 2.1 is completed. Q.E.D.

Analogously we may have a completely continuous part of the equation which is not necessarily homogeneous.

THEOREM 2.2 - *Let X be a reflexive Banach space. Let A be an odd (K, L, a)-homeomorphism of X onto Y which is a-quasihomogeneous with respect to J and let F be a completely continuous operator from X into Y which is a-strongly quasihomogeneous with respect to the operator $x \mapsto \mu S(x^+) - \nu S(x^-)$. Then if $(\mu, \nu) \in T_\lambda \subset A_0$, T_λ is some component containing the point (λ, λ) then the equation*

$$A(x) - F(x) = f$$

has at least one solution for arbitrary right-hand side $f \in Y$.

Proof. With respect to the previous theorem it is sufficient to prove that there exists a sufficiently large ball $B_x(r) \subset X$ such that

$$\tilde{H}(x, \tau) \neq O_Y,$$

for all $x \in \partial B_X(r)$, $\tau \in [0, 1]$, where

$$\tilde{H}(x, \tau) = A(x) - (1 - \tau) F(x) - \tau\mu S(x^+) + \tau\nu S(x^-) - (1 - \tau) f.$$

Then the assertion will follow from the homotopy invariance property and the basic property of the Leray-Schauder degree.

Let us suppose via contradiction that there are

$$\tau_n \in [0, 1], \ \|x_n\|_X \to \infty$$

such that

$$\text{(2.12)} \quad \tilde{H}(x_n, \tau_n) = O_Y.$$

Then, at least for some subsequence,

$$\tau_n \to \tau_0 \in [0, 1], \frac{x_n}{\|x_n\|_X} = v_n \rightharpoonup v_0 \in X$$

and

$$\frac{F(\|x_n\|_X v_n)}{\|x_n\|_X^a} \to \mu S(v_0^+) - \nu S(v_0^-), \ S(v_n^+) \to S(v_0^+), \ S(v_n^-) \to S(v_0^-).$$

Hence dividing (2.12) by $\|x_n\|_X^a$ we obtain that

$$\frac{A(\|x_n\|_X v_n)}{\|x_n\|_X^a} \to \mu S(v_0^+) - \nu S(v_0^-), \ \text{i.e.}$$

$$\text{(2.13)} \quad J(v_0) - \mu S(v_0^+) + \nu S(v_0^-) = O_Y.$$

Using (2.11) we obtain $v_0 \neq O_X$ which contradics (2.13) and our assumption on (μ, ν). Q.E.D.

3. - Boundary value problems for ordinary differential equations.

In this section we shall deal with some special cases of boundary value problems for ordinary differential equations, the solutions of which give a more or less exhaustive answer to the questions formulated in Section 2.

Let $L_p(0, \pi)$ and $C^k([0, \pi])$ denote the function spaces of Lebesque integrable functions and k-times continuously differentiable functions, respectively, with the norms defined as usual, where $p \in [1, \infty[$ is a real number and k is a non-negative integer. Let $\overset{0}{W}{}^{k,p}(0,\pi)$ denote the Sobolev space (see e.g. Kufner, John, Fučík [16]) with the norm

$$\|f\|_{k,p,0} = (\int_0^\pi |f^{(k)}(t)|^p \, dt)^{1/p}.$$

Let us suppose in the sequel $p \geqslant 2$, $q = \dfrac{p}{p-1}$. Let a and b be real functions defined on $[0,\pi]$. Suppose that $a(t) > 0$, for all $t \in [0,\pi]$ and $a \in C^1([0,\pi])$, $b(t) > 0$, for all $t \in [0,\pi]$, $b \in C([0,\pi])$. Put

$$X = Z = \overset{0}{W}{}^{1,p}(0,\pi), \quad Y = W^{-1,q}(0,\pi)$$

and denote

$$(3.1) \quad \begin{cases} \langle J(u), v\rangle = \int_0^\pi a(t)\,|u'(t)|^{p-2}u'(t)\,v'(t)\,dt \\ \langle S(u), v\rangle = \int_0^\pi b(t)\,|u(t)|^{p-2}u(t)\,v(t)\,dt \\ \langle f, v\rangle = \int_0^\pi h(t)\,v(t)\,dt \end{cases}$$

$h \in L_1(0,\pi)$, for all $v \in \overset{0}{W}{}^{1,p}(0,\pi)$, where $\langle \cdot,\cdot \rangle$ is used for the duality between X and Y.

Then it is possible to verify that the operators J and S satisfy the conditions (J1)-(J3), (S1)-(S3) from Section 2 (see [6]). The equation (1.1) is the operator representation of the boundary value problem

$$(3.2) \quad \begin{cases} -(a(t)\,|u'(t)|^{p-2}u'(t))' - \mu b(t)\,|u(t)|^{p-2}u^+(t) + \\ \quad + \nu b(t)\,|u(t)|^{p-2}u^-(t) = h(t), \; t \in]0,\pi[, \\ u(0) = u(\pi) = 0. \end{cases}$$

DEFINITION 3.1 - Let $h \in L_1(0,\pi)$ and let

$$\langle J(u), v\rangle - \mu\langle S(u^+), v\rangle + \nu\langle S(u^-), v\rangle = \langle f, v\rangle$$

holds for all $v \in \overset{0}{W}{}^{1,p}(0,\pi)$. Then $u \in \overset{0}{W}{}^{1,p}(0,\pi)$ is called the *weak solution* of boundary value problem for the nonlinear Sturm-Liouville equation of the second order (3.2) with the right-hand side h.

REMARK 3.1 - It is possible to prove that the weak solution u of (3.2) satisfies $u \in C^1([0,\pi])$. Moreover, if $h \in C([0,\pi])$ then

$$a(t)\,|u'(t)|^{p-2}u'(t) \in C^1([0,\pi])$$

(for the proof see [6, Th. 3.3]).

The following assertion is proved in [9], [20], [21].

