

**BOUND SETS, PERIODIC SOLUTIONS
AND FLOW-INVARIANCE
FOR ORDINARY DIFFERENTIAL EQUATIONS
IN \mathbf{R}^n : SOME REMARKS (*)**

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SOMMARIO. - Si considera un concetto generale di insieme limitante per le soluzioni di sistemi differenziali del primo e secondo ordine. Si esamina in particolare il caso di insiemi che si ottengono come intersezione di sottolivelli di funzioni del tipo di Liapunov non necessariamente differenziabili. Si forniscono infine applicazioni al problema periodico ed al problema della persistenza.

SUMMARY. - A generalized concept of bound set for first and second order differential systems is considered. Conditions on non-smooth bounding functions are given in order to obtain bound sets as intersection of sublevel sets of Liapunov-like functions. Some applications to the periodic boundary value problem and to persistence are presented.

0. - Introduction and notation.

In the application of topological methods to various boundary value problems associated to the ordinary differential system

$$(0.1) \quad x' = f(t, x),$$

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some existence results can be obtained through the verification of suitable transversality conditions concerning the behaviour of the nonlinear field f at the boundary $\text{fr} M$ of a subset M of \mathbf{R}^n .

For instance, let us recall a classical application of Mawhin's continuation theorem [24] to the periodic BVP associated to (0.1):

THEOREM 0 ([21]) - *Let $f = f(t, x; \lambda): \mathbf{R} \times \mathbf{R}^n \times [0, 1] \rightarrow \mathbf{R}^n$ be continuous and p -periodic in t . Let $G \subset \mathbf{R}^n$ be an open bounded set and assume*

(b) *if x is a p -periodic solution of $x' = \lambda f(t, x; \lambda)$, $\lambda \in]0, 1[$, such that $x(t) \in \text{cl} G$, for each t , then $x(t) \in G$, for each t ;*

(d) *$0 \notin \bar{f}(\text{fr} G)$ and $\deg(\bar{f}, G, 0) \neq 0$, for $\bar{f}(z) := (1/p) \int_0^p f(s, z; 0) ds$.*

Then there is a p -periodic solution of $x' = f(t, x; 1)$, with values in $\text{cl} G$.

The assumption (b) in Theorem 0 is a kind of boundary condition: no p -periodic solution of $x' = \lambda f(t, x; \lambda)$, lying in the closure of G for all the time, can touch the boundary of G anywhere.

As another example, let us consider the autonomous system

$$(0.2) \quad x' = f(x),$$

with f continuous, and suppose that f defines a flow in \mathbf{R}^n . Let $N \subset \mathbf{R}^n$ be a compact set and let $S = I(N)$ be the maximal invariant set for (0.2) which is contained in N . The following definition is a basic concept in the Conley's theory of the (generalized) Morse index [8]:

(b') *N is an isolating neighbourhood for S if $S = I(N) \subset \text{int} N$.*

In this case, S is an *isolated invariant set* for (0.2). Hence, a homotopy index $h(S)$ can be defined, which generalizes the Morse index of a nondegenerate critical point of a gradient flow. The existence of isolating neighbourhoods implies the existence of nearby isolating blocks by which the Conley index can be explicitly computed. Bifurcation results for the equation $x' = f(x; \lambda)$, as well as existence theorems of certain distinguished solutions of (0.2), are also obtained by applying a continuation principle involving isolating neighbourhoods (see [8] or [33], for the general theory and for applications).

Finally, let us recall that a key hypothesis (concerning the behaviour of a flow in \mathbf{R}^n at the boundary of an open set Ω) for the application of the Ważewski retract theorem [34] is that «every egress point of Ω is a strict egress point» [16, Ch. X, Th. 2.1]: again a type of «transversality/boundary condition», like (b) and (b').

In this article we propose (suggested by [10], [24]) a definition of «bound set» suitable to be applied in the cases previously mentioned and we exploit it using transversality conditions which are expressed by means of some tangent cones at the boundary of the bound sets which are considered. The assumptions employed here appear, in some sense, complementary to classical tangency conditions coming from the theory of flow invariant sets. In the final part of the paper we briefly discuss various concepts of invariance for (0.1) and (0.2) which are related to some of the tangent cones previously defined.

