ON FORCED OSCILLATIONS OF LAGRANGIAN SYSTEMS (*)

by CARLO GRECO (in Bari) (**)

SOMMARIO. - *Si studia il problema dell'esistenza di infinite soluzioni periodiche per il sistema Lagrangiano* $\frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{q}} - \frac{\partial \mathfrak{L}}{\partial q} + f(t) = 0$ *(ove* $f(t)$ *è un termine «forzante» periodico). Si assume che il potenziale «cresca» in modo sopraquadratico all'infinito.*

SUMMARY. - *We study the existence of infinitely many periodic solutions of the Lagrangian system* $\frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{q}} - \frac{\partial \mathfrak{L}}{\partial q} + f(t) = 0$ *(where* $f(t)$ *is a periodic «forcing» term). We assume that the potential «grows» superquadratically at infinity.*

1. - Introduction

Let us consider a constrained mechanical system (with holonomous, bilateral, smooth constraints) embedded in a conservative field of forces; this situation has been studied in [5] (c. [6[for a particular case): these authors showed that, if the potential energy possesses a suitable superquadratic growth at infinity, then there are infinitely many (free) oscillations of any fixed period T.

(*) Pervenuto in Redazione l'8 maggio 1986. Lavoro eseguito col contributo dello GNAFA (CNR), fondi 40-60%.

(**) Indirizzo dell'Autore: Dipartimento di Matematica dell'Università degli Studi - Via G. Fortunato - 70125 Bari (Italy).

In this paper we study the case of forced oscillations, i. e. we assume that the mechanical system it also subject to a T-periodic external force.

The main result (c. Th. 1.1) shows, roughly speaking, that, if the potential energy grows in a controlled superquadratic way, and suitable symmetry assumptions hold, then there are again infinitely many forced T-periodic oscillations.

Let us now introduce some notations. Let

$$\mathcal{L}(q,\xi) = \tfrac{1}{2}\sum_{i,j=1}^{N} a_{ij}(q)\,\xi_i\xi_j + \sum_{i=1}^{N} b_i(q)\,\xi_i + c(q) - V(q) \quad q,\xi \in \mathbf{R}^N$$

be the Lagrangian function (a_{ij}, b_i, c and V are C^1 real functions) and let $f:\mathbf{R}\to\mathbf{R}^N$ be continuous and T-periodic; then we look for T-periodic, C^2-solutions $q = q(t)$ of the following system of ordinary differential equations:

$$\text{(L)} \qquad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\xi}(q,\dot{q}) - \frac{\partial\mathcal{L}}{\partial q}(q,\dot{q}) + f(t) = 0\,.$$

We indicate by pq and by $|p|$ $(p, q \in \mathbf{R}^N)$ the scalar product and the norm in $\mathbf{R}^N$ and we set

$$a(q)(\xi) = \Big(\sum_{j=1}^{N} a_{ij}(q)\,\xi_i\Big)_{i=1,2,\dots,N}$$

$$b(q) \quad = (b_i(q))_{i=1,2,\dots,N}\,.$$

We suppose that the functions $a(q)$, $c(q)$ and $V(q)$ are even, and the function $b(q)$ is odd. Mereover we assume that:

(1.1) there exist $\vartheta \in \,]0, \tfrac{1}{2}[$ and $R > 0$ such that $V(q) \le \vartheta V'(q)q$ for every $|q| \ge R$

(1.2) $V(q) \le c_1|q|^\nu + c_2$, where $c_1, c_2 \ge 0$ and $2 < \nu < 4/\vartheta - 2$

(1.3) $a(q)$ is symmetric and positive-defined: $a(q)(\xi)\,\xi \ge \lambda|\xi|^2$, $\lambda > 0$.

(1.4) there exists $\beta \in \,]2 - \frac{1}{\vartheta}, 0[$ such that
$a'(q)[q,\xi]\,\xi + \beta a(q)(\xi)\,\xi \le 0$

(1.5) there exists $M > 0$ such that $|b(q)| \le M$, $|b'(q)(\xi)| \le M$

(1.6) $c(q) \ge 0$ and there exists $\hat{\vartheta} > \vartheta$ such that $\hat{\vartheta} c'(q)\,q \le c(q)$ for every $|q| \ge R$.