THEOREM 3.1 - *It is* $\sigma = \{\lambda_n\}_{n=1}^{\infty}$, *where*

$$0 < \lambda_1 < \lambda_2 < \dots, \lim_{n\to\infty} \lambda_n = \infty .$$

The eigenvectors are isolated and to each λ_n *there corresponds a finite number of normed eigenvectors.*

Let us remark that the previous assertion gives the partial answer to the Question 1 in the special case of the ordinary differential operator of second order presented above.

Using the abstract theory from Section 2 we obtain the following two assertions, the proofs of which may be found in [6].

THEOREM 3.2 - *Let* $\lambda \notin \sigma$ *and* $(\mu, \nu) = (\lambda + \alpha , \lambda + \beta)$, *where* $|\alpha| + |\beta| < \dfrac{C_1(\lambda,\lambda)}{s}$ *for* $C_1(\lambda,\lambda)$ *and* s *see Section 2). Then the boundary value problem* (3.2) *has at least one weak solution* $u \in W_0^{1,p}(0,\pi)$ *for an arbitrary right-hand side* $h \in L_1(0,\pi)$.

THEOREM 3.3 - *Let* $(\mu,\nu) \in A_1$ *and let* $g(t,z)$ *be the real function defined on* $[0,\pi] \times \mathbf{R}$. *Let the function* $g(t,z)$ *satisfy Caratheodory's conditions and, moreover, let there exist a function* $r(t) \in L_1(0,\pi)$ *such that*

$$|g(t,z)| \leqslant r(t) + C_1(\mu,\nu)|z|^{p-1}$$

holds for each $z \in \mathbf{R}$ *and for almost all* $t \in]0,\pi[$. *Then the boundary value problem*

$$\begin{cases} -(a(t)|u'(t)|^{p-2}u'(t))' - \mu b(t)|u(t)|^{p-2}u^+(t) + \\ \quad + \nu b(t)|u(t)|^{p-2}u^-(t) + g(t,u(t)) = h(t), \ t \in]0,\pi[, \\ u(0) = u(\pi) = 0 \end{cases}$$

has at least one weak solution for an arbitrary right-hand side $h \in L_1(0,\pi)$.

REMARK 3.2 - Let us remark that even in this special case of an ordinary differential operator of second order the exhaustive answers to the questions formulated in the previous section remain open.

Something more may be done in the case $a(t) = b(t) = 1$, for all $t \in [0,\pi]$. The constant coefficients in the ordinary differential operator allow us to give the precise description of the set A_{-1}.

On the basis of nodal properties of corresponding initial value problems it is possible to prove the following assertion.

THEOREM 3.4 - *Boundary value problem*

$$\begin{cases} -(|u'|^{p-2}u')' - \mu|u|^{p-2}u^+ + \nu|u|^{p-2}u^- = 0 , \\ u(0) = u(\pi) = 0 \end{cases} \tag{3.3}$$

has a nontrivial weak solution if and only if one of the following conditions holds:

(i) $\mu = \lambda_1$, ν *is arbitrary;*

(ii) μ *is arbitrary,* $\nu = \lambda_1$;

(iii) $\mu > \lambda_1$, $\nu > \lambda_1$,

$$w_1(\mu, \nu) = \frac{(\mu)^{1/p}(\nu)^{1/p}}{((\mu)^{1/p} + (\nu)^{1/p})(\lambda_1)^{1/p}} = k,$$

$$w_2(\mu, \nu) = \frac{((\mu)^{1/p} - (\lambda_1)^{1/p})(\nu)^{1/p}}{((\mu)^{1/p} + (\nu)^{1/p})(\lambda_1)^{1/p}} = k,$$

$$w_3(\mu, \nu) = \frac{((\nu)^{1/p} - (\lambda_1)^{1/p})(\nu)^{1/p}}{((\mu)^{1/p} + (\nu)^{1/p})(\lambda_1)^{1/p}} = k,$$

$k = 1, 2, 3, \ldots$

The proof of this assertion may be found in [6].

REMARK 3.3 - The conditions (i), (ii) and (iii) give us the precise description of the set A_{-1}. We obtain the system of curves which may be drawn in the plane (μ, ν) (for the picture see [6, p. 181]) and which intersect diagonal $\mu = \nu$ in the points (λ_1, λ_1), $(\lambda_2, \lambda_2), \ldots$, where $0 < \lambda_1 < \lambda_2 < \ldots \to \infty$ are the eigenvalues of the problem

$$(3.4) \quad \begin{cases} -(|u'|^{p-2}u')' - \lambda |u|^{p-2}u = 0, \\ u(0) = u(\pi) = 0. \end{cases}$$

Moreover these curves decompose the plane (μ, ν) into some components which may be divided into two groups. The first group contains the components which have nonempty intersection with the diagonal. Then according to the assertion (vii) from Section 2 this group of components belongs to the set A_1. Particularly, if the couple (μ, ν) lies in some component of this kind, we obtain that the boundary value problem

$$(3.5) \quad \begin{cases} -(|u'|^{p-2}u')' - \mu |u|^{p-2}u^+ + \nu |u|^{p-2}u^- = h, \\ u(0) = u(\pi) = 0 \end{cases}$$

has the weak solution for the arbitrary right-hand side $h \in L_1(0, \pi)$.

On the other hand the second group contains the components which have empty intersection with the diagonal $\mu = \nu$. In this case it is possible to prove using the so-called shooting argument (for the proof see [6, pp. 179-181]) that the union of components of this kind belongs to A_2. Particularly, if (μ, ν) belongs to some component which has empty intersection with the diagonal then there exists such $h \in L_1(0, \pi)$ that the boundary value problem (3.5) has no weak solution.

REMARK 3.4 - In this special case of the $(p-1)$-homogeneous ordinary differential operator of second order with constant coefficients we have just obtained the answers to Questions 3 and 4 from Section 2. In fact A_1 contains only the components of A_0 which have non-empty intersection with the diagonal and the set K defined in Section 2 always satisfies $K \subset A_3$. Note that even if we consider such a special case of an ordinary differential operator we have no exhaustive answers to the Questions 1, 2, 5 and 6.

