SEQUENTIAL CONVERGENCE IN FREE GROUPS (*)

by ROMAN FRIČ (in Košice) and FABIO ZANOLIN (in Trieste) (**)

SOMMARIO. - *Dato uno spazio di convergenza sequenziale, si costruisce il gruppo di convergenza libero sullo spazio assegnato. Si esaminano varie proprietà che si preservano nel passaggio dallo spazio di convergenza al gruppo libero di convergenza. La teoria svolta è utilizzata per la costruzione di vari esempi e controesempi necessari alla risoluzione di problemi riguardanti la teoria dei gruppi di convergenza sequenziali.*

SUMMARY. - *The free sequential convergence group generated by a given sequential convergence space is constructed. Various significant properties of the original space are proved to be valid for the generated free group, too. Such technique is exployted in the construction of some examples of peculiar sequential convergence groups which are needed for solving various problems.*

We study compatible sequential convergence structures for groups. In particular, we investigate free groups and convergence structures in which the elements of a fixed set of sequences converge to the neutral element and consider continuous extensions of mappings from the space of generators over the generated free group. Generalizing earlier results obtained by the second author for the commutative case, we construct the free sequential convergence group

(*) Pervenuto in Redazione il 16 luglio 1986.
(**) Indirizzo degli Autori: R. Frič, Matematicky ústav SAV, Karpatska 5, 040 01, Košice (Czechoslovakia); F. Zanolin, Dipartimento di Scienze Matematiche, Università, P.le Europa 1, 34100 Trieste (Italy).

and investigate its properties. Background information is provided and extensive list (by no means complete) of references is added.

1. - Introduction

The notion of a convergence on an arbitrary nonvoid set was introduced in terms of sequences by M. Fréchet [18]. Fundamental contributions to the sequential convergence were made by P. S. Alexandroff and P. Urysohn [1], [69], F. Hausdorff [33] and K. Kuratowski [48]. The theory of sequential convergence structures was further developed in [53], [15], [39], [56], [17], [10], [43], [31], [9], [34], [58], [7], [45], [28], etc., mostly within the realm of general topology. The interested reader will find more information in a survey paper by F. Siwiec [68] and in the Proceedings of various conferences devoted to convergence structures (e. g. Reno 1976, Frankfurt/Oder 1978, Szczyrk 1979, Lawton 1980, Schwerin 1983, Katowice 1983, Bechyně 1984). Sequential structures also provide an abstract framework in which many problems in abstract analysis (involving limits of sequences) can be formulated and solved. They appear in connection with measure theory and probability ([54], [47], [4], [37]), abstract analysis ([38], [36], [16], [52], [5], [3], [30], [40], [64], [29], [6]) and dynamical systems and differential equations ([32], [67], [50], [8], [14]).

In many cases the space in question is equipped with an algebraic structure (i. e., it is a group, ring, linear space, ordered space) and at the same time with a sequential structure, the two structures being linked together by a compatibility condition. As far as we know, groups equipped with a sequential convergence were introduced by 0. Schreier in [65]. The theory was further developed in [61], [55], [57], [59], [2], [19], [70], [71], [62], [21], [41], [44], [27]. To a certain extent, recent results in this area are covered by two survey papers [22] and [23]. Sequential convergence in free groups is investigated in [70], [71], [60], [47], [72], [24], [25], [26], and basic properties of free sequential convergence commutative groups are estabilished in [71]. In the present paper we develop the theory of noncommutative free sequential convergence groups. As shown in [24], [25], [26], the free group technique is very useful for constructing sequential convergence groups having some prescribed properties. It is to be noted that although sequential convergence groups resemble in many aspects topological groups and k-groups ([49]), Remark 2.1 indicates that sequential structures provide finer invariants than other continous structures compatible with a group structure.

In section 2 we recall some basic notions and introduce a suitable notation. Section 3 is devoted to sequences converging to the neutral element of a group. In section 4 we construct the free sequential convergence group and describe some of its properties. The last section contains miscellaneous remarks and illustrative examples.

2. - Basics

Denote by $\mathbf{N}$ the set of all natural numbers and by S the set of all increasing mappings of $\mathbf{N}$ into $\mathbf{N}$. If X is a nonvoid set, $S = \langle x_n \rangle \in X^{\mathbf{N}}$ a sequence of points of X (i.e. a mapping of $\mathbf{N}$ into X) and $s \in \mathsf{S}$, then $S \circ s$ denotes the subsequence of S the n-th term of which is $S(s(n)) = x_{s(n)}$. For $x \in X$ the constant sequence each term of which is x is denoted by $\langle x \rangle$. For $\mathfrak{L} \subset X^{\mathbf{N}} \times X$ and $x \in X$ put $\mathfrak{L}^{\leftarrow}(x) = \{\langle x_n \rangle \in X^{\mathbf{N}} : (\langle x_n \rangle, x) \in \mathfrak{L}\}$; we say that $\langle x_n \rangle$ $\mathfrak{L}-$ converges to x whenever $\langle x_n \rangle \in \mathfrak{L}^{\leftarrow}(x)$. For $\mathfrak{L} \subset X^{\mathbf{N}} \times X$ we consider the following axioms of convergence:

(F) *If* $(S, x) \in \mathfrak{L}$, *then* $(S \circ s, x) \in \mathfrak{L}$ *for all* $s \in \mathsf{S}$;

(U) *If* $(S, x) \notin \mathfrak{L}$, *then there exists* $s \in \mathsf{S}$ *such that* $(S \circ s \circ t, x) \notin \mathfrak{L}$ *for all* $t \in \mathsf{S}$;

(S) $(\langle x \rangle, x) \in \mathfrak{L}$ *for all* $x \in X$;

(H) *If* $(S, x) \in \mathfrak{L}$ *and* $(S, y) \in \mathfrak{L}$, *then* $x = y$.

If, e.g., $\mathfrak{L} \subset X^{\mathbf{N}} \times X$ satisfies axioms (F), (U) and (S), then we say that $\mathfrak{L}$ is a FUS-convergence for X. A set equipped with a FUS-convergence is said to be a *FUS-convergence space.* Similar convenction will be used for other systems of axioms.

Let f be a mapping of a set X into a set Y, let X be equipped with a FUS-convergence $\mathfrak{L}$ and Y with a FUS-convergence $\mathfrak{M}$. If $(\langle f(x_n) \rangle, f(x)) \in \mathfrak{M}$ whenever $(\langle x_n \rangle, x) \in \mathfrak{L}$, then f is said to be *sequentially continuous* or, for short, continuous.

If X is a group and $S, T \in X^{\mathbf{N}}$, then $(ST)(n) = S(n)\,T(n)$ and $S^{-1}(n) = S(n)^{-1}$ for all $n \in \mathbf{N}$. For a commutative group the additive notation will be used correspondingly.

Recall that a *FLUS-convergence group* is a group G equipped with a FUS-convergence $\mathfrak{G}$ satisfying the following compatibility axiom:

(L) *If* $(S, x), (T, y) \in \mathfrak{G}$, *then* $(ST^{-1}, xy^{-1}) \in \mathfrak{G}$.

REMARK 2.1 - It is known that if G is a topological group and $\mathfrak{G}$ is the convergence of sequences in G (a sequence $\langle x_n \rangle$ $\mathfrak{G}$-converges to $x \in G$ whenever each neighborhood of x contains x_n for all but finitely many $n \in \mathbf{N}$), then $\mathfrak{G}$ is a FLUS-convergence for G. A similar assertion holds if G is a k-group (the proof is analogous to that of Lemma 1.3(h) in [63]). As pointed out by A. Kamiński and P. Antosik at the

Conference on convergence held in Szczyrk in 1979, there are FLUS-convergence groups the convergence in which fails to be topological. E. g., let $\mathbf{R}$ be the group of all real numbers and let $(\langle x_n \rangle, x) \in \mathfrak{R}$ whenever the range of $\langle x_n \rangle$ is a finite subset of $\mathbf{R}$ and x is any real number. Then $\mathfrak{R}$ is a FLUS-convergence for $\mathbf{R}$ and the antidiscrete topology is the only topology for $\mathbf{R}$ in which all $\mathfrak{R}$-convergent sequences converge. Clearly, $\mathfrak{R}$ differs from the sequential convergence of the antidiscrete topology. Note that it is not difficult to construct a FLUS-convergence in a free group which fails to be topological.