The following notations are used: \mathbf{R}^n is the n -dimensional real euclidean space with inner product $(\cdot | \cdot)$ and norm $|\cdot| = (\cdot | \cdot)^{1/2}$; \mathbf{R}_+ is the set of the non-negative reals. $B(x, r)$ and $B[x, r] = \text{cl} B(x, r)$ denote, respectively, the open and the closed ball centered at x , with radius r ($r > 0$); $\text{fr} C$, $\text{cl} C$ and $\text{int} C$ are the boundary, the closure and the interior of a set $C \subset \mathbf{R}^n$. For fixed u , $\eta \in \mathbf{R}^n$, with $|\eta| = 1$ and $\alpha \in]0, \pi[$, we define the open convex cone $K_u(\eta, \alpha)$ of vertex u , direction η and angle α , as

$$K_u(\eta, \alpha) := \{x \in \mathbf{R}^n : (x - u | \eta) > |x - u| \cos \alpha\}.$$

1. - Bound sets: the smooth case.

Let $J \subset \mathbf{R}$ be an interval (not necessarily bounded) whose interior is denoted by I . Let us consider the differential equation

$$(1.1) \quad x' = f(t, x),$$

where $f: J \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a continuous map. (Without loss of generality for the following discussion we confine ourselves to this case instead of the more common one $f: J \times A \rightarrow \mathbf{R}^n$, $A \subset \mathbf{R}^n$, open).

Let $M \subset \mathbf{R}^n$, $M \neq \emptyset$. We say that M is a *bound set* for (1.1) if

(BS) *there is no solution $x(t)$ of (1.1) with $x(\bar{t}) \in \text{fr} M$, for some $\bar{t} \in I$, such that $x(t) \in \text{cl} M$, for all $t \in]\bar{t} - \varepsilon, \bar{t} + \varepsilon[$, $\varepsilon > 0$, holds.*

EXAMPLES.

- 1) Let $M = G$. If M is a bound set for $x' = \lambda f(t, x; \lambda)$, for every $\lambda \in]0, 1[$, then (b) holds.
- 2) Let $M = N$. If M is a bound set for (0.2), then N is an isolating neighbourhood for $I(N)$ (according to (b')).

REMARK 1 - For many applications, the definition of bound set proposed above could be unnecessarily restrictive. Having some knowledge about the solution set of (1.1), or looking for solutions

of a particular boundary value problem associated to the equation, we can require (BS) to be verified only for a restricted class of functions $x(\cdot) \in \Gamma$ which satisfy equation (1.1) (Γ being a suitable set of functions taking into account of the boundary conditions (like periodicity) and other possible informations — e.g. a priori bounds — we possess about the solutions of (1.1)).

An useful tool in order to locate bound sets consists into define them by means of an intersection of sublevel sets of scalar functions (actually, this is the original definition of bound set as presented in [10], [11], [24]). Precisely, let $C \subset \mathbf{R}^n$ be nonempty and assume that

for each $u \in \text{fr } C$, there is a function $V_u: \mathbf{R}^n \rightarrow \mathbf{R}$ such that

$$(i) \quad \text{cl } C \subset [V_u \leq 0] := \{x \in \mathbf{R}^n : V_u(x) \leq 0\},$$

$$(ii) \quad V_u(u) = 0.$$

The functions V_u will be called *bounding functions* for C (from [22]). Then, restating from [24, Ch. VII] and [23], we have

PROPOSITION 1 - Let $\{V_u : u \in \text{fr } C\}$ be a family of C^1 -bounding functions for C such that

$$(iii) \quad (\text{grad } V_u(u) | f(t, u)) \neq 0$$

holds, for every $u \in \text{fr } C$ and $t \in I$. Then, for each $\lambda \neq 0$, C is a bound set for the equation

$$(1.2) \quad x' = \lambda f(t, x).$$

(For a proof, see [24, Ch. VII], [23, Th. 7.3]).

The choice of the functions V_u is very natural if the set C is convex [24, Ch. VII] or if it possesses at each point of its boundary, an outer normal according to Bony [27]. We recall from [5] that a vector $\nu_u \neq 0$ is a *Bony's outer normal* to C at u if

$$B[u + \nu_u, |\nu_u|] \cap \text{cl } C = \{u\}.$$

We denote by $N_B(C; u)$ the set (possibly empty) of Bony's outer normals to C at u , and by

$$T_B(C; u) := \{w \in \mathbf{R}^n : (w | \nu) \leq 0, \text{ for every } \nu \in N_B(C; u)\},$$

the corresponding *Bony tangent cone* [29] ($T_B = \mathbf{R}^n$ if $N_B = \emptyset$). With the above positions, Proposition 1 yields (for an arbitrary set $\phi \neq C \subset \mathbf{R}^n$),

COROLLARY 1 ([27]) - Let us suppose that for each $u \in \text{fr } C$, there is $\nu_u \in N_B(C; u)$ such that

$$(iv) \quad (f(t, u) | \nu_u) \neq 0$$

holds, for every $t \in I$. Then C is a bound set for (1.2), for any $\lambda \neq 0$. The assumption (iv) can be written, in equivalent but geometrical fashion, as

$$(iv') \quad f(t, u) \notin T_B(C; u) \cap -T_B(C; u),$$

(for each $(t, u) \in I \times \text{fr} C$) so that it actually appears like a *transversality condition* for f at $\text{fr} C$: in fact, neither f , nor $-f$ can be tangent to C (in the Bony's sense) at any point of $\text{fr} C$.