Let us suppose, now, that $\Phi : \mathbf{R} \to \mathbf{R}$ is a continuous function which has finite limits

$$(\Phi_{\pm\infty})\ \Phi_{+\infty} = \lim_{s\to+\infty} \frac{\Phi(s)}{|s|^{p-2}s} \text{ and } \Phi_{-\infty} = \lim_{s\to-\infty} \frac{\Phi(s)}{|s|^{p-2}s}.$$

Then we way define the operator $F : \overset{0}{W}{}^{1,p}(0,\pi) \to W^{-1,q}(0,\pi)$ in the following way:

$$(3.6)\qquad \langle F(u), v\rangle = \int_0^\pi \Phi(u(t))\, v(t)\, dt, \text{for all } v \in \overset{0}{W}{}^{1,p}(0,\pi).$$

Define the operator $A : \overset{0}{W}{}^{1,p}(0,\pi) \to W^{-1,q}(0,\pi)$ by

$$(3.7)\qquad \langle A(u), v\rangle = \int_0^\pi (1 + |u'|^{p-2})\, u' v'\, dt, \text{ for all } v \in \overset{0}{W}{}^{1,p}(0,\pi).$$

Then proceeding exactly as in [6, Lemma 3.2] we can prove that A is odd, $(K, L, p-1)$-homeomorphism $\overset{0}{W}{}^{1,p}(0,\pi)$ onto $W^{-1,q}(0,\pi)$ which is $(p-1)$-quasihomogeneous with respect to J (defined in (3.1) with $a(t) = 1$ for all $t \in [0,\pi]$). It may be also shown that the operator F defined by (3.6) is completely continuous operator from $\overset{0}{W}{}^{1,p}(0,\pi)$ into $W^{-1,q}(0,\pi)$ which is $(p-1)$-strongly quasihomogeneous with respect to the operator

$$u \mapsto \Phi_{+\infty} S(u^+) - \Phi_{-\infty} S(u^-),$$

where S is defined in (3.1) and we consider $b(t) = 1$ for all $t \in [0,\pi]$. Then using the description of the set A_{-1} (see Th. 3.4 of this section) and applying Theorem 2.2 we obtain the following existence result.

THEOREM 3.5 - *Let us suppose that one of the following conditions is fulfilled:*

(i) $\Phi_{+\infty} < \lambda_1$, $\Phi_{-\infty} < \lambda_1$;

(ii) $\Phi_{+\infty} > \lambda_1$, $\Phi_{-\infty} > \lambda_1$ *and*

$$\frac{((\Phi_{+\infty})^{1/p} - (\lambda_1)^{1/p})\,(\Phi_{-\infty})^{1/p}}{((\Phi_{+\infty})^{1/p} + (\Phi_{-\infty})^{1/p})\,(\lambda_1)^{1/p}} < 1, \quad \frac{((\Phi_{-\infty})^{1/p} - (\lambda_1)^{1/p})\,(\Phi_{+\infty})^{1/p}}{((\Phi_{+\infty})^{1/p} + (\Phi_{-\infty})^{1/p})\,(\lambda_1)^{1/p}} < 1,$$

or

$$k - 1 < \frac{((\Phi_{+\infty})^{1/p} - (\lambda_1)^{1/p})\,(\Phi_{-\infty})^{1/p}}{((\Phi_{+\infty})^{1/p} + (\Phi_{-\infty})^{1/p})\,(\lambda_1)^{1/p}} < k, \ k - 1 <$$

$$< \frac{((\Phi_{-\infty})^{1/p} - (\lambda_1)^{1/p})(\Phi_{+\infty})^{1/p}}{((\Phi_{+\infty})^{1/p} + (\Phi_{-\infty})^{1/p})(\lambda_1)^{1/p}} < k$$

with some $k \in \mathbf{N}$, $k \geqq 2$. *Then the boundary value problem*

$$(3.8) \quad \begin{cases} -[(1 + |u'(t)|^{p-2})\, u'(t)]' - \Phi(u(t)) = h(t), \ t \in]0, \pi[, \\ u(0) = u(\pi) = 0 \end{cases}$$

has at least one weak solution $u \in W_0^{1,p}(0,\pi)$ *for arbitrary right-hand side* $h \in L_1(0,\pi)$.

Proof. If the couple $(\Phi_{+\infty}, \Phi_{-\infty})$ fulfils one of the conditions stated above then it belongs to some component of the set A_1. According to Remark 3.3 each such component has nonempty intersection with the diagonal. Because (3.8) allows the operator representation

$$A(u) - F(u) = f$$

we may apply directly Theorem 2.2 to obtain the conclusion. Q.E.D.

REMARK 3.5 - Using the shooting method and the properties of the initial value problem for ordinary differential equations of second order we may obtain the existence results under the more general assumptions on the nonlinearity Φ. Precise proofs of the assertions which will be presented in the sequal the reader may find in Boccardo, Giachetti, Kučera, Drábek [2].

Let us suppose that $\Phi(t,s): [0,\pi] \times \mathbf{R} \to \mathbf{R}$ is a Carathédory function, there is some constant $c > 0$ and a function $m \in L_\alpha(0,\pi)$ $(\alpha > 1)$ such that

$$(3.9) \quad |\Phi(t,s)| < m(t) + c|s|^{p-1}$$

for all $s \in \mathbf{R}$ and almost all $t \in [0,\pi]$. We shall suppose that there exist functions $X^{+\infty}, X^{-\infty}, X_{+\infty}, X_{-\infty} \in L_\infty(0,\pi)$ such that

$$(3.9') \quad \begin{cases} \limsup\limits_{s \to \pm\infty} \dfrac{\Phi(t,s)}{|s|^{p-2}s} = X^{\pm\infty}(t), \\ \liminf\limits_{s \to \pm\infty} \dfrac{\Phi(t,s)}{|s|^{p-2}s} = X_{\pm\infty}(t), \end{cases}$$

for almost all $t \in [0,\pi]$. Then using the description of the set A_1 (for the boundary value problem (3.3)) we obtain the following existence result for the boundary value problem

$$(3.10) \quad \begin{cases} -(|u'(t)|^{p-2}u'(t))' = \Phi(t,u(t)) + h(t), \ t \in]0,\pi[, \\ u(0) = u(\pi) = 0. \end{cases}$$

THEOREM 3.6 - *Let us suppose that either*

(i) *There exists some* $\delta > 0$ *such that*
$X_{\pm\infty}(t), X^{\pm\infty}(t) \leqslant \lambda_1 - \delta$,

for a.a. $t \in]0, \pi[$, *or*

(ii) *there are two couples* (μ_1, ν_1) *and* (μ_2, ν_2) *lying in the same component of* A_1 *and*

$$\mu_1 \leqslant X_{+\infty}(t) \leqslant X^{+\infty}(t) \leqslant \mu_2,$$

$$\nu_1 \leqslant X_{-\infty}(t) \leqslant X^{-\infty}(t) \leqslant \nu_2$$

holds for a.a. $t \in]0, \pi[$.