REMARK 2.2 - It is easy to prove (cf. [57]) that in order to equip a group G with a FLUS-convergence it suffices to start with $\mathfrak{S} \subset G^{\mathbf{N}} \times G$ satisfying axioms (F), (S) and

(L') *If* (S, x), $(T, y) \in \mathfrak{S}$, *then for some* $s \in \mathrm{S}$ *we have*
$((S \circ s)(T^{-1} \circ s), xy^{-1}) \in \mathfrak{S}$;

and then to pass to the so-called *Urysohn modification* $\mathfrak{S}^*$ *of* $\mathfrak{S}$ (i. e. $(S, x) \in \mathfrak{S}^*$ whenever for each $s \in \mathrm{S}$ there exists $t \in \mathrm{S}$ such that $(S \circ s \circ t, x) \in \mathfrak{S}$). Then $\mathfrak{S}^*$ is a FLUS-convergence for G (note that $\mathfrak{S}$ and $\mathfrak{S}^*$ induce the same sequential closure operator for G). Further, if $\mathfrak{S}$ satisfies (H), then $\mathfrak{S}^*$ satisfies (H), too. Also, in a FLUS-convergence group axiom (H) is equivalent to the following one:

(H_0) *If* S *is the constant sequence generated by the neutral element* e *of* G *and* $(S, x) \in \mathfrak{S}$, *then* $x = \mathrm{e}$.

3. - Neutral sequences

If G is a topological group and $\mathrm{N}(\mathrm{e})$ is a neighborhood base at the neutral element e of G, then $x\mathrm{N}(\mathrm{e}) = \mathrm{N}(\mathrm{e})x$ is a neighborhood base at x, for all $x \in G$. Further, a certain system of normal subgroups of a group G can be used to equip G with a group topology so that the system becomes a neighborhood base at e (see e.g. [35]). Similar assertions are true for FLUS-convergence groups. The commutative case is covered by Theorem 2 in [57] and by Corollary in [70]. We are going to extend these results to the noncommutative FLUS-convergence groups.

Let G be a group equipped with a FLUS-convergence $\mathfrak{S}$ and let $G^{\mathbf{N}}$ be the group of all sequences in G. Recall that

$$\mathfrak{S}^{\leftarrow}(x) = \{S \in G^{\mathbf{N}} : (S, x) \in \mathfrak{S}\}, \quad x \in G.$$

LEMMA 3.1 - $\mathfrak{S}^{\leftarrow}(\mathrm{e})$ *has the following properties:*

(i) $\mathfrak{S}^{\leftarrow}(\mathrm{e})$ *is a subgroup of* $G^{\mathbf{N}}$;

(ii) *If* $(S, x) \in \mathfrak{S}$, *then* $S\,\mathfrak{S}^{\leftarrow}(\mathrm{e})\,S^{-1} = \mathfrak{S}^{\leftarrow}(\mathrm{e})$;

(iii) $\mathfrak{S}^{\leftarrow}(x) = \langle x \rangle \mathfrak{S}^{\leftarrow}(e) = \mathfrak{S}^{\leftarrow}(e) \langle x \rangle$ *for all* $x \in G$;

(iv) *If* $S \in \mathfrak{S}^{\leftarrow}(e)$ *and* $s \in S$, *then* $S \circ s \in \mathfrak{S}^{\leftarrow}(e)$;

(v) *Let* $S \in G^{N}$. *If for each* $s \in S$ *there exists* $t \in S$ *such that* $S \circ s \circ t \in \mathfrak{S}^{\leftarrow}(e)$, *then* $S \in \mathfrak{S}^{\leftarrow}(e)$;

(vi) $\mathfrak{S}$ *satisfies axiom (H) iff* $\langle e \rangle$ *is the only constant sequence in* $\mathfrak{S}^{\leftarrow}(e)$.

The easy proof is omitted.

A sequence belonging to $\mathfrak{S}^{\leftarrow}(e)$ is said to be *neutral*. Motivated by Lemma 3.1, we introduce some additional terminology and notation.

Let G be a group. Identifying $x \in G$ with $\langle x \rangle \in G^{N}$, we can consider G as a subgroup of G^{N}. A subgroup H of G^{N} is said to be *normal with respect to G* if $gSg^{-1} = \langle gS(n)g^{-1} \rangle \in H$ whenever $g \in G$ and $S \in H$. Let A be a subset of G^{N}. Let δA be the set of all sequences $S \circ s$ such that $S \in A$ and $s \in S$ and let ζA be the set of all sequences $S \in G^{N}$ such that for each $s \in S$ there is $t \in S$ such that $S \circ s \circ t \in A$. Finally, consider the set of all subgroups of G^{N} containing A and normal with respect to G. Denote by $[A]_G$ the intersection of all such subgroups. Then G^{N} is the largest and $[A]_G$ is the smallest element of the set.

Note that if $\mathfrak{S}$ is a FLUS-convergence for G, then by Lemma 3.1 $\mathfrak{S}^{\leftarrow}(e)$ is a subgroup of G^{N} normal with respect to G and closed with respect to ζ and δ. We shall see that each such subgroup of G^{N} is precisely the set of all neutral sequences of some FLUS-convergence for G.

LEMMA 3.2 - *(i)* $[A]_G$ *consists precisely of finite products of sequences of the form* $gS^{\varepsilon}g^{-1} = \langle gS(n)^{\varepsilon}g^{-1} \rangle$, *where* $g \in G$, $S \in A$ *and* $\varepsilon = \pm 1$.

(ii) $\zeta[\delta A]_G$ *is the smallest subgroup of* G^{N} *containing* A, *closed with respect to* δ *and* ζ *and normal with respect to* G.

(iii) If $A = \zeta[\delta A]_G$, *then* A *is the set of all neutral sequences of a FLUS-convergence for* G.

Proof. *(i)* and *(ii)* follow directly from the definitions of the notions involved.

(iii). Define $\mathfrak{S} \subset G^{N} \times G$ as follows: $(S, x) \in \mathfrak{S}$ whenever $Sx^{-1} \in A$. It follows immediately from *(ii)* that $\mathfrak{S}$ is a FLUS-convergence for G. Clearly, $A = \mathfrak{S}^{\leftarrow}(e)$. This completes the proof.

THEOREM 3.3 - *Let* G *be a group and let* A *be a subset of* G^{N}.

(i) *There is a FLUS-convergence* $\mathfrak{S}_A$ *for* G *such that* $A \subset \zeta[\delta A]_G = \mathfrak{S}_A^{\leftarrow}(e)$.

(ii) *If* $\mathfrak{S}$ *is a FLUS-convergence for G such that* $A \subset \mathfrak{S}^{\leftarrow}(e)$, *then* $\mathfrak{S}_A \subset \mathfrak{S}$.

(iii) $\mathfrak{S}_A$ *satisfies axiom (H) iff* $\zeta[\delta A]_G$ *contains no constant sequence except* $\langle e \rangle$.

Proof. (i). The existence of $\mathfrak{S}_A$ follows from *(iii)* of Lemma 3.2.

(ii). Let $\mathfrak{S}$ be a FLUS-convergence for G such that $A \subset \mathfrak{S}^{\leftarrow}(e)$. By Lemma 3.1, $\mathfrak{S}^{\leftarrow}(e)$ is a subgroup of G^N which is closed with respect to δ and ζ and normal with respect to G. By *(iii)* of Lemma 3.2, $\zeta[\delta A]_G = \mathfrak{S}_A^{\leftarrow}(e)$ is the smallest of all subgroups of G^N containing A, slosed with respect to δ and ζ and normal with respect to G. Hence $\mathfrak{S}_A^{\leftarrow}(e) \subset \mathfrak{S}^{\leftarrow}(e)$ and also $\mathfrak{S}_A \subset \mathfrak{S}$.