As an application of Corollary 1, we consider the following example. Let $D := \{x \in \mathbf{R}^n : x_i \geq 0 \ (\forall i = 1, \dots, n), r \leq |x| \leq R\}$. For $x = (x_1, \dots, x_n)$, we set $\hat{x}_i := x - x_i e_i$ (e_i being the i -th element of the canonical orthonormal basis in \mathbf{R}^n).

Then D is a bound set for (1.2), for each $\lambda \neq 0$, provided that

$$(f(t, x)|x) \neq 0 \text{ for } |x| = r \text{ or } |x| = R, \text{ and } f_i(t, \hat{x}_i) \neq 0,$$

hold for every $x \in D$ and $t \in I$.

This result, in connection with Theorem 0 and a standard perturbation argument (in order to relax some inequalities), can be straightforwardly employed for improving an existence theorem by Gaines and Santanilla [12, Th. 3.1], concerning the existence of positive periodic solutions for (1.1) in a conical shell. For more recent results in this direction, see [43], [37], [39], [40].

2. - Bound sets: the nonsmooth case.

It is not difficult to find examples of a set M and of a nonlinear field $f(t, x)$ such that M is a bound set for $x' = f(t, x)$ but M cannot be defined in terms of smooth bounding functions V_u with $V'_u(u) \neq 0$ ($V' = \text{grad } V$) — an assumption which is implicitly required in order to apply Proposition 1 —. In these cases, even if the set M can be slightly deformed to another bound set with a piecewise smooth boundary defined by means of a finite number of C^∞ -bounding functions with $V'_u(u) \neq 0$ (and, indeed, this always happens for M compact and $f = f(x)$ inducing a flow [35, Th. 2.4]), nevertheless, the transversality condition can be easily checked in a direct way as well, by a suitable choice of nondifferentiable bounding functions.

In this section we propose some examples in which nonsmooth bounding functions are involved and we show that an extension of Proposition 1 can be still derived. Moreover, the employed tools (which are of common use in stability and optimization theory) allow us to shed more light on the geometrical meaning of the transversality conditions which are required in the «production» of bound sets.

At a first instance, we restrict ourselves to the locally lipschitzian case. Let $\{V_u : u \in \text{fr } C\}$ be a family of bounding functions for the (nonempty) set $C \subset \mathbf{R}^n$, according to the preceding definition (i.e. the V_u satisfy (i), (ii), and suppose that for each $u \in \text{fr } C$, V_u is (locally) lipschitzian at u , that is

$$(l) \quad |V_u(x) - V_u(y)| \leq L_u |x - y|,$$

holds, for each x, y in a neighbourhood of u .

Let us define, (for each $u \in \text{fr } C$ and $t \in I$),

$$a^+(u, t) := \limsup_{h \rightarrow 0^+} V_u(u + hf(t, u))/h = D^+ V_u(u)(f(t, u))$$

$$a_-(u, t) := \liminf_{h \rightarrow 0^-} V_u(u + hf(t, u))/h = D_- V_u(u)(f(t, u)),$$

(where D^+, D_- are directional Dini derivates).

Under (i), (ii) and (l) we can prove

PROPOSITION 2 - Let us suppose that for each $(t, u) \in I \times \text{fr } C$,

$$(w) \quad 0 \notin [a^+(u, t), a_-(u, t)]$$

holds. Then C is a bound set for $x' = \lambda f(t, x)$, for every $\lambda \neq 0$.

Proof. We consider only the case $\lambda = 1$. The general situation easily follows from (w) and obvious homogeneity properties of the Dini derivates. Let us suppose that C is not a bound set for the equation (1.1): $x' = f(t, x)$. Then there is a solution $x(t)$ to (1.1) such that, for some $\bar{t} \in I$, $x(\bar{t}) = \bar{u} \in \text{fr } C$, and $x(\bar{t} + h) \in \text{cl } C$, for $|h| < \varepsilon$. Hence, by (i), (ii), $v(h) := V_{\bar{u}}(x(\bar{t} + h)) \leq 0$ for $|h| < \varepsilon$ and $v(0) = 0$. Let us set

$$x(\bar{t} + h) = x(\bar{t}) + hx'(\bar{t}) + h\Delta(h) = \bar{u} + hf(\bar{t}, \bar{u}) + h\Delta(h),$$

with $|\Delta(h)| \rightarrow 0$, as $|h| \rightarrow 0$. Then, for $h \neq 0$ in a neighbourhood of 0, we have

$$\begin{aligned} 0 &\geq v(h)/|h| = V_{\bar{u}}(\bar{u} + hf(\bar{t}, \bar{u}) + h\Delta(h))/|h| \\ &\geq V_{\bar{u}}(\bar{u} + hf(\bar{t}, \bar{u}))/|h| - L_{\bar{u}} |\Delta(h)| \end{aligned}$$

(recalling (l) too).