Then the boundary value problem (3.10) *has at least one weak solution for arbitrary right-hand side* $h \in L_1(0, \pi)$.

REMARK 3.6 - We shall not give the precise proof of this assertion because the reader may find it in [2]. We shall sketch here only the main idea of it.

We may define the operator $J : \overset{0}{W}{}^{1,p}(0, \pi) \to W^{-1,q}(0, \pi)$ and an element $f \in W^{-1,q}(0, \pi)$ as in (3.1) (we suppose $a \equiv 1$). Because of (3.9) we are allowed also to define an operator

$$\hat{F} : \overset{0}{W}{}^{1,p}(0, \pi) \to W^{-1,q}(0, \pi)$$

by

$$\langle \hat{F}(u), v \rangle = \int_0^{\pi} \Phi(t, u(t))\, v(t)\, dt,$$

for all $v \in \overset{0}{W}{}^{1,p}(0, \pi)$. Then the equation

$$J(u) = \hat{F}(u) + f \tag{3.11}$$

is an operator representation of the boundary value problem (3.10) and $u \in \overset{0}{W}{}^{1,p}(0, \pi)$ is the solution of (3.11) if and only if it is the weak solution of (3.10). The main idea to solve (3.11) is to use the homotopy invariance property of the Leray-Schauder degree and the description of the set A_1. We take some $(\mu, \nu) \in A_1$ such that $\mu, \nu \leqslant \lambda_1 - \delta$ in the case (i) and $\mu_1 \leqslant \mu \leqslant \mu_2$, $\nu_1 \leqslant \nu \leqslant \nu_2$ in the case (ii) and consider

$$H(u, \tau) = J(u) - (1 - \tau)\, \hat{F}(u) - (1 - \tau)\, f - \tau\mu S(u^+) + \tau\nu S(u^-),$$
$u \in \overset{0}{W}{}^{1,p}(0, \pi)$, $\tau \in [0, 1]$ (for $S : \overset{0}{W}{}^{1,p}(0, \pi) \to W^{-1,q}(0, \pi)$ see (3.1); we suppose $b \equiv 1$). Our aim is to show that

$$H(u, \tau) \neq 0, \tag{3.12}$$

for all $\tau \in [0, 1]$ and $\| u \|_{1,p} = r$, for sufficiently large $r > 0$. To have this the assertion will follow from the fact that J is $(K, L, p-1)$-homeomorphism, homotopy invariance property of the Leray-Schauder degree and $(\mu, \nu) \in A_1$.

In order to prove (3.12) we proceed via contradiction. We suppose that the converse of (3.12) is true and on the basis of the assumption (i), resp. (ii), of Theorem 3.6 we get that there are some functions $X_+(t), X_-(t) \in L_\infty(0,\pi)$ such that either

(a) $X_+(t), X_-(t) \leqslant \lambda_1 - \delta$, for a.a. $t \in]0,\pi[$ or

(b) $\mu_1 \leqslant X_+(t) \leqslant \mu_2$, $\nu_1 \leqslant X_-(t) \leqslant \nu_2$,

for a.a. $t \in]0,\pi[$, and the function $v_0 \in W_0^{1,p}(0,\pi)$ such that $\|v_0\|_{1,p} = 1$, $v_0(0) = v_0(\pi) = 0$ and the equation

$$(3.13) \quad -(|v_0'(t)|^{p-2} v_0'(t))' = = X_+(t)|v_0(t)|^{p-2} v_0^+(t) - X_-(t)|v_0(t)|^{p-2} v_0^-(t)$$

holds for a.a. $t \in]0,\pi[$. But an investigation of the initial value problem for the equation (3.13) with initial conditions $v(0) = 0$, $v'(0) = \alpha$, $\alpha \in \mathbf{R}$, implies that if X_+, X_- satisfy (a), resp. (b), then either $v = 0$ in $[0,\pi]$ (in the case $\alpha = 0$) or $v(\pi) \neq 0$ (in the case $\alpha \neq 0$) and this is a contradiction which proves that (3.12) is true.

REMARK 3.7 - For the definition of the operator

$$\hat{F} : W_0^{1,p}(0,\pi) \to W^{-1,q}(0,\pi)$$

it is sufficient to suppose in (3.9) that $m(t) \in L_1(0,\pi)$. But in order to obtain the limit equation (3.13) with X_+, X_- satisfying (a), resp. (b), we need more restrictive assumption $m(t) \in L_d(0,\pi)$ with some $d > 1$ (see [2]).

REMARK 3.8 - Since the proof of the previous assertion is based on the homotopy invariance property of the Leray-Schauder degree and the properties of the limit equation (3.13) we obtain the same result also if we consider, instead of J in (3.11), some odd, $(K, L, p-1)$-homeomorphism A which is $(p-1)$-quasihomogeneous with respect to J (because in this case using the same argument we obtain again the limit equation (3.13)). Hence for the special choice of the operator A as in (3.7) we obtain that if the function ψ satisfies the assumptions of Theorem 3.6 then the boundary value problem

$$(3.14) \quad \begin{cases} -[(1 + |u'(t)|^{p-2}) u'(t)]' = \Phi(t, u(t)) + h(t), \; t \in]0,\pi[, \\ u(0) = u(\pi) = 0, \end{cases}$$

has at least one weak solution.