Then the following theorem holds:

THEOREM 1.1 - *If* (1.1)-(1.6) *hold, then the system* (L) *has infinitely many T-periodic, C^2, solutions.*

We take $T = 2\pi$ to simplify the notations; $\|\cdot\|_p$ is the norm in the space $L^p = L^p((0,2\pi),\mathbf{R}^N)$ and $E = H^{1,2}(S^1,\mathbf{R}^N)$ is the Sobolev space of 2π-periodic absolutely continuous functions $u:\mathbf{R}\to\mathbf{R}^N$ with square integrable derivative. The E-norm is $\|u\|^2 = \|u\|_2^2 + \|\dot u\|_2^2$.

Let us consider the functional $I: E\to\mathbf{R}$

(1.7) $\quad I(u) = \frac{1}{2}\int a(u)(\dot u)\,\dot u + \int b(u)\,\dot u + \int c(u) - \int V(u) - \int fu$

(here and in the sequel the integrals are extended to $(0,2\pi)$). The functional I is C^1 and we have:

(1.8) $$\langle I'(u),v\rangle = \int a(u)(\dot u)\,\dot u + \frac{1}{2}\int a'(u)[v,\dot u]\,\dot u + \int b(u)\,\dot v + \int b'(u)(v)\,\dot u + \int c'(u)\,v - \int V'(u)\,v - \int fv.$$

The critical points of I (i.e. the zeros of I') are the 2π-periodic C^2 solutions of (L).

If $f\equiv 0$, then the functional I is invariant for the $\mathbf{Z}_2$-action, i.e. $I(u) = I(-u)$.

In our case the forcing term $f(t)$ «destroys» this symmetry.

In recent years some autors have studied perturbed symmetric functionals (c. [1]-[4], [7]-[10]); they applied their results to elliptic problems, second order Hamiltonian systems, and general Hamiltonian systems. Our proof of Theorem 1.1 is based on some tools introduced in [9] (1).

REMARK - Hypothesis (1.1) is the classical superquadratic condition; it implies

(1.9) $\quad V(q) \geq c_3|q|^{\mu} + c_4,$

where $\mu = 1/\vartheta$ and $c_3, c_4 > 0$. (1.2) is a rather technical assumption; it probably can be removed (on this subject cf. [3], [4]). (1.3) has an obvious physical meaning, while (1.4) is similar to (A_2) in [5]. The bilinear form $a'(q)[\cdot,\cdot]$ in (1.4) is, of course, the Fréchet-derivative of $q\to a(q)$.

Notice that (1.4) implies:

(1.10) $\quad a(q)(\xi)\,\xi \leq c_5|q|^{-\beta}|\xi|^2 + c_6.$

Finally (1.5) and (1.6) are technical limitations to the growth of the kinetic energy with respect to Lagrangian coordinates. In particular (1.6) implies

(1.11) $\quad c(q) \leq c_7|q|^{\hat\mu} + c_8,$

where $\hat\mu = 1/\hat\vartheta$.

(1) Pisani and Tucci have recently presented a result like theorem 1.1 (c. [8]); without symmetry assumptions on $\mathcal{L}$. On the other hand, if we think, for instance, of the fixed constraints case, then A_3) and A_4) in [8] appear unnecessary; moreover (1.1) is weaker then V_2) (because $4/\vartheta - 2 > 6 > 4$). Finally we should point out that the physical meaning of (1.1) - (1.6) is rather transparent.

2. - Proof of the result

Following [9] we consider a modified functional $\hat{I}$ which is, unlike I, almost invariant in a suitable sense (c. Lemma 2.4).

The critical points of $\hat{I}$ are related to those of I by Lemma 2.5.

Let $\chi \in C^\infty(\mathbf{R})$ such that $0 \leq \chi \leq 1, \chi' \leq 0$, $\chi(s) = 1$ if $s \leq 1$ and $\chi(s) = 0$ if $s \geq 2$. We set

$$\varphi(u) = (I(u)^2 + 1)^{1/2}\ ,\ \psi(u) = \chi(\frac{1}{\varphi(u)} \| u \|_\mu^\mu)$$

and

$$\hat{I}(u) = \tfrac{1}{2} \int a(u)(\dot{u})\,\dot{u} + \int b(u)\,\dot{u} + \int c(u) - \int V(u) - \psi(u) \int fu. \tag{2.1}$$