Passing in the above inequality, respectively, to the «lim sup» as $h \rightarrow 0^+$, and to the «lim inf» as $h \rightarrow 0^-$, we get $a^+(\bar{u}, \bar{t}) \leq 0 \leq a_-(\bar{u}, \bar{t})$ and contradict (w). Thus the conclusion follows.

REMARK 2 - We note that Proposition 1 and 2 are still true if we suppose that each bounding function is defined only in a nbd $B(u, r_u)$ of u and we change (i) into

$$(i') \quad \text{cl } C \cap B(u, r_u) \subset [V_u \leq 0] \cap B(u, r_u).$$

As a simple example of application of the above result to a case of nondifferentiable (but possessing directional derivatives) bounding functions, let us consider the following situation.

Let $C \subset \mathbf{R}^n$ be a nonempty set verifying the *cone condition*:

- (c) for each $u \in \text{fr } C$, there are an open convex cone $K_u = K_u(\eta_u, \alpha_u)$, of vertex u , direction η_u and angle α_u , and an open ball $B(u, r_u)$, such that $K_u \cap B(u, r_u) \cap C = \emptyset$;

that is, locally, K_u lies in $\mathbf{R}^n \setminus C$. Then, the assumption (w) reads

$$(cc) \quad |(f(t, u) | \eta_u)| > |f(t, u)| \cos \alpha_u, \text{ for every } (t, u) \in I \times \text{fr } C.$$

Indeed, let us define $V_u(x) := (x - u | \eta_u) - |x - u| \cos \alpha_u$. It is immediate to check that (c) implies $\{V_u : u \in \text{fr } C\}$ is a family of lipschitzian bounding functions for the set C (use the Remark 2). Then a computation of the directional derivatives for the V_u gives, from (cc), that (w) holds for every $(t, u) \in I \times \text{fr } C$.

Remark that for those points $u \in \text{fr } C$ for which $\pi/2 < \alpha_u < \pi$, (cc) simply requires $f(t, u) \neq 0$ for each $t \in I$. Geometrically, (cc) means that $f(t, u) \in (K_u - u) \cup -(K_u - u)$. Thus we have proved

COROLLARY 2 - Let us assume that (c) and (cc) are fulfilled. Then C is a bound set for the equation (1.2), for any $\lambda \neq 0$.

As a second example, we consider, for a given (nonempty) set C ,

$$V_u(x) = V(x) := d(x; C),$$

where $d(x; C)$ is the distance of the point x from the set C .

As well known ([7]), $d(\cdot; C) : \mathbf{R}^n \rightarrow \mathbf{R}$ is a 1-lipschitzian function. Moreover, for the family $\{V_u = V, \forall u \in \text{fr } C\}$, all the basic hypotheses (i), (ii) are satisfied. The assumption (w) becomes now

$$(ww) \quad \lim_{h \rightarrow 0} d(u + hf(t, u); C) / h \neq 0,$$

(remark: we allow also the case in which the limit does not exist:

(ww) has to be interpreted as the negation of « $\lim_{h \rightarrow 0} \frac{d}{h} = 0$ »), which, actually, is a condition of «nontangentiality» (i.e. transversality) for f at the boundary of C , as we discuss below. In order to precise the geometrical meaning of (ww), let us consider the following definitions:

$$T_{DM}(C; x) := \{v \in \mathbf{R}^n : \lim_{h \rightarrow 0^+} d(x + hv; C)/h = 0\}.$$

$T_{DM}(C; x)$ is the *Dubovickii-Miljutin tangent cone* to C at x [29], used in the theory of extrema in nonsmooth analysis. A vector $v \in T_{DM}(C; x)$ is said to be a *tangent direction* to C at x [13].