REMARK 3.9 - Let us remark that if we suppose that there exists some $\delta > 0$ such that

$$(3.15) \quad \begin{cases} \lambda_k + \delta \leqslant X_{+\infty}(t) \leqslant X^{+\infty}(t) \leqslant \lambda_{k+1} - \delta, \\ \lambda_k + \delta \leqslant X_{-\infty}(t) \leqslant X^{-\infty}(t) \leqslant \lambda_{k+1} - \delta, \end{cases}$$

holds for a.a. $t \in]0,\pi[$ (where λ_k, λ_{k+1} are two successive eigenvalues of (3.4)) then the boundary value problem (3.10), resp. (3.14), has at least one weak solution $u \in \overset{0}{W}{}^{1,p}(0,\pi)$ for arbitrary right-hand side $h \in L_1(0,\pi)$ (see [7] for the proof of this assertion).

This is easy to see because the assumptions (3.15) are the special case of the assumptions formulated in Theorem 3.6.

In the following part of this section we shall deal with $p = 2$ because in this case the more exhaustive answers to the Questions 1-6 from Section 2 may be given.

Let us remark, at first, that since this is the special case of $p \geqslant 2$, we have (according to Remark 3.4) the answers to Questions 3 and 4 (see also [3], [4], [5] and [12] and the classical Fredholm alternative gives the complete answer to the Question 1.

Some answers to the Questions 2 and 5 are contained in the prepared paper Ruf [23]. We shall briefly recall the main results contained in this paper. Let us remark, at first, that

$$X = Z = \overset{0}{W}{}^{1,2}(0,\pi), \; Y = W^{-1,2}(0,\pi)$$

are Hilbert spaces and the operators $J, S : X \to Y$,

$$\langle J(u), v\rangle = \int_0^\pi u'(t)\, v'(t)\, dt,$$

$$\langle S(u), v\rangle = \int_0^\pi u(t)\, v(t)\, dt,$$

are linear operators satisfying the assumptions (J1)-(J3), (S1)-(S3) from the beginning of Section 3. The equation (1.1) is now the operator representation of the boundary value problem

$$(3.16) \quad \begin{cases} -u''(t) - \mu u^+(t) + \nu u^-(t) = h(t), \; t \in]0,\pi[, \\ u(0) = u(\pi) = 0. \end{cases}$$

REMARK 3.10 - Using the standard regularity argument it is possible to show that if $h \in L_2(0,\pi)$ then the weak solution u of (3.16) satisfies $u'' \in L_2(0,\pi)$.

THEOREM 3.7 - *Let $v_{\mu,\nu} \in \overset{0}{W}{}^{1,2}(0,\pi) \cap W^{2,2}(0,\pi)$ be the normed nontrivial solution of the boundary value problem*

$$\begin{cases} -u''(t) - \mu u^+(t) + \nu v^-(t) = 0, \; t \in]0,\pi[, \\ u(0) = u(\pi) = 0, \end{cases}$$

Then for given $h_1 \in [v''_{\mu,\nu}]^\perp$ (an orthogonal complement in the space $L_2(0,\pi)$) there exists an $\alpha(h_1) \in \mathbf{R}$ such that (3.16) *has at least one weak solution for $h = h_1 + \alpha(h_1)\, v''_{\mu,\nu}$.*

REMARK 3.11 - Note that for $(\mu,\nu) \to (\lambda_k, \lambda_k)$, where $\lambda_k = k^2$, the assertion of Theorem 3.7 reduces to the classical Fredholm alternative. In particular, we obtain that $\{h_1 + \alpha(h_1)\, v''_{\mu,\nu}\}$ becomes $[-\sin kt]^{\perp}$.

Let us suppose that $(\mu,\nu) \in A_2$. Then the following statement about the weak solvability of (3.16) may be proved (see [23, Th. 1.4(b)]).

THEOREM 3.8 - *There exists $w_{\mu,\nu} \in L_2(0,\pi)$ such that for any given $h_1 \in [w_{\mu,\nu}]^{\perp}$ there exists a constant $T(h_1)$ such that* (3.16) *has at least two weak solutions for $h = h_1 + t w_{\mu,\nu}$ provided that $t > T(h_1)$. Furthermore, there exists $h \in L_2(0,\pi)$ such that* (3.16) *has no solution.*

REMARK 3.12 - Let us note that for $(\mu,\nu) \in A_2$ the equation (3.16) is of Ambrosetti-Prodi type (cf. Ambrosetti, Prodi [1]).

REMARK 3.13 - In [25], [26] we study the periodic boundary value problem for forced Duffing equation

$$\begin{cases} u'' + cu' + \Phi(t,u) = h \text{ in }]0,\pi[\,, \\ u(\pi) = u(0),\ u'(\pi) = u'(0), \end{cases}$$

$h \in L_2(0,\pi)$, $c \in \mathbf{R}$, where the Carathéodory's function ψ satisfies conditions of the type (3.9), (3.9'). Using the description of the set A_1 for (3.16) we have proved solvability of this problem for each right-hand side $h \in L_2(0,\pi)$.

The general theory from Section 2 may be applied also to the boundary value problems for ordinary differential equations of the fourth order.

Let a, b be the real functions as at the beginning of this section. Put $X = \overset{0}{W}{}^{2,p}(0,\pi)$, $Z = L_p(0,\pi)$, $Y = W^{-2,q}(0,\pi)$ (where again $p \geq 2, \dfrac{1}{p} + \dfrac{1}{q} = 1$). Let us denote

$$(3.17) \quad \begin{cases} \langle J(u), v\rangle = \int_0^\pi a(t)\,|u''(t)|^{p-2} u''(t)\, v''(t)\, dt \\ \langle S(u), v\rangle = \int_0^\pi b(t)\,|u(t)|^{p-2} u(t)\, v(t)\, dt \\ \langle f, v\rangle \quad = \int_0^\pi h(t)\, v(t)\, dt, \end{cases}$$

$h \in L_1(0,\pi)$ for all $v \in \overset{0}{W}{}^{2,p}(0,\pi)$.

