ON LOCAL T-TIGHTNESS (*)

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SOMMARIO. - *In questa nota si dà una nozione di T-tightness locale e si studiano alcune sue proprietà.*

SUMMARY. - *In this note a notion of local T-tightness is given and some properties of it are studied.*

I. Juhàsz in [8] introduced the notion of T-tightness as a variation on the classical notion of tightness. The definition of this new cardinal function given in [8] is of global nature, while it is well known that the tightness can be considered as a function of local character. In this note we present a notion of local T-tightness that agrees with the global one i.e., for any space $X, T(X) = \sup_{x \in X} T(x, X)$ (see definitions below).

For notations not explicitly mentioned here we refer to [5], [6]. m, ρ will denote cardinal numbers and α, β ordinal numbers. m^+ is the successor cardinal of m and a cardinal number is assumed to be an initial ordinal. The cardinality of a set S is denoted by $|S|$. All topological spaces considered here are assumed to be T_1. We recall the following:

DEFINITION 1. - *Let X be a topological space and A a subset of X. The tightness of A with respect to X, denoted by $t(A,X)$, is the*

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smallest cardinal number m with the property that for any set $C \subset X$, *for which* $A \cap \bar{C} \neq \phi$, *there exists a set* $C_0 \subset C$ *such that* $A \cap \bar{C}_0 \neq \phi$ *and* $|C_0| \leq m$. *If* $A = \{x\}$ *we write* $t(x,X)$ *instead of* $t(\{x\},X)$. *The tightness of X, denoted by* $t(X)$, *is defined as* $t(X) = \sup_{x \in X} t(x,X)$.

DEFINITION 2. (see [8] - *Let X be a topological space. The T-tightness of X, denoted by* $T(X)$, *is the smallest cardinal number m with the property that, for any increasing sequence* $\{F_\alpha\}_{\alpha \in \rho}$ *of closed subsets of X such that* $\mathrm{cf}(\rho) > m$, *the set* $\bigcup_{\alpha \in \rho} F_\alpha$ *is closed.*

For a discussion on the behaviour of the T-tightness under the usual topological operations see [3]. Various relations between the T-tightness and other cardinal functions can be found in [4] and [8].

We now pose the following:

DEFINITION 3. - *Let X be a topological space and A a subset of X. The T-tightness of A with respect to X, denoted by* $T(A,X)$, *is the smallest cardinal number m with the property that, for any increasing sequence* $\{F_\alpha\}_{\alpha \in \rho}$ *of closed subsets of X such that* $\mathrm{cf}(\rho) > m$, $A \cap \overline{\bigcup_{\alpha \in \rho} F_\alpha} = \phi$ *whenever* $A \cap \bigcup_{\alpha \in \rho} F_\alpha = \phi$.

It is easy to see that for any space $X, T(A,X) \leq t(A,X)$.

If $A = \{x\}$ we write $T(x,X)$ instead of $T(\{x\},X)$.

PROPOSITION 1. - *If X is a topological space then* $T(X) = \sup_{x \in X} T(x,X)$.

Proof. It is clear from the definitions that, for any

$$x \in T, T(x,X) \leq T(X).$$

So we need to show that $T(X) \leq \sup_{x \in X} T(x,X)$. Let $\{F_\alpha\}_{\alpha \in \rho}$ be an increasing sequence of closed subsets of X such that $\mathrm{cf}(\rho) > m$ where

$$m = \sup_{x \in X} T(x,X).$$

If $x \notin \bigcup_{\alpha \in \rho} F_\alpha$ then $x \notin \overline{\bigcup_{\alpha \in \rho} F_\alpha}$, by the fact that $T(x,X) \leq m$. This implies that the set $\bigcup_{\alpha \in \rho} F_\alpha$ is closed and therefore $T(X) \leq m$.

PROPOSITION 2. - *Let X be a topological space and* A_1, A_2 *two subspaces of X such that* $A_1 \subset A_2$. *If for any set* $F \subset A_2 \setminus A_1$, *that is closed in* A_2, *there exist two disjoint open sets in X containing respectively* A_1 *and F, then* $T(A_1,X) \leq T(A_1,A_2)\, T(A_2,X)$.

Proof. Let $m = T(A_1, A_2)\, T(A_2, X)$. Let $\{F_\alpha\}_{\alpha\in\rho}$ be an increasing sequence of closed subsets of X such that $A_1 \cap \left(\bigcup_{\alpha\in\rho} F_\alpha\right) = \phi$ and $\mathrm{cf}(\rho) > m$. The family $\{F_\alpha \cap A_2\}_{\alpha\in\rho}$ is an increasing sequence of closed subsets of A_2 and so $A_1 \cap \overline{\bigcup_{\alpha\in\rho} (F_\alpha \cap A_2)}^{A_2} = \phi$, by the fact that $T(A_1, A_2) \leq m$. Thanks to the hypothesis there exist two disjoint open sets U and V such that $A_1 \subset U$ and $\overline{\bigcup_{\alpha\in\rho} (F_\alpha \cap A_2)}^{A_2} \subset V$. The family $\{F_\alpha - V\}_{\alpha\in\rho}$ is an increasing sequence of closed subsets of X and $A_2 \cap \left[\bigcup_{\alpha\in\rho} (F_\alpha - V)\right] = \phi$. As $T(A_2, X) \leq m$ we have $A_2 \cap \overline{\bigcup_{\alpha\in\rho} (F_\alpha - V)} = \phi$, i. e. there exists an open set W such that $A_2 \subset W$ and