Then the hypothesis (ww) can be rewritten as

$$(ww') \quad f(t, u) \notin T_{DM}(C; u) \cap -T_{DM}(C; u).$$

For the sake of completeness, we recall the following chain of inclusion (taken from [28], [29]) which explain the role of T_{DM} among other tangent cones employed in the literature

$$(2.1) \quad T_B(C; x) \supset T(C; x) \supset T_{DM}(C; x) \supset I(C; x),$$

where T_B is the Bony tangent cone seen in Section 1,

$$T(C; x) := \{v \in \mathbf{R}^n : \liminf_{h \rightarrow 0^+} d(x + hv; C)/h = 0\}$$

is the *Bouligand contingent cone* [1], [2] (which plays an important role in invariance results for ODEs and more general differential inclusions), and

$$I(C; x) := \{v \in \mathbf{R}^n : (\exists r, \varepsilon > 0) x + hw \in C, \forall 0 < h < \varepsilon, w \in B(v, r)\} = \\ = \mathbf{R}^n \setminus T(\mathbf{R}^n \setminus C; x),$$

is the *inner tangent cone* to C at x (the cone of *feasible directions*, according to [13]). Remark that T_{DM} (as well as the other objects above defined) is actually a cone, that is $\lambda v \in T_{DM}$, provided $\lambda > 0$ and $v \in T_{DM}$. Although $T(\dots)$ is closed and $I(\dots)$ is open, T_{DM} is in general neither closed nor open. We refer, for instance, to [1], [2], [7], [17], [29], [31], for an investigation of the main properties of these and others cones introduced in nonsmooth analysis, with applications to various problems.

According to the above remarks, Proposition 2 implies

PROPOSITION 3 - *Let us suppose (ww') (or, equivalently (ww)) holds, for every $(t, u) \in I \times \text{fr} C$. Then C is a bound set for the equation (1.2), for any $\lambda \neq 0$.*

It seems to be worthy to notice that Proposition 3 is equivalent to Proposition 2 (and not only a consequence of it). Namely, if $\text{cl} C \subset [V_u \leq 0]$, $V_u(u) = 0$ and V_u is locally lipschitzian at $u \in \text{fr} C$, then

$$T_{DM}(C; u) \subset T_{DM}([V_u \leq 0]; u) \\ \subset \{v \in \mathbf{R}^n : \limsup_{h \rightarrow 0^+} V_u(u + hv)/h \leq 0\}$$

and

$$\begin{aligned}
-T_{DM}(C; u) &\subset -T_{DM}([V_u \leq 0]; u) \\
&\subset \{v \in \mathbf{R}^n : \liminf_{h \rightarrow 0^-} V_u(u + hv)/h \geq 0\},
\end{aligned}$$

so that (ww') is implied by (w) . Hence we see that (w) and (ww') are two different aspects (viz, the analytic and the geometrical one) of the same transversality/boundary condition.

3. - Remarks and related results.

The case of the V_u not locally lipschitzian can be easily studied along the same lines as in the previous section.

For instance, it can be proved that, setting

$$b_+(u, t) := \liminf_{\substack{h \rightarrow 0^+ \\ v \rightarrow f(t, u)}} V_u(u + hv)/h, \quad b^-(u, t) := \limsup_{\substack{h \rightarrow 0^- \\ v \rightarrow f(t, u)}} V_u(u + hv)/h,$$

then, the assumption

$$(3.1) \quad 0 \notin [b_+(u, t), b^-(u, t)], \text{ for each } (t, u) \in I \times \text{fr } C,$$

implies, for a set C (with not necessarily locally lipschitzian bounding functions V_u) to be a bound set for $x' = \lambda f(t, x)$, for any $\lambda \neq 0$. (b_+ and $-b^-$ are the *upper contingent derivates* [1] of V_u at u , in the direction of $f(t, u)$ and $-f(t, u)$, respectively). The condition (3.1) can be obtained either in a direct way, from the proof of Proposition 2, or (following [1, Prop. 2, p. 180]) as a consequence of the geometric hypothesis $f(t, u) \notin T(C; u) \cap -T(C; u)$, which, by (2.1), implies (ww') .

We also observe that the foregoing statements on bound sets can be extended (following, for instance, [24]) to more general type of differential equations, like functional differential equations as $x' = f(t, x, x_t)$, with $f: J \times \mathbf{R}^n \times C([-r, 0], \mathbf{R}^n) \rightarrow \mathbf{R}^n$ continuous, or as $x'(t) = f(t, x(t), (Sx)(t))$, with $f: J \times \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ continuous and S a suitable operator acting between spaces of continuous functions (the definition of bound set for such FDEs is an obvious modification of the one given in Section 1 for ODEs). In the latter case, in particular, there are many concrete situations in which we know a priori that

$$(3.2) \quad \sup \{ |(Sx)(t) - x(t)| : t \in I \} \leq r$$

holds for any pertinent (in the sense of Remark 1) solution of $x' = f(t, x, Sx)$, or of $x' = \lambda f(t, x, Sx)$. Then, under (3.2), it is not difficult to prove that all the sufficient criteria (for a set C to be a bound set) given in the preceding sections, can be restated here, provided that the assumptions are made on $f(t, u, y)$, with $t \in I$,

$u \in \text{fr } C$ and $y \in B[u, r]$. More detailed applications of this remark to forced periodic BVPs, using bound sets, will be discussed in a future paper.