Then $\hat{I} \in C^1$ and

$$\begin{aligned} \langle \hat{I}'(u), v \rangle &= \int a(u)(\dot{u})\,\dot{v} + \tfrac{1}{2} \int a'(u)[v, \dot{u}]\,\dot{u} + \int b(u)\,\dot{v} + \\ &+ \int b'(u)(v)\,\dot{u} + \int c'(u)\,v - \int V'(u)\,v - \int (\psi(u)\,v + \\ &+ \langle \psi'(u), v \rangle\, u)\, f, \end{aligned} \tag{2.2}$$

where, of course,

$$\begin{aligned} \langle \psi'(u), v \rangle &= \chi'(\frac{1}{\varphi(u)} \| u \|_\mu^\mu)\,(\mu \varphi(u)^{-1} \int | u |^{\mu - 2} uv - \\ &- \varphi(u)^{-3} I(u)\,\langle I'(u), v \rangle\, \| u \|_\mu^\mu). \end{aligned} \tag{2.3}$$

In the sequel $a_1, a_2, \ldots$ will denote positive constants.

LEMMA 2.1 - *$\varphi(u) \neq 0$ implies $| \int fu | \leq a_1(I(u)^\vartheta + 1)$.*

LEMMA 2.2 - *There exists $\alpha > 0$ such that if $c \geq \alpha, \varphi(u) \neq 0$, and $\hat{I}(u) \geq c$, then $I(u) \geq a_2 c$.*

The proof of Lemmas 2.1 and 2.2 is as in [9], Lemmas 1.13 and 1.25. Now, let us check the well-known Palais-Smale condition for the functional $\hat{I}$.

LEMMA 2.3 - *There exists $\bar{\beta} > 0$ such that if*

$$(u_n)_n \subset E,\ \hat{I}(u_n) \to c \geq \bar{\beta} \text{ and } \hat{I}'(u_n) \to 0,$$

then $(u_n)_n$ has a converging subsequence.

Proof. Let $\bar{\beta} > 0$ free for the moment, and $c \geq \bar{\beta}$. Let $(u_n)_n$ be as i nthe statement of the lemma. Then we can suppose (we write for simplicity u instead of u_n) $|\langle \hat{I}'(u), u \rangle| \leq \| u \|$; it gives (because of (2.3) and (1.8)):

$$-(1 + T_1(u))\,(\int a(u)(\dot{u})\,\dot{u} + \tfrac{1}{2} \int a'(u)[u, \dot{u}]\,\dot{u} + \int b(u)\,\dot{u} +$$

$$(2.4)\qquad + \int b'(u)(u)(\dot u) + \int c'(u)\, u - \int V'(u)u) + \\ + (\psi(u) + T_1(u)) \int fu + T_2(u) \| u \|_\mu^\mu \le \| u \|,$$

where

$$T_1(u) = \chi'(\frac{1}{\varphi(u)} \| u \|_\mu^\mu) \| u \|_\mu^\mu \varphi(u)^{-3} I(u) \int fu$$

$$T_2(u) = \mu\chi'(\frac{1}{\varphi(u)} \| u \|_\mu^\mu)\, \varphi(u)^{-1} \int fu .$$

Set $\bar\vartheta = \vartheta(1+\varepsilon)$; we may choose $\varepsilon > 0$ such that $\bar\vartheta < \hat\vartheta$ and $\frac12 - \bar\vartheta(1 - \frac{\beta}{2}) > 0$. Using (1.1), (1.3)-(1.6) and (1.9), the (2.3) becomes:

$$(\tfrac12 - \bar\vartheta(1 - \frac{\beta}{2}))\, \lambda \| \dot u \|_2^2 \le \frac{\bar\vartheta}{1 + T_1(u)} \| u \| -$$

$$(2.5)\qquad - \frac{\bar\vartheta(\psi(u) + T_1(u))}{1 + T_1(u)} \int fu -$$

$$- \frac{\bar\vartheta T_2(u)}{1 + T_1(u)} \| u \|_\mu^\mu + a_3 \| \dot u \|_2 - a_4 \| u \|_\mu^\mu + a_5 .$$