(iii) follows from *(vi)* of Lemma 3.1.

Theorem 3.3 provides a very efficient way how to equip a group with a FLUS-convergence (resp. FLUSH-convergence). We shall say that $\mathfrak{S}_A$ is the *FLUS-convergence for G generated by* A.

Our next result is a construction dealing with convergence quotient groups. The construction has an auxiliary character. Once we have developed the theory of noncommutative free convergence groups, it will enable us to obtain theorems for commutative free convergence groups.

LEMMA 3.4 - *Let h be a homomorphism of a group H equipped with a FLUS-convergence* $\mathfrak{H}$ *into a group G equipped with a FLUS-convergence* $\mathfrak{S}$. *Then we have*

(i) *h is continuous iff* $h \circ S = \langle h(S(n)) \rangle \in \mathfrak{S}^{\leftarrow}(e_G)$ *for all* $S \in \mathfrak{H}^{\leftarrow}(e_H)$, *and*

(ii) *if* $\mathfrak{H}^{\leftarrow}(e_H) = \zeta[\delta B]_H$ *for some* $B \in H^N$ *and* $h \circ S \in \mathfrak{S}^{\leftarrow}(e_G)$ *for all* $S \in B$, *then h is continuous.*

We omit the proof of these simple assertions.

Let H be a group and let K be a normal subgroup of H. Denote by G the quotient group of H by K and denote by h the natural homomorphism of H onto G defined by $h(x) = xK$, $x \in H$. Let B a subset of H^N and let $\mathfrak{H}$ be the FLUS-convergence for H generated by B. Denote by $\mathfrak{S}_A$ the FLUS-convergence for G generated by $A = \{h \circ S : S \in B\}$ (i.e. $\mathfrak{S}_A^{\leftarrow}(e_G) = \zeta[\delta A]_G$).

LEMMA 3.5 - $\mathfrak{S}_A^{\leftarrow}(e_G) = \zeta\{h \circ S : S \in \mathfrak{H}^{\leftarrow}(e_H)\}$.

The proof is straightforward and is omitted.

THEOREM 3.6 - *(i)* $\mathfrak{S}_A$ *is the smallest FLUS-convergence for G rendering the natural homomorphism h of H equipped with* $\mathfrak{H}$ *into G continuous.*

(ii) Let f be a homomorphism of G equipped with $\mathfrak{S}_A$ into a group F equipped with a FLUS-convergence $\mathfrak{F}$. Then f is continuous iff $f \circ h$ is continuous.

(iii) $\mathfrak{S}_A$ satisfies axiom (H) iff K is a sequentially closed subgroup of H.

Proof. *(i)* and *(ii)* follow from Lemma 3.4 and Lemma 3.5. *(iii)* follows from axiom (H_0) and from the construction of $\mathfrak{S}_A$.

DEFINITION 3.7 - Under the above notations, $\mathfrak{S}_A$ is said to be the *quotient convergence* for G; it will be denoted by $h(\mathfrak{H})$.

4. - Free convergence groups

Free topological groups have been investigated for over four decades (see [35]). The study of sequential convergence in free commutative groups was initiated by F. Zanolin in [70] (see also [71], [72]). The noncommutative case was first considered by J. Novák in [60]. He investigated the so-called pointed free group, i.e., given an infinite set Y equipped with a convergence of sequences, a point $p \in Y$ is singled out and the free group G generated by $Y \setminus \{p\}$ is equipped with a compatible convergence of sequences subjected to the following restriction: if a sequence of points of $Y \setminus \{p\}$ converges to a point x in K, then the sequence converges in G to x provided $x \neq p$ and to the neutral element e of G provided $x = p$ (elements of $Y \setminus \{p\}$ are considered as one-letter words in G). In fact, he considered the case when p is the only nonisolated point of Y. Fine FLUSH-convergences in free groups (commutative, noncommutative, pointed commutative and pointed noncommutative) are studied in [26]. Using some results from [26], we develop in this paper a general theory of free convergence groups.

In this section we construct the noncommutative free convergence group over a FUS-convergence space and describe some of its properties.

Let X be any nonvoid set. Recall (cf. [35]) that a *word* is either void, written e, or a finit formal product $x_1^{\varepsilon_1} \ldots x_k^{\varepsilon_k}$ of elements of X, where $\varepsilon_i = \pm 1$, $i = 1, \ldots, k$. A word is *reduced* if it is void or if $\varepsilon_i = \varepsilon_{i+1}$ whenever $x_i = x_{i+1}$. The *length* $l(w)$ of a reduced word w is defined as follows: $l(w) = 0$ if $w = e$ and $l(w) = k$ if $w = x_1^{\varepsilon_1} \ldots x_k^{\varepsilon_k}$, $k > 0$. Denote by $F(X)$ the set of all reduced words of X. If $w_1 = x_1^{\varepsilon_1} \ldots x_k^{\varepsilon_k}$ and $w_2 = y_1^{\delta_1} \ldots y_l^{\delta_l}$ belong to $F(X)$ then their product $\sigma(w_1, w_2)$ is defined as follows. Consider $x_1^{\varepsilon_1} \ldots x_k^{\varepsilon_k} y_1^{\delta_1} \ldots y_l^{\delta_l}$. If this word is reduced we define it to be $\sigma(w_1, w_2)$ and write $\sigma(w_1, w_2) = w_1 w_2$. If it is not reduced, then $x_k = y_1$ and $\varepsilon_k = -\delta_1$. Then consider

$x_1^{\varepsilon_1} \ldots x_{k-1}^{\varepsilon_{k-1}} y_2^{\delta_2} \ldots y_l^{\delta_l}$ and if it is reduced we define it to be $\sigma(w_1, w_2)$. If it is not reduced, we continue in the indicated way until a reduced word is obtained and define it to be $\sigma(w_1, w_2)$. With this multiplication, $F(X)$ is a group and is called the *free group generated by* X. The neutral element of $F(X)$ is e and the inverse of $x_1^{\varepsilon_1} \ldots x_k^{\varepsilon_k}$ is $x_k^{-\varepsilon_k} \ldots x_1^{-\varepsilon_1}$.

Throughout the remainder of this section, X is a nonvoid set, $\mathfrak{L}$ is a FUS-convergence for X, $F(X)$ is the free group generated by X, $\mathfrak{F}_0$ is a subset of $F(X)^{\mathbf{N}} \times F(X)$ defined as follows:

(C) $(W, w) \in \mathfrak{F}_0$ whenever
$W = \sigma(S_1^{\varepsilon_1}, \ldots, S_k^{\varepsilon_k})$ and $w = \sigma(x_1^{\varepsilon_1}, \ldots, x_k^{\varepsilon_k})$,
where $k \in \mathbf{N}$, $(S_i, x_i) \in \mathfrak{L}$ and $\varepsilon_i = \pm 1$, $i = 1, \ldots, k$, and, for each $n \in \mathbf{N}$, either $\sigma(S_1^{\varepsilon_1}(n), \ldots, S_k^{\varepsilon_k}(n)) = \mathrm{e}$ or $S_1^{\varepsilon_1}(n) \ldots S_k^{\varepsilon_k}(n)$ is a reduced word;

and $\mathfrak{F} \subset F(X)^{\mathbf{N}} \times F(X)$ is defined as follows:

(C*) $(W, w) \in \mathfrak{F}$ whenever for each $s \in \mathrm{S}$ there exists $t \in \mathrm{S}$ such that $(W \circ s \circ t, w) \in \mathfrak{F}_0$;

i.e., $\mathfrak{F}$ is the Urysohn modification of $\mathfrak{F}_0$.

THEOREM 4.1 - *(i)* $\mathfrak{F}_0$ *restricted to* X *is equal to* $\mathfrak{L}$.