$$W \cap \left[\bigcup_{\alpha\in\rho} (F_\alpha - V)\right] = \phi.$$

The set $U \cap W$ is an open neighborhood of A_1 and it is easy to see that $U \cap W \cap \left(\bigcup_{\alpha\in\rho} F_\alpha\right) = \phi$. This means that $A_1 \cap \overline{\bigcup_{\alpha\in\rho} F_\alpha} = \phi$ and therefore $T(A_1, X) \leq m$.

COROLLARY 1.

a. *If X is a Hausdorff space and F_1, F_2 two compact subspaces of X such that $F_1 \subset F_2$ then $T(F_1, X) \leq T(F_1, F_2)\, T(F_2, X)$;*

b. *If X is a regular space and F is a closed subspace of X then, for any $x \in F, T(x, X) \leq T(x, F)\, T(F, X)$.*

PROPOSITION 3. - *Let X and Y be topological spaces. If $f : X \to Y$ is a closed map and A is a subset of Y then $T(f^{-1}(A), X) \leq T(A, Y)$.*

Proof. It easily follows from the definitions.

From Prop. 1, Cor. 1b. and Prop. 3 we can deduce the following:

PROPOSITION 4. - *Let X a regular space and Y a topological space. If $f : X \to Y$ is a closed map then* $T(X) \leq \max\left[T(Y), \sup_{y\in Y} T(F^{-1}(y))\right]$.

REMARK - Prop. 4 as been already proved in [3] Th. 3.2.

In the next Propositions some sufficient conditions in order to guarantee the equality between the local T-tightness and the local tightness are given.

PROPOSITION 5. - *Let X be a topological space and A a subset of X. If $t(A, X)$ is a successor cardinal then $T(A, X) = t(A, X)$.*

Proof. Since $t(A, X)$ is a successor cardinal, say m^+, there must exist a set $C \subset X$ such that $|C| = m^+$, $A \cap \overline{C} \neq \phi$, and for any $B \subset C$

for which $|B| \leq m$, $A \cap \bar{B} = \phi$. Well ordering the set C we can write $C = \{x_\alpha : \alpha \in m^+\}$. Let $F_\alpha = \overline{\{x_\beta : \beta \in \alpha\}}$. The family $\{F_\alpha\}_{\alpha \in m^+}$ is an increasing sequence of closed subsets of X such that $A \cap \left(\bigcup_{\alpha \in m^+} F_\alpha\right) = \phi$ but $A \cap \overline{\bigcup_{\alpha \in m^+} F_\alpha} \neq \phi$. This implies $T(A, X) \geq \mathrm{cf}(m^+) = m^+$ and therefore $T(A, X) = t(A, X)$.

PROPOSITION 6. - *Let X be a topological space and A a subset of X. If, for any set $C \subset X$, $A \cap \bar{C} = \phi$ whenever $A \cap (C)_{\aleph_\omega} = \phi$, then $T(A, X) = t(A, X)$. $(C)_{\aleph_\omega}$ is the set $\bigcup\{\bar{B} : B \subset C, |B| < \aleph_\omega\}$.*

Proof. We proceede by contraddiction. Let us assume

$$T(A, X) < t(A, X)$$

and let $T(A, X) = m$. Since $t(A, X) > m$ there is some $C \subset X$ for which $A \cap \bar{C} \neq \phi$ but, for any $B \subset C$ such that $|B| \leq m$, $A \cap \bar{B} = \phi$. By the hypothesis we have $A \cap (C)_{\aleph_\omega} \neq \phi$ and so there is some $C_0 \subset C$ such that $A \cap \bar{C}_0 \neq \phi$ and $|C_0| < \aleph_\omega$. We can assume that C_0 has minimal cardinality, i.e. A does not intersect the closure of any subset of C_0 whose cardinality is less than $|C_0|$. Let $|C_0| = \rho$. Clearly ρ is a successor cardinal and an argument similar to that used in the proof of Prop. 5 shows that $T(A, X) \geq \rho$. This is a contradiction because $\rho > m$.

In [8] it is proved that $T(X) = t(X)$ for a compact Hausdorff space X. We do not know if the same holds locally, i.e. $T(A, X) = t(A, X)$ for any $A \subset X$, or, in particular, $T(x, X) = t(x, X)$ for any $x \in X$.

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