Then it is possible to verify that the operators J and S satisfy the conditions (J1)-(J3), (S1)-(S3) from Section 2. The equation (1.1) is then the operator representation of the boundary value problem

$$(3.18)\quad \begin{cases} (a(t)|u''(t)|^{p-2}u''(t))'' - \mu b(t)|u(t)|^{p-2}u^+(t) + \\ \quad + \nu b(t)|u(t)|^{p-2}u^-(t) = h(t),\ t \in]0,\pi[\,, \\ u(0) = u'(0) = u(\pi) = u'(\pi) = 0\,. \end{cases}$$

REMARK 3.14 - Analogously as in Definition 3.1 it is possible to define the weak solution of (3.18). It is proved in Fučík, Nečas, Souček, Souček [9] and Kratochvíl, Nečas [13] that the same statement as in Theorem 3.1 is true also for the nonlinear Sturm-Liouville equation of the fourth order (3.18). It means that also in this case we have a partial answer to the Question 1.

Using the abstract theory from Section 2 we obtain also for this type of boundary value problem the assertions of Theorems 3.2 and 3.3.

REMARK 3.15 - Let us remark that in distinction from the case of the operator of the second order it was not possible, up to now, to give the precise description of the set A_{-1} even if we suppose the constant coefficients, i.e. $a(t) = b(t) = 1$, for all $t \in [0,\pi]$. That is why the analogous assertions to Theorems 3.5 and 3.6 have only local character.

Let us suppose that the real continuous function $\Phi : \mathbf{R} \to \mathbf{R}$ satisfies $(\Phi_{\pm\infty})$ and the operator $F : \overset{0}{W}{}^{2,p}(0,\pi) \to W^{-2,q}(0,\pi)$ is defined by

$$(3.19)\quad \langle F(u), v\rangle = \int_0^\pi \Phi(u(t))\, v(t)\, dt, \text{ for all } v \in \overset{0}{W}{}^{2,p}(0,\pi).$$

Define also the operator $A : \overset{0}{W}{}^{2,p}(0,\pi) \to W^{-2,q}(0,\pi)$ by

$$(3.20)\quad \langle A(u), v\rangle = \int_0^\pi (1 + a(t)|u''(t)|^{p-2})\, u''(t)\, v''(t)\, dt,$$

for all $v \in \overset{0}{W}{}^{2,p}(0,\pi)$. Then we may check that A is odd, $(K, L, p-1)$-homeomorphism $\overset{0}{W}{}^{2,p}(0,\pi)$ onto $W^{-2,q}(0,\pi)$ which is $(p-1)$-quasihomeogeneous with respect to J (see (3.17)) and that F is completely continuous operator from $\overset{0}{W}{}^{2,p}(0,\pi)$ into $W^{-2,q}(0,\pi)$ which is $(p-1)$-strongly quasihomogeneous with respect to the operator

$$u \mapsto \Phi_{+\infty} S(u^+) - \Phi_{-\infty} S(u^-)$$

(for S see (3.17)). Then using the assertion (v) of Section 2 and Theorem 2.2 we obtain the following local result.

THEOREM 3.9 - *Let us suppose that $\lambda \notin \sigma$, where σ is the set of real eigenvalues of*

$$(3.21)\quad \begin{cases} (a(t)|u''(t)|^{p-2}u''(t))'' - \lambda|u(t)|^{p-2}u(t) = 0,\ t \in]0,\pi[\,, \\ u(0) = u'(0) = u(\pi) = u'(\pi) = 0\,. \end{cases}$$

Let one of the following conditions be fulfilled

(i) $\Phi_{+\infty} < \lambda_1$, $\Phi_{-\infty} < \lambda_1$;

(ii) $\Phi_{+\infty} > \lambda_1$, $\Phi_{-\infty} > \lambda_1$ *and*

$$\max\{|\lambda - \Phi_{+\infty}| \, , \, |\lambda - \Phi_{-\infty}|\} < \frac{c_1(\lambda,\lambda)}{s}$$

(for the definition of $c_1(\lambda,\lambda)$ *and* s *see* (ii), *Section 2). Then the boundary value problem*

$$(3.22) \quad \begin{cases} [(1+a(t)|u''(t)|^{p-2})\, u''(t)]'' - \Phi(u(t)) = h(t), t \in]0,\pi[, \\ u(0) = u'(0) = u(\pi) = u'(\pi) = 0 \end{cases}$$

has at least one weak solution for arbitrary right-hand side $h \in L_1(0,\pi)$.

The proof of this assertion follows directly from the fact that (3.22) allows the operator representation

$$A(u) - F(u) = f,$$

from the application of the assertion (ii) (Section 2) and from Theorem 2.2.

Let us suppose, now, that we have the Carathéodory's function $\Phi(t,s)\colon [0,\pi] \times \mathbf{R} \to \mathbf{R}$ which satisfies (3.9), (3.9'). Then we obtain the following local analogy to Theorem 3.6.

THEOREM 3.10 - *Let us suppose that either*

(i) *there exists some* $\delta > 0$ *such that*

$$X_{\pm\infty}(t) \, , \, X^{\pm\infty}(t) \leqslant \lambda_1 - \delta \, ,$$

for a.a. $t \in]0,\pi[$, *or*

(ii) $$\lambda - \frac{c_1(\lambda,\lambda)}{s} + \delta \leqslant X_{\pm\infty}(t), \; X^{\pm\infty}(t) \leqslant \lambda + \frac{c_1(\lambda,\lambda)}{s} - \delta \, ,$$

for a.a. $t \in]0,\pi[$, *with some* λ *which is not an eigenvalue of* (3.21), $\delta > 0$. *Then the boundary value problem*

$$(3.23) \quad \begin{cases} [a(t)|u''(t)|^{p-2} u''(t)]'' = \Phi(t,u(t)) + h(t), \; t \in]0,\pi[\, , \\ u(0) = u'(0) = u(\pi) = u'(\pi) = 0 \, , \end{cases}$$

has at least one weak solution $u \in \overset{0}{W}{}^{2,p}(0,\pi)$ *for arbitrary right-hand side* $h \in L_1(0,\pi)$.