(ii) $\mathfrak{F}_0$ *satisfies axioms (F), (S) and (L').*

(iii) $\mathfrak{F}_0$ *satisfies axiom (H) whenever* $\mathfrak{L}$ *does.*

(iv) $\mathfrak{F}$ *is a FLUS-convergence for* $F(X)$ *and* $\mathfrak{F}$ *restricted to* X *is equal to* $\mathfrak{L}$. *Further,* $\mathfrak{F}$ *is the finest of all FLUS-convergences for* $F(X)$ *the restriction of which to* X *is equal to* $\mathfrak{L}$.

(v) *Put* $\mathrm{A} = \{\langle \sigma(x_n, x^{-1}) \rangle \in F(X)^{\mathbf{N}} : (\langle x_n \rangle, x) \in \mathfrak{L}\}$. *Then* $\mathfrak{F}^{\leftarrow}(\mathrm{e}) = \zeta[\delta\mathrm{A}]_{F(X)}$.

(vi) *Let* h *be a continuous mapping of the FUS-convergence space* X *into a FLUS-convergence group* G. *Then* h *can be uniquely extended to a continuous homomorphism of* $F(X)$ *equipped with* $\mathfrak{F}$ *into* G.

(vii) X *is a (sequentially) closed subset of* $F(X)$ *iff* $\mathfrak{L}$ *satisfies axiom (H).*

Proof. The fact that $\mathfrak{F}_0$ satisfies axioms (F) and (S), as well as *(i)* and *(iii)*, follows directly from (C). The proof that $\mathfrak{F}_0$ satisfies (L') is virtually the same as the proof of the corresponding assertion of Theorem 3 in [26] and is omitted.

(iv). According to Remark 2.2, $\mathfrak{F}$ is a FLUS-convergence for F(X) and, clearly, its restriction to X is equal to $\mathfrak{L}$. Let $\mathfrak{G}$ be a FLUS-con-

vergence for $F(X)$ such that $(S,x) \in \mathfrak{G}$ whenever $(S,x) \in \mathfrak{L}$. Let $(W,w) \in \mathfrak{F}_0$. By (C), W is a certain product of $\mathfrak{L}$-convergent sequences and w is the corresponding product of their $\mathfrak{L}$-limits. Using axiom (L) we get $(W,w) \in \mathfrak{G}$ and from $\mathfrak{F}_0 \subset \mathfrak{G}$ we infer $\mathfrak{F} \subset \mathfrak{G}^* = \mathfrak{G}$.

(v) follows from *(iv)* and Theorem 3.3.

(vi). Since $F(X)$ is the free group generated by X, h can be uniquely extended to a homomorphism of $F(X)$ into G. Now *(v)* and condition *(ii)* of Lemma 3.4 imply that the homomorphism is continuous.

(vii). Assume that $\mathfrak{L}$ satisfies (H). If S is a sequence of points of X and $(S,w) \in \mathfrak{F}$, then, for some $s \in S$ we have $(S \circ s, w) \in \mathfrak{F}_0$. From (C) we infer that $1(w) = 1$ and hence $w \in X$. Thus X is a closed subset of $F(X)$. Conversely, assume that $\mathfrak{L}$ does not satisfy (H). Then some $S \in X^N$ $\mathfrak{L}$-converges in X to two different points x and y. By axiom (L), the constant sequence $SS^{-1} = \langle e \rangle$ $\mathfrak{F}$-converges to the reduced word xy^{-1} and the constant sequence $\langle x \rangle$ $\mathfrak{F}$-converges in F(X) to the reduced word xxy^{-1}. Hence X is not closed in $F(X)$. This completes the proof.

COROLLARY 4.2 - *For every FUSH-convergeence space X, there exists a FLUSH-convergence group F with the following properties:*

(i) *X is a closed subspace of F;*

(ii) *Algebraically, F is the free group generated by X;*

(iii) *Every continuous mapping of X into a FLUSH-convergence group G can be uniquely extended to a continuous homomorphism of F into G;*

(iv) *Let $\tilde{F}$ be a FLUSH-convergence group such that X is a subspace of $\tilde{F}$, the smallest closed subgroup of $\tilde{F}$ that contains X is $\tilde{F}$ itself, and every continuous mapping of X into a FLUSH-convergence group G can be extended to a continuous homomorphism of $\tilde{F}$ into G. Then there is a homeomorphic isomorphism of F onto $\tilde{F}$ leaving X pointwise fixed.*

Proof. *(i)*, *(ii)* and *(iii)* follow directly from Theorem 4.1. The last assertion follows by the so-called *extension of the identity principle* (for FUSH-convergence spaces, if two continuous mappings agree on a dense subset of the domain, then they agree on the whole domain).

DEFINITION 4.3 - $F(X)$ equipped with $\mathfrak{F}$ is said to be the *free convergence group over the FUS-convergence space X*.

Let us turn to properties of $F(X)$. Some of them depend on the additional properties of X and some of them are valid for every FUS-convergence space X. First, we shall prove that $F(X)$ equipped with $\mathfrak{F}$ is always complete.

Recall that in a FLUS-convergence group a sequence S is said to be *Cauchy* (two-sided) if both $(S \circ s)(S^{-1} \circ t)$ and $(S^{-1} \circ s)(S \circ t)$ are neutral sequences for all $s, t \in \mathrm{S}$, and the group is said to be *complete* if every Cauchy sequence converges.

LEMMA 4.4 - *Let S be a Cauchy sequence in $F(X)$.*

(i) *$S \circ s$ is a Cauchy sequence for all $s \in \mathrm{S}$.*

(ii) *If for some $s \in \mathrm{S}$ the sequence $S \circ s$ converges in $F(X)$ to some point x, then $S \circ t$ converges to x for all $t \in \mathrm{S}$.*

(iii) *There exists $k \in \mathbf{N}$ such that for each $n \in \mathbf{N}$ we have $\mathrm{l}(S(n)) < k$.*

(iv) *For some $s \in \mathrm{S}$, $S \circ s$ is either a finite reduced product of one-to-one and constant sequences of points of X resp. X^{-1}* (i.e. there are a natural number k and sequence $S_i^{\varepsilon_i} \in X^{\mathbf{N}}$, $\varepsilon_i = \pm 1, i = 1, \ldots, k$, such that each S_i is either constant or one-to-one and for each $n \in \mathbf{N}$ $S_1(n) \ldots S_k(n)$ is a reduced word equal to $(S \circ s)(n)$) *or $S \circ s$ is the constant sequence $\langle \mathrm{e} \rangle$.*

Proof. *(i)* and *(ii)* hold in every FLUS-convergence group and follow easily from the definition of a Cauchy sequence. Now assume, on the contrary, that *(iii)* does not hold. Then there are $s, t \in \mathrm{S}$ such that $\langle \mathrm{l}(((S \circ s)(n))((S^{-1} \circ t)(n)))\rangle$ is an increasing sequence of natural numbers. Since $(S \circ s)(S^{-1} \circ t)$ is a neutral sequence, we have a contradiction with (C). *(iv)* is a straightforward consequence of *(iii)*. This completes the proof.

THEOREM 4.5 - *$F(X)$ is complete.*

Proof. Let S be a Cauchy sequence in $F(X)$. We have to prove that S converges in $F(X)$ to some point x. By *(ii)* of Lemma 4.4, it suffices to prove that any subsequence of S converges in $F(X)$. If for some $s \in \mathrm{S}$ we have $(S \circ s)(n) = \mathrm{e}$, for all $n \in \mathbf{N}$, we are done. In the opposite case, by *(iv)* of Lemma 4.4, there are $s \in \mathrm{S}$, $k \in \mathbf{N}$, $S_i^{\varepsilon_i} \in X^{\mathbf{N}}$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, k$, such that each S_i is either constant or one-to-one and for each $n \in \mathbf{N}$ $S_1(n) \ldots S_k(n)$ is a reduced word equal to $(S \circ s)(n)$. If all sequences S_i are constant, then $S \circ s$ is a convergent sequence. So, assume that at least one of the sequences S_i is one-to-one. Put $m = \max\{i \in \{1, \ldots, k\} : S_i$ is a one-to-one sequence$\}$ and define $t \in \mathrm{S}$ by $t(n) = s(n+1)$, $n \in \mathbf{N}$. Then for each $n \in \mathbf{N}$, $S_1(n) \ldots S_m(n) S_m^{-1}(n+1) \ldots S_1^{-1}(n+1)$ is a reduced word equal to $\sigma(((S \circ s)(n))((S^{-1} \circ t)(n)))$. On the other hand, by (C), there exists $u \in \mathrm{S}$ such that the sequence $(S \circ s \circ u)(S^{-1} \circ t \circ u)$ is a finite reduced product of sequences T_i such that for some $\delta_i = \pm 1$ each $T_i^{\delta_i}$ is a convergent sequence in the original FUS-convergence space X. Since $S_{m+1}, \ldots, S_k$ are constant sequences, the subsequence $S \circ s \circ u$ of S converges in $F(X)$. This completes the proof.