Finally we note that, similarly as in the smooth case [10], the theory can be extended to *nonautonomous bound sets* [10], [11]. Namely, the case $M = M(t)$ (i.e. the set M depends on t , or, equivalently $M \subset \mathbf{R} \times \mathbf{R}^n = \{(t, x)\}$) can be considered as well in the definition and in the relative examples of Sections 1, 2. In particular, it is possible to apply (following [10, Ch. V]) an extension of the previously given results in the nonsmooth case to scalar differential equations of the first order, by considering nonautonomous bound sets in $\mathbf{R} \times \mathbf{R}$ defined by $a \leq t \leq b$, $\alpha(t) \leq x \leq \beta(t)$; $\alpha, \beta : [a, b] \rightarrow \mathbf{R}$ not necessarily differentiable. In this manner, for instance, generalizations of some classical theorems of existence of periodic solutions, via (not C^1) strict lower and upper solutions, can be given. In the same framework, analogous results can be obtained for differential systems too, by means of nonautonomous bound sets of the form $a \leq t \leq b$, $\alpha_i(t) \leq x_i \leq \beta_i(t)$, $i = 1, \dots, n$. For a recent survey on upper and lower solutions, see [20].

For the sake of simplicity, we have confined our discussion about bound sets to the consideration of differential equations with a continuous function at the right hand side. The case of a vector field f satisfying the Carathéodory conditions [15] can be treated as well, by modifying suitably the argument proposed in [32] for «smooth» bound sets of RFDEs.

4. - Second order systems.

Let us consider the second order vector differential equation

$$(4.1) \quad x'' = f(t, x, x'),$$

with $f : J \times \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ continuous and $J \subset \mathbf{R}$ an interval. Let $I = \text{int } J$. Here we give just a brief account about bound sets in \mathbf{R}^n for the equation (4.1), in the nonsmooth case. Although it is possible to replace the second order equation (4.1) into an equivalent first order system in \mathbf{R}^{2n} like

$$(4.2) \quad x' = y, \quad y' = f(t, x, y)$$

and then apply the framework considered in the preceding sections, we prefer to deal with equation (4.1) directly, and obtain more explicit conditions on the vector field f ensuring the existence of a bound set. The choice of this approach is justified also by the fact that, in many applications of the theory to BVPs for second order differential systems [3], [18], [22], a priori bounds for x' are ob-

tained by means of suitable Bernstein-Nagumo growth restrictions on f provided that $x(t)$ belongs to a bounded set. In these cases, the existence of a bounded bound set to (4.2), containing $(x(t), x'(t))$, is soon reached if a bounded bound set to (4.1), containing $x(t)$, exists according to the definition given below. (See, for instance, [25], for a general treatment of this subject).

Let $\phi \neq M \subset \mathbf{R}^n$. Following [10], [11] and Section 1, we say that M is a bound set for (4.1) if (BS) holds with respect to the solutions $x(t)$ of such equation.

We consider now some sufficient criteria for a set C , determined by bounding functions, to be a bound set (for a second order differential equation). Referring to [10], [22], for the smooth case, we confine ourselves, for the sake of simplicity to the lipschitzian case (the general situation should be developed as in Section 3).

Let $\{V_u : u \in \text{fr } C\}$ be a family of bounding functions for a non-empty subset C of \mathbf{R}^n , with V_u locally lipschitzian at u , for each $u \in \text{fr } C$ (i.e. (i), (ii) and (l) are satisfied).

We define

$$A(u) := \{v \in \mathbf{R}^n : \limsup_{h \rightarrow 0^+} V_u(u + hv)/h \leq 0 \leq \liminf_{h \rightarrow 0^-} V_u(u + hv)/h\}$$

and let

$$B(v, u, t) := \limsup_{h \rightarrow 0} V_u(u + hv + (h^2/2) f(t, u, v))/h^2.$$

Remark that $v \in A(u)$ implies $\lambda v \in A(u)$, for every $\lambda \in \mathbf{R}$.