Proof. Let us define the operator $\hat{F}\colon \overset{0}{W}{}^{2,p}(0,\pi) \to W^{-2,q}(0,\pi)$ by the relation

$$\langle \hat{F}(u), v \rangle = \int_0^\pi \Phi(t,u(t))\, v(t)\, dt ,$$

for all $v \in \overset{0}{W}{}^{2,p}(0,\pi)$. Then the boundary value problem (3.23) has the operator representation

(3.24) $J(u) = \hat{F}(u) + f$.

Put

$$H(u,\tau) = J(u) - \tau\hat{F}(u) - \tau f - (1-\tau)\lambda S(u),$$

for $\tau \in [0,1]$, $u \in \overset{0}{W}{}^{2,p}(0,\pi)$, where $\lambda = \lambda_1 - \delta$ in the case (i). Then there exists $r > 0$ such that

(3.25) $H(u,\tau) \neq 0$,

for all $\tau \in [0,1]$, $u \in \overset{0}{W}{}^{2,p}(0,\pi)$, $\|u\|_{2,p} \geqq r$. To prove this we proceed via contradiction. Let us suppose that there are

$$u_n \in \overset{0}{W}{}^{2,p}(0,\pi),\ \|u_n\|_{2,p} = n,\ \tau_n \in [0,1]$$

such that

(3.26) $H(u_n, \tau_n) = 0$,

for all $n \in \mathbf{N}$. Dividing (3.26) by $\|u_n\|_{2,p}^{p-1}$ we obtain

(3.27) $$J(v_n) - \tau_n \frac{\hat{F}(u_n)}{\|u_n\|_{2,p}^{p-1}} - \tau_n \frac{f}{\|u_n\|_{2,p}^{p-1}} - (1-\tau_n)\lambda S(v_n) = 0,$$

for all $n \in \mathbf{N}$, where $v_n = u_n / \|u_n\|_{2,p}$. But the left-hand side of (3.27) may be written in the form

$$L_n = J(v_n) - \tau_n\left(\frac{\hat{F}(u_n)}{\|u_n\|_{2,p}^{p-1}} - \lambda S(v_n)\right) - \tau_n \frac{f}{\|u_n\|_{2,p}^{p-1}}.$$

For sufficiently large n we have

$$\tau_n \left\langle \frac{\hat{F}(u_n)}{\|u_n\|_{2,p}^{p-1}} - \lambda S(v_n),\ v_n \right\rangle \leqq 0,$$

if (i) is satisfied, or

$$\tau_n \left\| \frac{\hat{F}(u_n)}{\|u_n\|_{2,p}^{p-1}} - \lambda S(v_n) \right\|_{-2,q} \leqq c_1(\lambda,\lambda) - \delta s,$$

in the case (ii). Taking into account that

$$\|J(v_n) - \lambda S(v_n)\|_{-2,q} \geqq c_1(\lambda,\lambda),$$

in both cases there is some $n_0 \in \mathbf{N}$ such that for all $n \geqq n_0$ it is $L_n \neq 0$. Hence (3.25) is true and using the fact that J is $(K, L, p-1)$-homeomorphism we obtain from the homotopy invariance property of the Leray-Schauder degree and the Borsuk theorem that (3.24) has a solution for arbitrary f. This proves that the boundary value problem (3.23) has at least one weak solution $w \in \overset{0}{W}{}^{2,p}(0,\pi)$ for arbitrary right-hand side $h \in L_1(0,\pi)$. Q.E.D.

REMARK 3.16 - Let us note (as in the Remark 3.8) that since the proof of Theorem 3.10 is based on the homotopy invariance property of the Leray-Schauder degree we obtain the same existence result also for the houndary value problem

$$(3.28)\quad \begin{cases} -[(1+a(t)|u''(t)|^{p-2})\,u''(t)]''=\Phi(t,u(t))+h(t),\, t\in]0,\pi[, \\ u(0)=u'(0)=u(\pi)=u'(\pi)=0\,, \end{cases}$$

or for any other boundary value problem where instead of J there is some odd, $(K, L, p-1)$-homeomorphism which is $(p-1)$-quasi-homogeneous with respect to J.

Some more general global results for ordinary differential equations of the fourth order including the description of the set A_{-1} one obtains in the case $a(t)=b(t)=1$, $t\in[0,\pi]$, $p=2$ and periodic boundary conditions, resp. boundary conditions of the type

$$u(0)=u''(0)=u(\pi)=u''(\pi)=0\,.$$

Since the regularity argument may be applied we restrict ourselves to finding all pairs $(\mu,\nu)\in \mathbf{R}^2$ such that there exists a non-constant 2π-periodic solution $u\in C^4(\mathbf{R})$ solving the ordinary differential equation of the fourth order

$$(3.29)\quad u^{\mathrm{IV}}=\mu u^+ - \nu u^-\,.$$

We shall give here the survey of the results obtained in the paper [14]. For the sake of brevity it is useful to put $\mu=a^4$, $\nu=b^4$, $(a,b)\in]0,+\infty[\times]0,+\infty[$. Then the equation (3.29) may be written in the form

$$(3.30)\quad u^{\mathrm{IV}}=a^4u^+ - b^4u^-\,.$$

Let us denote by $\psi\in]\frac{3}{4}\pi,\pi[$ the smallest positive root of the equation

$$\tan(x)+\mathrm{th}(x)=0\,,$$

and for $z\in]0,\psi[$

$$g(z)=\frac{\mathrm{ch}(z)\sin(z)-\mathrm{sh}(z)\cos(z)}{\mathrm{ch}(z)\sin(z)+\mathrm{sh}(z)\cos(z)}\,.$$

Then it is possible to prove (see [14, Th. 2.7]) the following assertion.