THEOREM 4.6 - *If points of the space X are separated by continuous functions, then points of $F(X)$ (equipped with $\mathfrak{F}$) can be separated by continuous functions.*

Proof. Assume that points of X are separated by continuous functions. Then X satisfies axiom (H). Let $\check{X}$ be the completely regular modification of X ($\check{X}$ is the underlying set of X equipped with the weak topology with respect to the set of all continuous functions on X, cf. [43]), let $\tilde{F}$ be the free topological group generated by $\check{X}$ ($\tilde{F}$ is the underlying group of $F(X)$ equipped with the weakest of all (Hausdorff) group topologies the restriction of which to X is the topology of $\check{X}$), and let $s\tilde{F}$ be the corresponding associated FLUSH-convergence group (see Remark 2.1). The identity mapping of the FUSH-convergence space X into the FLUSH-convergence group $s\tilde{F}$ is continuous and hence it can be uniquely extended to a continuous homomorphism of $F(X)$ into $s\tilde{F}$. It is easy to see that the extended mapping is in fact a continuous isomorphism, viz. the identity mapping. Since points of $\tilde{F}$ are separated by continuous functions and each continuous function on F is a continuous function on $s\tilde{F}$, points of $F(X)$ are also separated by continuous functions.

Recall that each FS-convergence space is equipped with a closure operator: for each subset A, its closure clA is defined to be the set of all limits of convergent sequences of points of A. The operator need not be idempotent and if it happens to be, then the space is said to be *Fréchet*. It is known that a FLUSH-convergence group is a Fréchet space iff the following diagonal condition holds (cf. [61], [28]):

(PSD) If for each $n \in \mathbf{N}$, S_n is a sequence converging to the neutral element e, then there is a mapping f of $\mathbf{N}$ into $\mathbf{N}$ and $s \in \mathrm{S}$ such that the corresponding subdiagonal sequence $\langle S_{s(n)}(f(s(n)))\rangle$ converges to e.

THEOREM 4.7 - *Let X be a FUSH-convergence space. Then $F(X)$ is a Fréchet space iff X is discrete.*

Proof. By *(iii)* of Theorem 4.1, $\mathfrak{F}_0$ satisfies axiom (H). Consequently, by Remark 2.2, $\mathfrak{F}$ also satisfies axiom (H). If X is discrete, then from (C) we easily infer that $F(X)$ is discrete, and hence a Fréchet group. On the other hand, assume that X is not discrete. Then there is a one-to-one sequence $\langle x_n \rangle$ converging in X to a point x such that $x_n \neq x$ for all $n \in \mathbf{N}$. Define $S = \langle x_n x^{-1} \rangle$ and $S_n = S^n, n \in \mathbf{N}$. Since each S_n converges in $F(X)$ to e and for each $k \in \mathbf{N}$ we have $1(S_n(k)) = 2n$, it follows from *(iii)* of Lemma 4.4 that no subdiagonal of $\langle S_n \rangle$ converges to e. Consequently, $F(X)$ does not satisfy (PSD), and hence it fails to be a Fréchet group.

REMARK 4.8 - Let Y be a FUS-convergence space and let p be any point of Y. Using some results from [26], the *pointed free noncommutative convergence group $F_p(Y)$ over Y* can be constructed parallel to the construction of $F(X)$ equipped with $\mathfrak{F}$. Namely, the free group $F_p(Y)$ generated by the set $Y \setminus \{p\}$ can be equipped with a FLUSH-convergence $\mathfrak{F}_p$ such that:

(i) $Y \setminus \{p\}$ is a subspace of $F_p(Y)$;

(ii) if S is a sequence of points of $Y \setminus \{p\}$ converging in Y to p, then S $\mathfrak{F}_p$-converges to the neutral element of $F_p(Y)$;

(iii) for every continuous mapping h of Y into a FLUSH-convergence group G, carring p into the neutral element of G, there exists a unique continuous homomorphism $\bar{h}$ of $F_p(Y)$ into G such that $\bar{h}(x) = h(x)$ for all $x \in Y \setminus \{p\}$.

The explicit construction of $\mathfrak{F}_p$ can be found in [26]. Note that in [26] only FUSH-convergences are considered, but the generalization to FUS-convergences involves no difficulty.

5. - Miscellanea

Let X be a nonvoid set, let $F(X)$ be the free group generated by X, and let K be the commutator subgroup of $F(X)$ (i.e. the subgroup generated by all elements of the form $\sigma(a, b, a^{-1}, b^{-1})$, where $a, b \in F(X)$). It is known that K is a normal subgroup of $F(X)$ and that the factor group of $F(X)$ by K is the free commutative group $FC(X)$ generated by X. Denote by h the natural homomorphism of $F(X)$ onto $FC(X)$. In the additive notation, elements of $FC(X)$ are represented by formal finite linear combinations of the form $\sum_{i=1}^{k} z_i x_i$, where z_i is a nonzero integer, $x_i \in X$, $i = 1, \dots, k$ and $x_i \neq x_j$ whenever $i \neq j$; the neutral element 0 of $F(X)$ is represented by the void combination $(k = 0)$ and the sum $\sigma_c(w_1, w_2)$ of w_1 and w_2 is defined in the obvious way. We shall identify each $x \in X \subset F(X)$ with $h(x) \in FC(X) = F(X) / K$.

Let X be equipped with a FUS-convergence $\mathfrak{L}$. Symbols $\mathfrak{F}_0$ and $\mathfrak{F}$ have the same meaning as in the previous section. We shall show that $FC(X)$, as a quotient of $F(X)$ by K, can be equipped with the quotient convergence $h(\mathfrak{F})$ so that it becomes the free commutative convergence group.

LEMMA 5.1 - *If* $\mathfrak{L}$ *satisfies axiom* (H) *then* K *is sequentially closed in* $F(X)$.

Proof. Let S be a sequence of points of K $\mathfrak{F}$-converging in $F(X)$ to a point w. According to (C*) there is a subsequence W of S $\mathfrak{F}_0$-converging to w. By (C), there are $k \in \mathbf{N}$, $\varepsilon_i = \pm 1$ and $(S_i, x_i) \in \mathfrak{L}$, $i = 1, \dots, k$, such that W is the product of the sequences $S_i^{\varepsilon_i}$ and w is the corresponding product of the points $x_i^{\varepsilon_i}$. Since $\mathfrak{L}$ satisfies axiom (H), the limit w is uniquely determined and it easy to see that since each $W(n) = \sigma(S_1^{\varepsilon_1}(n), \dots, S_k^{\varepsilon_k}(n))$, $n \in \mathbf{N}$, belongs to K, then also $w = \sigma(x_1^{\varepsilon_1}, \dots, x_k^{\varepsilon_k})$ belongs to K. Thus K is sequentially closed in $F(X)$.

THEOREM 5.2 - *(i) The quotient convergence* $h(\mathfrak{F})$ *for* $FC(X) = F(X)/K$ *is generated by* $\{\langle \sigma_c(x_n, x^{-1}) \rangle \in FC(X)^{\mathbf{N}} : (\langle x_n \rangle, x) \in \mathfrak{L}\}$.

(ii) $h(\mathfrak{F})$ *restricted to* X *is equal to* $\mathfrak{L}$.