Let $x(\cdot)$ be a solution of (4.1) such that $x(\bar{t}) \in \text{fr } C$ for some $\bar{t} \in I$ and $x(t) \in \text{cl } C$ for every t in a neighbourhood of \bar{t} . From the same argument as in the proof of Proposition 2, we get $x'(\bar{t}) \in A(\bar{u})$, with $\bar{u} = x(\bar{t})$. On the other hand,

$V_u(x(\bar{t} + h)) = V_u(\bar{u} + hx'(\bar{t}) + (h^2/2) f(\bar{t}, \bar{u}, x'(\bar{t})) + h^2 \Delta(h)) \leq 0$,
for h small, with $\Delta(h)/h^2 \rightarrow 0$ as $h \rightarrow 0$. Then, dividing the above inequality by h^2 and passing to the «lim sup» as h goes to zero, we obtain, in virtue of (l) that $B(x'(\bar{t}), \bar{u}, \bar{t}) \leq 0$. Therefore we have proved that the validity of the assumption

(k) $0 < B(v, u, t)$, for every $(t, u) \in I \times \text{fr } C$ and $v \in A(u)$,

is a sufficient condition for C to be a bound set for the equation (4.1). Let us observe that in the case of a collection of C^2 -bounding functions for the set C , a straightforward computation shows that $A(u) = \{v \in \mathbf{R}^n : |\text{grad } V_u(u)|v| = 0\}$ and

$$2B(v, u, t) = (\text{Hess } V_u(u) \cdot v | v) + (\text{grad } V_u(u) | f(t, u, v)).$$

In this way, a classical hypothesis in existence results for the periodic BVP to second order differential systems [3], [10], [18], [22] is fitted by (k).

In terms of transversality conditions (using tangent cones), this assumption can be enunciated as follows.

For a set $C \subset \mathbf{R}^n$, let

$$P(C; x)(v) := \{w \in \mathbf{R}^n : \lim_{h \rightarrow 0} d(x + hv + (h^2/2)w; C)/h^2 = 0\}$$

and set

$$D(C; x) := T_{DM}(C; x) \cap -T_{DM}(C; x).$$

Then, restating (k) to the case $V_u(x) = d(x; C)$, we get

(kk) $f(t, u, v) \notin P(C; u)(v)$, for every $(t, u) \in I \times \text{fr } C$ and $v \in D(C; u)$.

Thus we have

PROPOSITION 4 - *Let C be a nonempty subset of \mathbf{R}^n and suppose that (kk) holds. Then C is a bound set for (4.1).*

As we have just observed, Proposition 4 contains many results for second order ODEs in which the concept of bound set appears more or less explicitly. We also remark that the case of nonsmooth nonautonomous (i.e. $M = M(t)$) bound sets can be considered too, e.g. working in a parallel way as in [11], [14].

If the vector field f is p -periodic in the first variable, Proposition 4, in connection with Mawhin's continuation theorem (namely using an analogue of Theorem 0 for second order systems) can be employed in order to get the existence of p -periodic solutions to equation (4.1), by assuming a suitable Bernstein-Nagumo-type growth restriction on the behaviour of f with respect to x' or, more generally, supposing that (4.1) is a *Nagumo equation* [22], with respect to bounded sets (i.e., for each R , there is $k = k(R)$ such that $|x'|_\infty \leq k$, for any p -periodic solution x of $x'' = \lambda f(t, x, x')$, with $\lambda \in]0, 1[$ and $|x|_\infty \leq R$). We refer to [4], [10], [18], [22], [3], and the references therein quoted, for a more complete exposition of the classical results. See [11], for general BVPs using bound sets.

Finally, let us note that the remarks in Section 3, extend (with obvious changes) to equation (4.1) as well.

5. - Flow invariance and related topics.

We briefly recall in this section some results on flow invariant sets which are related to the use of the tangent cones considered

in the foregoing exposition. For the sake of simplicity, we restrict ourselves to the case of the autonomous system

$$(5.1) \quad x' = f(x),$$

with $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$, continuous. (The statements listed below, straightforwardly extend to nonautonomous differential equations: only (δ) needs a more complicated formulation). For the various definitions of invariance, we follow [36].

Let M be a (nonempty) *closed* subset of \mathbf{R}^n . Then we have:

$\alpha)$ Let $f(u) \in T_B(M; u)$, for each $u \in \text{fr } M$, with f locally lipschitzian.

Then M is a positively invariant set for (5.1) [5], [30].

Actually (α) can be obtained as a consequence of the more general claim (β) below, by taking into account also that [29, Th. 3.9], under our assumptions on M and f , $f(x) \in T(M; x)$, for each $x \in \text{fr } M$, if and only if, $f(x) \in T_i(M; x)$, for each $x \in \text{fr } M$, with $i = \langle B \rangle, \langle DM \rangle, \langle C \rangle$ ($T_C(M; x)$ being the Clarke tangent cone to M at x [7]).

$\beta)$ Let $f(u) \in T(M; u)$, for each $u \in \text{fr } M$. Then M is a positively weakly invariant set for (5.1) [26], [36], [2].

Further, we have

$\gamma)$ Let $f(u) \in I(M; u)$, for each $u \in \text{fr } M$. Then M is a positively invariant set for (5.1).