THEOREM 3.11 - *The set $\tilde{A}_{-1}$ of all $(a,b)\in]0,+\infty[\times]0,+\infty[$, for which there exists a nontrivial periodic solution of* (3.30) *of period 2π, is the system $\{S_k, k\in\mathbf{N}\}$ of C^∞-curves, where S_1 is a curve $(a,b(a))$; $b(a)$ is a decreasing C^∞-function defined in $]\frac{\Phi}{\pi},+\infty[$ with*

$$\lim_{a+\infty} b(a) = \frac{\Phi}{\pi} .$$

The curve S_1 *is symmetrical with respect to the straight line* $b = a$ *and fulfils* $S_1 \subset G_1$, *where* G_1 *is the set of all pairs*

$$(a, b) \in]0, +\infty[\times]0, +\infty[$$

such that

$$b \geqq a, \ (\frac{b}{a})^2 - g(\pi a(1 - \frac{1}{2b})) \geqq 0 \geqq (\frac{a}{b})^2 - g(\pi b(1 - \frac{1}{2a})),$$

or

$$b \leqq a, \ (\frac{a}{b})^2 - g(\pi b(1 - \frac{1}{2a})) \geqq 0 \geqq (\frac{b}{a})^2 - g(\pi a(1 - \frac{1}{2b})).$$

For $k \geqq 2$ *it is* $S_k = \{(a, b) \in]0, +\infty[\times]0, +\infty[; \ (\frac{a}{k}, \frac{b}{k}) \in S_1\}$

and $S_k \subset G_k$, *where*

$$G_k = \{(a, b) \in]0, +\infty[\times]0, +\infty[; \ (\frac{a}{k}, \frac{b}{k}) \in G_1\}.$$

In particular, $\tilde{A}_{-1} \subset \bigcup_{k=1}^{\infty} G_k$. *For* $(a, b) \in S_k$ *the corresponding* 2π*-periodic solution has exactly* $2k$*-semi-waves in an interval of length* 2π. *This solution is unique if translations and positive multiples are not considered.*

REMARK 3.17 - See [14, p. 35] for the sketch of the picture of the system $\{G_k\}_{k=1}^{\infty}$.

Similarly as the periodic problem for the equation (3.30) it is possible to treat the boundary value problem for (3.30) with the boundary conditions

(3.31) $u(0) = u''(0) = u(\pi) = u''(\pi) = 0$.

THEOREM 3.12 - *The set* $\tilde{A}_{-1}$ *of all* $(a, b) \in]0, +\infty[\times]0, +\infty[$ *such that there exists a nontrivial solution* u *of the boundary value problem* (3.30), (3.31) *is a system of continuous curves*

$$\{S_i^+, S_i^- ; i \in \mathbf{N}\}$$

such that

(i) *for* $(a, b) \in S_i^+$, *resp.* S_i^-, *the solution* u *satisfies* $u'(0) > 0$, *resp.* $u'(0) < 0$. *This solution is uniquely determined by the choice of* $u'(0)$ *and it has in* $[0, \pi]$ *exactly* $i + 1$ *zeroes;*

(ii) *S_i^+ is symmetrical to S_i^- with respect to the straight line $a = b$. If i is even then $S_i^+ = S_i^-$;*

(iii) *for each $i \in \mathbf{N}$ we have $(S_i^+ \cup S_i^-) \cap (S_{i+1}^+ \cup S_{i+1}^-) = \phi$.*

For the proof of this assertion see [14, p. 39].

REMARK 3.18 - Using the description of the set A_{-1} (which is given by Theorems 3.11, 3.12) and the abstract Theorem 2.2 we may formulate the global existence results (analogous to that from Theorem 3.5) for the equation

$$u^{IV}(t) = \Phi(u(t)) + h(t),$$

with the periodic boundary conditions and the boundary conditions of the type (3.31), respectively.

4. - Some final remarks.

In this section we shall give some remarks about the multiplicity results and about the results for partial differential equations.

REMARK 4.1 -Since the purpose of this survey is to deal with the topological methods in the theory of boundary value problems for ordinary differential equations let us mention only briefly the results for partial differential equations which are related to the topic which is discussed above. In the papers Gallouet, Kavian [27, 28] there is studied the existence of the solution of the problem

(4.1) $u \in D(A),\ A(u) = \mu u^+ - \nu u^- + \gamma(\cdot\,, u) + h,$

under the following assumptions: $\Omega \subset \mathbf{R}^N$ is an open set, $h \in L_2(\Omega)$, A is a linear self-adjoint operator with compact resolvent, the domain of A is $D(A) \subset L_2(\Omega)$, and A maps $D(A)$ into $L_2(\Omega)$, $\gamma: \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory's function

$$\lim_{|s| \to \infty} \frac{\gamma(\cdot\,, s)}{s} = 0, \quad \sup_{s \in \mathbf{R}} \left| \frac{\gamma(\cdot\,, s)}{s} \right| \in L_\infty(\Omega).$$

Then there is proved in [27] that if $\mu \neq \nu$, say $\mu < \nu$, and interval $[\mu, \nu]$ does not contain an eigenvalue of A then (4.1) has at least one solution for every $h \in L_2(\Omega)$. Also, if $\mu = \nu = \lambda$ and λ is not an eigenvalue of A the problem (4.1) has at least one solution for every $h \in L_2(\Omega)$. It means that in this case we have a positive answer to the Question 4.

It is also proved in [27] that the set A_{-1} in the neighbourhood of the simple eigenvalue λ has character of a continuous curve, resp. two continuous curves, passing through the point (λ, λ).

In [28] the case is studied when $[\mu, \nu]$ contains one simple eigenvalue λ of operator A and $(\mu, \nu) \in A_{-1}$, (μ, ν) «near» (λ, λ). The authors have obtained sufficient conditions of Landesman-Lazer-type for the solvability of (4.1), i.e. in the neighbourhood of a simple eigenvalue they obtained the answer to the Questions 2 and 6.

REMARK 4.2 - Let us note that many other interesting results are proved in [27], [28]. For instance under the assumption that $s \mapsto \mu s^+ - \nu s^- + \gamma(\cdot, s)$ is monotone, the same results mentioned in Remark 4.1 may be proved also for the operator A with closed range and which has no compact resolvent.

REMARK 4.3 - Let us suppose that $\mu \neq \nu$, $\mu < \nu$. Then the number of eigenvalues of operator A lying in interval $[\mu, \nu]$ has a close relation not only to the existence results for (4.1) but it has also a connection with the multiplicity of the solutions. Let us mention here, at least, the papers Lazer, McKenna [17], [18] and Ruf [24] which deal with the ordinary differential operators and where the most precise results have been obtained.

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