Arguing as in [9], we can show that $T_1(u)$ and $T_2(u)$ are small if $\bar\beta$ is big enough. Then we can suppose that $\dfrac{\bar\vartheta T_2(u)}{1 + T_1(u)} + a_4 > 0$, so (2.5) gives $\| \dot u \|_2^2 \le a_6 \| u \| - a_7 \| u \|_\mu^\mu + a_8$, and therefore

$$\| u \|^2 \le a_6 \| u \| - a_7 \| u \|_\mu^\mu + \| u \|_2^2 + a_8 .$$

Fix $0 < \varepsilon_1 < a_7$ and set $s = \mu/(\mu - 1)$. From the Hölder-Young inequality there exists a_9 such that $\| u \|_2^2 \le \varepsilon_1 \| u \|_\mu^\mu + a_9 \| u \|_2^s$. Then we get $\| u \|^2 \le a_6 \| u \| - (a_7 - \varepsilon_1) \| u \|_\mu^\mu + a_9 \| u \|_2^s + a_8$. Because $s < 2$, it follows (remember that $u = u_n$)$: (u_n)_n$ is bounded in H.

At this point, standard arguments permit us to achieve the conclusion.

We can now state two lemmas. The proofs are as in [9], Lemma 1.18 and 1.29.

LEMMA 2.4 - *There exists $a_{10} > 0$ such that*

$$| \hat I(u) - \hat I(-u) | \le a_{10}(| \hat I(u) |^\mu + 1)$$

for every $u \in H$.

LEMMA 2.5 - *There exists* $\gamma > 0$ *such that* $\hat{I}(u) \geq \gamma, \hat{I}'(u) = 0$ *imply* $\hat{I}(u) = I(u)$ *and* $I'(u) = 0$.

Let $e_1, \ldots, e_N$ be the standard basis in $\mathbf{R}^N$; for $j \geq 1$, we set $V_{jm}(t) = (\sin jt)\, e_m$ if $1 \leq m \leq N$, and $V_{jm}(t) = (\cos jt)\, e_{m-N}$ if $N + 1 \leq m \leq 2N$. Moreover, for $k \geq 1$ and $1 \leq i \leq 2N$, we set

$$E_{ki} = \mathbf{R}^N \oplus \operatorname{span}\{V_{jm} \mid 1 \leq j \leq k,\ 1 \leq m \leq i\}.$$

It is easy to check that there exists $R_{ki} > 0$ such that $\hat{I}(u) < 0$ for every $u \in E_{ki}, \|u\| = R_{ki}$. Indeed, from (1.10), (1.5), (1.11) and (1.9), it follows:

$$\hat{I}(u) \leq \frac{c_5}{2} \int |u|^{-\beta} |\dot{u}|^2 + 2\pi c_6 + M \int |\dot{u}| + c_7 \int |u|^{\hat{\mu}} +$$

$$+ 2\pi c_8 - c_3 \int |u|^{\mu} - 2\pi c_4 \leq \frac{c_5}{2} \hat{c}\, \|u\|^{2-\beta} + a_{11} \|u\| +$$

$$+ a_{12} \|u\|^{\hat{\mu}}_{\hat{\mu}} - a_{13} \|u\|^{\mu}_{\mu} + a_{14}$$

where $\hat{c} = \max\{\int |u|^{-\beta} |\dot{u}|^2 \mid u \in E_{ki}, \|u\| = 1\}$. Since the norms on E_{ki} are equivalent, and $\hat{\mu} < \mu, 2 - \beta < \mu$, we have $\hat{I}(u) < 0$ if $\|u\|$ is large enough.

We suppose that

$$R_{ki} < R_{ki+1} (k \geq 1, 1 \leq i \leq 2N - 1), R_{k,2N} < R_{k+1\,1}.$$

Set $B_r = \{u \in E \mid \|u\| \leq r\}$ (for every $r > 0$), $D_{ki} = E_{ki} \cap B_{R_{ki}}$ and:

$$\Gamma_{ki} = \{h \in C(D_{ki}, E) \mid h \text{ is odd and } h(u) = u \text{ if } \|u\| = R_{ki}\}$$

$$b_{ki} = \inf_{h \in \Gamma_{ki}} \max_{u \in D_{ki}} \hat{I}(h(u))$$

LEMMA 2.6 - *There exist* a_{15}, a_{16} *such that* $b_{ki} \geq a_{15} K^{(\nu+2)/(\nu-2)} - a_{16}$.