(iii) $h(\mathfrak{F})$ *is the finest of all* FLUS*-convergences for* $FC(X)$ *the restriction of which to* X *is equal to* $\mathfrak{L}$.

(iv) Let g *be a continuous mapping of the FUS-convergence space* X *into a commutative FLUS-convergence group* G. *Then* g *can be uniquely extended to a continuous homomorphism of* $FC(X)$ *equipped with* $h(\mathfrak{F})$ *into* G.

(v) $h(\mathfrak{F})$ *satisfies axiom* (H) *whenever* $\mathfrak{L}$ *does.*

Proof. All the assertions are straightforward consequences of the properties of $F(X)$ equipped with $\mathfrak{F}$ and the properties of the quotient convergence $h(\mathfrak{F})$ for $FC(X) = F(X)/K$. The details are omitted.

COROLLARY 5.3 - *For every FUSH-convergence space X, there exists a FLUSH-convergence group F with the following properties:*

(i) X is a closed subspace of F ;

(ii) Algebraically, F is the free commutative group generated by X;

(iii) Every continuous mapping of X into a commutative FLUSH-convergence group G can be uniquely extended to a continuous homomorphism of F into G ;

(iv) Let $\tilde{F}$ *be a commutative FLUSH-convergence group such that X is a subspace of* $\tilde{F}$, *the smallest closed subgroup of* $\tilde{F}$ *that*

contains X is $\tilde{F}$ tself and every continuous mapping of X into a commutative FLUSH-convergence group G can be extended to a continuous homomorphism of $\tilde{F}$ into G. Then there is a homeomorphic isomorphism of F onto $\tilde{F}$ leaving X pointwise fixed.

DEFINITION 5.4 - $FC(X)$ equipped with $h(\mathfrak{F})$ is said to be the *free commutative convergence group over the FUS-convergence space X.*

REMARK 5.5 - Virtually in the same way as in the noncommutative case it can be proved that:

(i) $FC(X)$ equipped with $h(\mathfrak{F})$ is complete;

(ii) If points of X can be separated by continuous functions, then points of $FC(X)$ can be separated by continuous functions;

(iii) $FC(X)$ is a Fréchet space iff X is discrete.
Note that *(ii)* has been already proved (in a different way) in [25].

REMARK 5.6 - Let Y be a FUS-convergence space and let p be any point of Y. Then the free commutative group generated by $Y \setminus \{p\}$, denote it by $FC_p(Y)$, can be equipped with a FLUS-convergence $\mathfrak{F}_{pc}$ such that

(i) $Y \setminus \{p\}$ is a subspace of $FC_p(Y)$;

(ii) If S is a sequence of points of $Y \setminus \{p\}$ converging in Y to p, then S $\mathfrak{F}_{pc}$-converges to the neutral element of $FC_p(Y)$;

(iii) For every continuous mapping h of Y into a commutative FLUS-convergence group G, carring p into the neutral element of G, there exists a unique continuous homomorphism $\bar{h}$ of $FC_p(Y)$ into G such that $\bar{h}(x) = h(x)$ for all $x \in Y \setminus \{p\}$.

The convergence $\mathfrak{F}_{pc}$ can be straightforwardly obtained by modifying the construction of the fine convergence for $FC_p(Y)$ given in [26]. The group $FC_p(Y)$ equipped with $\mathfrak{F}_{pc}$ is said to be the *pointed free commutative group over Y.*

Free groups equipped with FLUSH-convergences are natural candidates when looking for a FLUSH-group having some prescribed properties (cf. [70], [71]). If we start with a FUSH-convergence space X, then $\mathfrak{F}$ and $h(\mathfrak{F})$ satisfy axiom (H). But, if G is a free group (either commutative or noncommutative) generated by a set X, $A \subset G^N$ is a set of sequences of points of G, and $\mathfrak{G}_A$ is the generated FLUS-convergence for G, then, according to *(iii)* of Theorem 3.3, $\mathfrak{G}_A$ satisfies axiom (H) iff $\zeta[\delta A]_G$ contains no constant sequence except $\langle e \rangle$. Our last theorem provides a simple sufficient condition for $\mathfrak{G}_A$ to satisfy axiom (H).

Let X be a nonvoid set and let $F(X)$ be the free group generated by X. For a reduced word $w = x_1^{\varepsilon_1} \dots x_k^{\varepsilon_k}$, $k \geqslant 1$, put $gen(w) = \{x \in X : x = x_i \text{ for some } i \in \{1, \dots, k\}\}$ and put $gen(e) = \phi$. Recall that if $\langle A_n \rangle$ is a sequence of subsets of X, then

$$\limsup A_n = \{x \in X : x \in A_n \text{ for infinitely many } n \in \mathbf{N}\}.$$

THEOREM 5.7 - *Let* $\mathfrak{G}_{\mathrm{A}}$ *be the FLUS-convergence for* $F(X)$ *generated by a set* $\mathrm{A} \subset F(X)^{\mathbf{N}}$. *If for each* $\langle w_n \rangle \in \mathrm{A}$ *we have* $\limsup gen(w_n) = \phi$, *then* $\mathfrak{G}_{\mathrm{A}}$ *satisfies axiom* (H).

Proof. The assertion follows from *(iii)* of Theorem 3.3 and *(i)* of Lemma 3.2. Indeed, for each $x \in X$ and each $S \in \delta\mathrm{A}$ we have $x \in gen(S(n))$ for at most finitely many $n \in \mathbf{N}$. Now, assume that for some $w \neq e$ we have $\langle w \rangle \in \zeta[\delta\mathrm{A}]_{F(X)}$. Then for infinitely many $n \in \mathbf{N}$ we have $gen(w) = gen(T(n))$, where $T \in F(X)^{\mathbf{N}}$ is a product of finitely many sequences of the form $\langle g S^{\varepsilon}(n) g^{-1} \rangle$ with $g \in F(X)$, $\varepsilon = \pm 1$, $S \in \delta\mathrm{A}$. Clearly, this is impossible. Consequently, $\zeta[\delta\mathrm{A}]_{F(X)}$ contains no nontrivial constant sequence and hence $\mathfrak{G}_{\mathrm{A}}$ satisfies axiom (H).

REMARK 5.8 - Note that the above criterion for $\mathfrak{G}_A$ to satisfy axiom (H) can be also used in the commutative case.

To illustrate the theory developed throughout the paper we present three examples.

EXAMPLE 5.9 - In [24] the following FLUSH-convergence group G having no (two-sided) completion has been constructed. Consider two disjoint countable infinite sets $A = \{a_n : n \in \mathbf{N}\}$ and $B = \{b_n : n \in \mathbf{N}\}$. Put $X = A \cup B$ and $S = \langle a_n \rangle$, $T = \langle b_n \rangle$. Let G be the free group generated by X. For $\mathrm{A} = \{T\} \cup \{(S \circ s)^{\varepsilon} (S \circ t)^{-\varepsilon} : s, t \in \mathrm{S}, \varepsilon = \pm 1\}$ let $\mathfrak{G}_{\mathrm{A}}$ be the generated FLUS-convergence for G. It follows by Theorem 5.7 that $\mathfrak{G}_{\mathrm{A}}$ satisfies axiom (H). Clearly, S is a (two-sided) Cauchy sequence no subsequence of which converges in G. Further, T is a neutral sequence, but no subsequence of STS^{-1} converges in G to the neutral element. Consequently, the sequence S cannot converge in any FLUSH-convergence group $\bar{G}$ containing G as a dense convergence subgroup, i.e., G has no completion.

The importance of the above result becomes clear when realizing that all topological groups (both commutative and noncommutative) have (two-sided) completions and that until [24] it was not clear whether all FLUSH-convergence groups have (two-sided) completions. As shown in [59], each commutative FLUSH-convergence group has a «categorical» completion and it can have several nonhomeomorphic completions. At present, however, it is still not known whether for any nontrivial class of noncommutative FLUSH-groups a «categorical» completion exists. Further, observe that each commutative filter convergence group has a categorical completion, it can have several nonhomeomorphic completions ([20]), but nothing is known about the completions of noncommutative filter convergence groups. For a categorical completion theory of various convergence structures (including filter and sequential convergence groups) the reader is referred to [42].