(The proof is almost immediate, recalling the definition of $I(M; u)$ [28]).

Finally, we state

$\delta)$ Let us suppose that, for each $u \in \text{fr } M$, there is a Bony's outer normal $v_u \in N_B(M; u)$ such that $(f(u) | v_u) < 0$ (or, equivalently, $f(u) \notin -T_B(M; u)$, for each $u \in \text{fr } M$). Moreover, let us assume M compact and with $\text{int } M \neq \emptyset$. Then M is a strongly flow invariant set according to Freedman and Waltman (that is, [9] for any $x_0 \in \text{int } M$, and any solution $x(t)$ of (5.1), with $x(0) = x_0$, $x(t) \in \text{int } M \forall t \geq 0$ and $\liminf_{t \rightarrow +\infty} d(x(t); \text{fr } M) > 0$).⁽¹⁾

(For a proof of (δ) , with more general results, see [37], [38]).

As well known, the use of flow invariant sets in connection with the properties of the operator of translation along the trajectories of the differential system (Poincaré map) is a basic tool in the study of the existence of periodic solutions to nonautonomous ODEs (see,

(1) This fact is also named as *persistence* for the system (5.1) with respect to the set M .

for instance, [19], [2, Ch. V]). In [19] this program is carried on to the computation of the (Brouwer) topological degree of some nonlinear vector fields. We end this paper with some elementary considerations along the lines of Krasnosel'skii's book. This gives us the opportunity to join together the theory of positively invariant sets and an argument based on another kind of transversality/boundary condition (i.e. the concept of ω -irreversibility point).

We say that $u \in \mathbf{R}^n$ is a *point of ω -irreversibility* ($\omega > 0$), for equation (5.1), [19] if there is no solution $x(t)$ of (5.1) such that $x(0) = u = x(\bar{t})$ for some $0 < \bar{t} \leq \omega$. Let us observe that if, for a compact set C , $0 \notin f(C)$, then there is an open neighbourhood W of C such that each point of W is of ω -irreversibility for a sufficiently small $\omega > 0$.

Let $G \subset \mathbf{R}^n$ be an *open bounded* set and suppose that $\text{cl } G$ is an ANR. In order to simplify the subsequent arguments, we confine ourselves to the hypothesis

(h) *the solutions of Cauchy problems for equation (5.1), with initial data in $\text{cl } G$, are uniquely determined in the future.*

Let $[0, k)$ be a common interval of existence for all the solutions x_u 's of the initial value problems $x'_u = f(x_u)$, $x_u(0) = u \in \text{cl } G$. Then, for any $0 \leq T < k$, the operator of T -translation

$$U(T) : \text{cl } G \rightarrow \mathbf{R}^n, \quad U(T)(u) := x_u(T)$$

is defined and it is continuous.

After these preliminaries, we can state

LEMMA 1 - *Let us assume (h) and suppose there is $0 < \omega < k$ such that each point of $\text{fr } G$ is of ω -irreversibility with respect to the equation (5.1). Finally, let $K \subset \text{cl } G$ be a contractible set such that $U(\omega)(\text{cl } G) \subset K$. Then $\deg(f, G, 0) = (-1)^n$.*

Sketch of the proof: First we note that $0 \notin f(\text{fr } G)$ (by the ω -irreversibility) and that $\text{cl } G$ is connected (actually, it is an AR). Then, it is sufficient to verify that the following equalities are well settled (recalling, in particular, [19, p. 80]) and apply [6, p. 43]:

$$\begin{aligned} (-1)^n \deg(f, G, 0) &= \deg(-f, G, 0) = \deg(\text{Id} - U(\omega), G, 0) = \\ &= i(U(\omega), G) = i(U(\omega), \text{cl } G) = L(U(\omega)) = 1, \end{aligned}$$

where $i(\cdot, \cdot)$ is the fixed point index (with respect to $\text{cl } G$) and $L(\cdot)$ is the Lefschetz number [6]. As an immediate corollary, we have an alternative proof of [19, Lemma 6.5] (under (h)), and

PROPOSITION 5 - *Let G be an open bounded set with $\text{cl } G$ homeo-*

morphic to (a retract of) $B[0, 1]$. Moreover, let us assume (h) and $0 \neq f(u) \in T(G; u)$, for each $u \in \text{fr } G$. Then $\deg(f, G, 0) = (-1)^n$.

The proof is a straightforward application of (β) , Lemma 1 and the equality $T(\text{cl } G; u) = T(G; u)$.

Analogous results can be obtained dropping (h) and assuming stronger conditions, either of invariance, or on the shape of G : this will be discussed elsewhere. Variants of Lemma 1 have been obtained and applied recently in [44], [42]. General theorems in this direction can be found in [45].

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APPENDIX

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