Proof. We follow a simple comparison argument. Let $\varepsilon > 0$ be such that $M\varepsilon < \lambda/2$. There exists, of course, $a_{17} > 0$ such that $|x| \leq \varepsilon |x|^2 + a_{17}$ and $|x| \leq V(x) + a_{17}$ for every $x \in \mathbf{R}^N$. Then (1.3), (1.5) and (1.6) give:

$$\hat{I}(u) \geq \frac{\lambda}{2} \int |\dot{u}|^2 - M \int |\dot{u}| - \int V(u) - \|f\|_\infty \int |u|^2 \geq \left(\frac{\lambda}{2} M\varepsilon\right) \int |\dot{u}|^2 -$$

$$- (1 + \|f\|_\infty) \int V(u) - a_{18} =$$

$$= (\frac{\lambda}{2} - M\varepsilon)\,(\int |\dot u|^2 - \frac{1 + \|f\|_\infty}{\lambda/2 - \varepsilon M} \int V(u)) - a_{18}.$$

Set $J(u) = \int |\dot u|^2 - \frac{1 + \|f\|_\infty}{\lambda/2 - \varepsilon M} \int V(u)$ and $b^*_{ki} = \inf_{h \in \Gamma_{ki}} \max_{u \in D_{ki}} J(h(u))$.

Arguing as in [9], we get $b^*_{ki} \geq a_{19}\, k^{\frac{\nu+2}{\nu-2}}$; since

$b_{ki} \geq (\frac{\lambda}{2} - \varepsilon)\, b^*_{ki} - a_{18}$, the lemma is proved.

Proof of Theorem 1.1 - Set (we replace $V_{k,2N+1}, R_{k,2N+1}$ with $V_{k+1\,1}$, $R_{k+1\,1}$)

$$U_{ki} = \{u = \tau v_{k,i+1} + Q \mid \tau \in [0, R_{k,i+1}], Q \in E_{ki} \cap B_{R_{k,i+1}},$$
$$\|u\| \leq R_{k,i+1}\}$$
$$\Lambda_{ki} = \{H \in C(U_{ki}, E) \mid H_{|D_{ki}} \in \Gamma_{ki} \text{ and } H(u) = u$$
$$\text{if } \|u\| = R_{k,i+1} \text{ or } u \in (B_{R_{k,i+1}} \setminus B_{R_{ki}}) \cap E_{ki}\}$$
$$c_{ki} = \inf_{H \in \Lambda_{ki}} \max_{u \in U_{ki}} \hat I(H(u)).$$

In general we have $b_{ki} \leq c_{ki}$. Suppose that $b_{ki} < c_{ki}$ for suitable (k, i), fix $\delta \in \mathbf{R}$ with $0 < \delta < c_{ki} - b_{ki}$ and set

$$\Lambda_{ki}(\delta) = \{H \in \Lambda_{ki} \mid \sup_{u \in D_{ki}} \hat I(H(u)) \leq b_{ki} + \delta\}$$

$$c_{ki}(\delta) = \inf_{H \in \Lambda_{ki}(\delta)} \sup_{u \in D_{ki}} \hat I(H(u)).$$

Then one can check that $c_{ki}(\delta)$ is a critical level of $\hat I$ (by standard deformation argument; c. Lemma 2.3).

Now we claim that there exists infinitely many pairs (k, i) (with $k \in N, i = 1, 2, \ldots, N$) such that $b_{ki} < c_{ki}$; for if not, arguing as in [9] Lemma 1.64 and using the Lemma 2.4, we get $b_{ki} \leq a_{20}\, k^{\mu/(\mu-1)}$. But this is impossible because of Lemma 2.6 and the assumption $\nu < 4\mu - 2$ in (1.2).

Then there exists infinitely many critical points of the functional $\hat I$. Finally we observe that $c_{ki}(\delta) \to +\infty$, therefore the Lemma 2.5 give us the result.

Added in proof. While this note was in press, we have known that Yiming Long (*«Multiple solutions of perturbed superquadratic second order Hamiltonian systems»*, Math. Research Center, Tech. Summ. Report # 2963, Univ. of Wisconsin - Madison, 1987) has proved the existence of infinitely many solutions of (L) under assumptions weaker than ours. In particular, the symmetry assumptions on the coefficients of $\mathcal{L}$ and (1.2) are unnecessary, and a weaker version of (1.5) is required.

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