The next two examples deal with *coarse* convergence in groups, a notion similar to minimal group topology (cf. [11], [12]). Recall that a FLUSH-convergence $\mathfrak{S}$ for a group G is coarse if there is no FLUSH-convergence for G properly larger than $\mathfrak{S}$. According to Theorem 1 in [27] (see also Theorem 2 in [25]), every FLUSH-convergence can be enlarged to a coarse one. For a more detailed discussion of the properties of coarse convergence groups we refer to [27] and [13].

EXAMPLE 5.10 - We are going to construct a coarse convergence group which fails to be complete. Consider a countable infinite set X, choose a point $p \in X$ and arrange $X \setminus \{p\}$ into a one-to-one sequence S. Let G be the free commutative group generated by X. Put $\mathrm{A} = \{2S - \langle p \rangle\} \cup \{S \circ s - S \circ t : s, t \in \mathrm{S}\}$ and equip G with the generated FLUS-convergence $\mathfrak{S}_{\mathrm{A}}$. Using Theorem 5.7 and Remark 5.8 it can be easily verified that $\mathfrak{S}_{\mathrm{A}}$ satisfies axiom (H). Clearly, S is a Cauchy sequence and $2S$ $\mathfrak{S}_{\mathrm{A}}$-converges to p. Let $\mathfrak{S}_C$ be a coarse FLUSH-convergence for G such that $\mathfrak{S}_{\mathrm{A}} \subset \mathfrak{S}_C$. Then no subsequence $S \circ s$ of S $\mathfrak{S}_C$-converges. Indeed, $w = \mathfrak{S}_C\text{-}\lim S(n)$ and $p = \mathfrak{S}_C\text{-}\lim 2S(n)$ would imply $2w = p$, a contradiction with the fact that p is a generator of the free group G.

In [66] U. Schwanengel proved that there is a minimal topological group with a closed (normal) subgroup which is not minimal. A closed subgroup of a coarse commutative group is coarse ([27]). We show that the assertion cannot be generalized to the noncommutative case.

EXAMPLE 5.11 - Let X' be a countable infinite set. Choose $p \in X'$ and put $X = X' \setminus \{p\}$. We denote by G and G' the free groups generated by X and X', respectively. Let $\mathfrak{S}'_{\mathrm{A}}$ be the FLUS-convergence on G' generated by the set of (neutral) sequences $\mathrm{A} = \{S\langle p \rangle S^{-1} : S \in G^{\mathbf{N}}$ and S is one-to-one$\}$. Using *(iii)* of Lemma 3.3 and *(i)* of Lemma 3.2 it can be easily verified that $\mathfrak{S}'_{\mathrm{A}}$ satisfies axiom (H), too. According to Theorem 1 in [27], let $\mathfrak{S}'_C$ be a coarse FLUSH-convergence for G' with $\mathfrak{S}'_C \supset \mathfrak{S}'_{\mathrm{A}}$ and let us denote by $\mathfrak{S}_C$ the restriction of $\mathfrak{S}'_C$ to G. We claim that $\mathfrak{S}_C$ is discrete. Indeed, if a sequence $T \in G^{\mathbf{N}}$, $\mathfrak{S}_C$-converges to a point $y \in G'$, then T must be eventually constant. Otherwise, for some one-to-one subsequence $S = T \circ s$ of T, we would have $\mathrm{e} = \mathfrak{S}_C\text{-}\lim S\langle p \rangle S^{-1} = y p y^{-1}$ and hence $p = \mathrm{e}$. Note that in this way we have also proved that G is a closed subgroup of G' equipped with $\mathfrak{S}'_C$. Since $G = F(X)$, from Corollary 1 in [27] it follows that $\mathfrak{S}_C$ is not coarse.

REFERENCES

[1] P. ALEXANDROFF, P. URYSOHN, *Une condition nécessaire et suffisante pour qu'une class (L) soit une classe (D)*, C. R. Acad. Sci. Paris, *177* (1923), 1274.

[2] P. ANTOSIK, *Mappings from L-groups into topological groups*, I., Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys., *21* (1973), 155-160.

[3] P. ANTOSIK, *A diagonal theorem for nonnegative matrices and equicontinuous sequences of mapping*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys., *24* (1976), 855-860.

[4] P. ANTOSIK, *The uniform approach to Nikodym and Vitali - Hahn - Saks type theorems*, in: Proc. of the Second Conference on Convergence, held in Szczyrk, Inst. of Math., Polish Acad. Sci., Warsaw, 1981, 7-25.

[5] P. ANTOSIK, J. MIKUSIŃSKI, R. SIKORSKI, *Theory of distributions. The sequential approach*, Elsevier - Amsterdam, PWN - Warszawa, 1973.

[6] P. ANTOSIK, C. SWARTZ, *Matrix methods in Analysis*, Lecture Notes in Math., *1113*, Springer - Verlag, Berlin, New York, 1985.

[7] A. V. ARHANGEL'SKII, *The frequency spectrum of a topological space and the product operation*, (Russian), Trudy Moskovskogo matematičeskogo obščestva, *40* (1970), 171-206.

[8] Z. ARTSTEIN, *Limiting equations and stability of nonautonomous ordinary differential equations*. (Appendix A in [50]).

[9] A. R. BEDNAREK, J. MIKUSIŃSKI, *Convergence and topology*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys., *17* (1969), 437-442.

[10] E. ČECH, *Topological spaces*, Academia, Prague, 1966.

[11] W. W. COMFORT, D. GRANT, *Cardinal invariants, pseudocompactness and minimality: some recent advances in the topological theory of topological groups*, Topology Proc. *6* (1981), 227-265.

[12] D. DIKRANJAN, *Minimal topologies on Abelian groups*, Seminar notes, Univ. Padova, Padova 1983.

[13] D. DIKRANJAN, R. FRIČ, F. ZANOLIN, *On convergence groups with dense coarse subgroups*, Czechoslovak Math. J. (to appear).

[14] S. DOLECKI, *Tangency and differentiation: Some application of convergence theory*, Ann. Mat. Pura Appl. (IV) *130* (1982), 223-256.

[15] M. DOLCHER, *Topologie e strutture di convergenza*, Ann. Scuola Norm. Sup. Pisa, (III), *14* (1960), 63-92.

[16] R. M. DUDLEY, *On sequential convergence*, Trans. Amer. Math. Soc., *112* (1964), 483-507.

[17] S. P. FRANKLIN, *Spaces in which sequence suffice*, Fund. Math., *57* (1965), 107-115.

[18] FRÉCHET, *Sur quelques points du calcul fonctionell*, Rend. Circ. Mat. Palermo, *22* (1906), 1-74.

[19] R. FRIČ, *On convergence spaces and groups*, Math. Slovaca, *27* (1977), 323-330.

[20] R. FRIČ, D. C. KENT, *Completion of pseudo-topological groups*, Math. Nachr., *99* (1980), 99-103.

[21] R. FRIČ, M. GAVALEC, *On the Novák completion of convergence groups*, Commènt. Math. Univ. Carolin., *24* (1983), 341-347.

[22] R. FRIČ, V. KOUTNÍK, *Sequential convergence since Kanpur Conference*, in: General Topology and its Relations to Modern Analysis and Algebra, V (Proc. Fifth Prague Topological Sympos., 1981). Heldermann Verlag, Berlin, 1983, 442-447.

[23] R. FRIČ, V. KOUTNÍK, *Recent development in sequential convergence*, in: Convergence Structures and Applications, II (Proc. Schwerin Conference),

Abh. Akad. Wiss. DDR, Abt. Math. - Naturwiss. - Technik, *2* (1984), Akademie-Verlag, Berlin 1984, 37-46.

[24] R. FRIČ, F. ZANOLIN, *A convergence group having no completion*, in: Convergence Structures and Applications, II (Proc. Schwerin Conference), Abh. Akad. Wiss. DDR, Abt. Math. - Naturwiss. - Technik, *2* (1984), Akademie-Verlag, Berlin 1984, 47-48.

[25] R. FRIČ, F. ZANOLIN, *Remarks on sequential convergence in free groups*, in: Colloquium on Topology (Proc. Eger Conference, 1983), J. Bolyai Math. Soc., *41* (1983), 283-291.

[26] R. FRIČ, F. ZANOLIN, *Fine convergence in free groups*, Czechoslovak Math. J., *36* (1986), 134-139.

[27] R. FRIČ, F. ZANOLIN, *Coarse convergence groups*, in: Convergence Structures 1984 (Proc. Conference on Convergence, Bechyne 1984), Akademie - Verlag, Berlin 1985, 107-114.

[28] R. FRIČ, P. VOJTÁŠ, *The space* $^{\omega}\omega$ *in sequential convergence*, in: Convergence Structures 1984 (Proc. Conference on Convergence, Bechyne 1984), Akademie - Verlag, Berlin 1985, 95-106.

[29] S. GÄHLER, *Discrete convergence*, in: Convergence Structures 1984 (Proc. Conference on Convergence, Bechyne 1984), Akademie - Verlag, Berlin 1985, 127-136.

[30] W. GÄHLER, *Grundstrukturen der Analysis*, Vol. I, Berlin - Basel 1977, Vol. II, Berlin - Basel 1979.

[31] A. GOETZ, *Proximity like structures in convergence spaces*, Topology Conf., 1967, Arizona State Univ., Tempe 1968, 104-137.

[32] O. HAJEK, *Dynamical systems in the plane*, Academic Press, London, 1968.

[33] F. HAUSDORFF, *Gestufte Räume*, Fund. Math. *25* (1935), 486-502.

[34] H. HERRLICH, *Limit operators and topological coreflections*, Trans. Amer. Math. Soc. *146* (1969), 203-210.

[35] E. HEWITT, K. A. ROSS, *Abstract harmonic analysis* (Vol. I), Springer-Verlag, Berlin, Heidelberg, 1963.

[36] J. JAKUBÍK, *On convergence in linear spaces* (Slovak, Russian summary), Mat. - Fyz. Casopis Slovensk. Akad. Vied *6* (1956), 57-67.

[37] A. KAMIŃSKI, *On the Rènyi theory of conditional probabilities*, Stud. Math. *79* (1984), 151-191.

[38] L. V. KANTOROVIČ, B. Z. VULIH, A. G. PINSKER, *Functional analysis in semiordered spaces* (Russian), Moscow 1950.

[39] J. KISIŃSKI, *Convergence du type L*, Colloq. Math. 7 (1960), 205-211.

[40] C. KLIŚ, *An example of a noncomplete normed (K)-space*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys., *26* (1978), 415-420.

[41] G. KNEIS, *Completion of sequential convergence groups*, in Proc. Conf. Topology and Measure IV, Univ. Greifswald 1984, Part 1, 125-132.

[42] G. KNEIS, *Eine allgemeine Theorie der Vervollständingung un Čech-Stone-Kompaktifizierung*, Dr. Sc. Nat. Dissertation, Ernst-Maritz-Arndt-Universität Greifswald, Greifswald, 1983.

[43] V. KOUTNÍK, *On sequentially regular convergence spaces*, Czechoslovak Math. J., *17* (1967), 232-247.

[44] V. KOUTNÍK, *Completeness of sequential convergence groups*, Studia Math. *77* (1984), 454-464.

[45] V. KOUTNÍK, *Closure and topological sequential convergence*, in: Convergence Structures 1984 (Proc. Conference on Convergence, Bechyne 1984), Akademie-Verlag, Berlin 1985, 199-204.

[46] P. KRATOCHVÍL, *Multisequence and measures*, in: General Topology and its Relations to Modern Analysis and Algebra, IV (Proc. Fourth Prague Topological Sympos., 1976), Part B. Soc. Czechoslovak Math. - Phys., Prague 1977, 237-244.

[47] P. KRATOCHVÍL, *Sequential convergence as a partial operation,* in: General Topology and its Relations to Modern Analysis and Algebra, V (Proc. Fifth Prague Topological Sympos., 1981). Heidermann Verlag, Berlin, 1983), 442-447.

[48] K. KURATOWSKI, *Topology* (Vol. I), Academic Press, New York, 1966.

[49] W. F. LA MARTIN, *On the foundations of k-group theory,* Diss. Math., *146* (1977).

[50] J. P. LA SALLE, *The stability of dynamical systems,* SIAM, Philadelfia, 1976.

[51] W. A. J. LUXEMBURG, A. C. ZAANEN, *Riesz spaces* (Vol. I), North Holland, Tmsterdam - London, 1971.

[52] J. MIKUSIŃSKI, *Operational calculus,* Pergamon Press, Oxford, 1967.

[53] J. NOVÁK, *Sur les espaces (L) et sur les produits cartésiens (L),* Publ. Fac. Sci. Univ. Masaryk, Brno, *273* (1939), 1-28.

[54] J. NOVÁK, *Über die eindeutigen stetigen Erweiterungen stetiger Funktionen,* Czechoslovak. Math. J., *8* (1958), 344-355.

[55] J. NOVÁK, *Eine Bemerkung zum Begriff der topolotgischen Konvergenz-gruppen,* Celebrazioni Archimedee del sec. XX, Simposio di Topologia, Siracusa, 1964, 71-74.

[56] J. NOVÁK, *On convergence spaces and their sequential envelopes,* Czechoslovak Math. J., *15* (1965), 74-99.

[57] J. NOVÁK, *On convergence groups,* Czechoslovak Math. J., *20* (1970), 357-374.

[58] J. NOVÁK, *On some problems concerning convergence spaces and groups,* in: General Topology and its Relations to Modern Analysis and Algebra (Proc. Kanpur Topological Conf. 1968). Academia, Prague, 1971, 219-229.

[59] J. NOVÁK, *On completion of convergence commutative groups,* in: General Topology and its Relations to Modern Analysis and Algebra, III (Proc. Third Prague Topological Sympos., 1971). Academia, Prague, 1972, 335-340.

[60] J. NOVÁK, *On a free convergence group,* in: Proc. Conf. on Convergence Structures (Cameron Univ., 1980). Cameron Univ., Lawton, Oklahoma, 1980, 97-102.

[61] J. NOVÁK, L. MIŠÍK, *On L-spaces of continuous functions,* (Slovak), Mat.-fys. sb. Slov. akad. vied a um. *1* (1951), 1-17.

[62] P. NYIKOS, *Metrizability and the Fréchet-Urysohn property in topological groups,* Proc. Amer. Math. Soc., *83* (1981), 793-801.

[63] E. T. ORDMAN, *Free k-groups and free topological groups,* Gen. Topology Appl., *5* (1975), 205-219.

[64] E. PAP, *Functional analysis* (Serbo-Croatian), Novi Sad, 1983.

[65] O. SCHREIER, *L-gruppen,* Abh. Math. Seminar (1926), 15.

[66] U. SCHWANENGEL, *An example of a q-minimal precompact topological group containing a non-minimal closed normal subgroup,* Manuscripta Math., *27* (1979), 323-327.

[67] G. R. SELL, *Topological dynamics and ordinary differential equations,* Van Nostand Reinhold Co.. London, 1971.

[68] F. SIWIEC, *Generalizations of the first axiom of countability,* Rocky Mountain J. Math., *5* (1975), 1-60.

[69] P. URYSOHN, *Sur les classes (L) de M. Fréchet,* L'Enseign. Math., *25* (1926), 77-83.

[70] F. ZANOLIN, *Solution of a problem of Josef Novák about convergence groups,* Bollettino Un. Mat. Ital. (5), *14*-A (1977), 375-381.

[71] F. ZANOLIN, *A note on convergences in commutative groups,* Rend. Ist. Mat. Univ. Trieste, *13* (1981), 79-86.

[72] F. ZANOLIN, *Example of a convergence commutative group which is not separated,* Czechoslovak Math. J., *34* (1984), 169-171.