

ON THE COUNTABILITY OF A SET OF REAL NUMBERS (*)

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SOMMARIO. - *Si dimostra che se ogni sottoinsieme numerabile di un insieme S di numeri reali è sommabile, allora S è numerabile. Poggiando su ciò si dà una originale dimostrazione della numerabilità di insiemi di sequenze ortonormali.*

SUMMARY. - *It is shown that if every countable subset of a set S of real numbers is summable then S is countable. Based on this an original proof of the countability of any set of orthonormal sequence is given.*

To the knowledge of the author Theorem 1 below which states a rather basic property of real numbers has not appeared in the literature. Based on Theorem 1 a new set-theoretical proof (in contradistinction to the known function-theoretic proofs) is given of the countability of any orthonormal set of infinite real sequences.

In what follows ω stands for the first infinite ordinal and ω_1 for the first uncountable ordinal. We need the following obvious lemma.

LEMMA 1. - *Let W be a set of real numbers such that W is well ordered by the usual order of the real numbers. Then W is countable.*

Proof. Let us assume to the contrary that W is uncountable. But

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then uncountably many rational numbers can be chosen each located between a pair of consecutive elements of W . This contradicts the fact that there are countably many rational numbers.

THEOREM 1. - *Let S be a set of real numbers such that every countable subset of S is summable. Then S is countable.*

Proof. Let us assume to the contrary that S is uncountable. But then S would have a subset S_1 all of whose elements are of the same sign, say, positive and such that S_1 is in one-to-one correspondence with ω_1 . Let us recall that ω_1 has uncountably many countable initial segments and that the set of these initial segments is well ordered by the set-theoretical inclusion. Since every countable subset of S_1 is a summable subset of positive real numbers we may consider the set W whose elements are the sums of those subsets of S_1 which (by virtue of the above one-to-one correspondence) are in one-to-one correspondence with an initial segment of ω_1 . But then, clearly, W would be an uncountable subset of the real numbers which is well ordered by the usual order of the real numbers, contradicting Lemma 1. Thus, our assumption is false and the Theorem is proved.

As shown below, Theorem 1 provides a novel (and perhaps quite original) proof of the following classical result [1, p. 237] of the Hilbert Space of the square summable (real) sequences.

THEOREM 2. - *Let N be an orthonormal set of (countably) infinite sequences of real numbers. Then N is countable.*

Proof. Let us assume to the contrary that N is uncountable. Let N_1 be a subset of N such that N_1 is in one-to-one correspondence with ω_1 . Let M be an ω_1 by ω matrix whose rows are the elements $r_1, r_2, r_3, \dots, r_i, \dots$ (for $i \in \omega_1$) of N_1 . Let M be given by:

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots \\ \dots & & & \dots & & \dots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} & \dots \\ \dots & & & \dots & & \dots \end{bmatrix}$$

for $i \in \omega_1$ and $j \in \omega$.

Let us denote by e_1 the ω by 1 matrix which is the transpose of the matrix $(1, 0, 0, \dots)$. Clearly, the inner product $r_i \cdot e_1$ is the length of the projection of e_1 on r_i . Moreover,

$$(1) \quad r_i \cdot e_1 = a_{i1} \text{ for } i \in \omega_1.$$

Now, let us consider the set S of the squares of the elements of the first column of M , i.e.,

$$(2) \quad S = \{a_{11}^2, a_{21}^2, a_{31}^2, \dots, a_{i1}^2, \dots\} \text{ for } i \in \omega_1.$$

We claim that every countable subset C of S is summable. This is because from (1) it follows that the elements of C are the squares of the lengths of the projections of e_1 on the elements of a denumerable orthonormal set C . Thus, by Bessel's inequality [2, pp. 15-17] the sum of the squares of the elements of C is less than or equal to 1 which is equal to the square of the length of e_1 . Hence, every countable subset of S given by (2) is summable and therefore, by Theorem 1, the set S is countable. But then since the first column of M has at most twice as many distinct elements as S does, we see that the first column of M has countably many distinct elements. Consequently, for a countable ordinal u_1 we have:

$$(3) \quad a_{u_1,1} = a_{i1} \text{ with } u_1 \in \omega_1 \text{ for } i > u_1.$$

Now, let us consider e_j instead of e_1 where e_j is the transpose of the matrix $(0, 0, \dots, 1, 0, 0, \dots)$ with 1 at the j -th coordinate. But then with a reasoning analogous to the case of e_1 we derive (in accordance with (3)) that for a countable ordinal u_j we have:

$$(4) \quad a_{u_j,j} = a_{ij} \text{ with } u_j \in \omega_1 \text{ and } j \in \omega \text{ for } i > u_j.$$

Finally, let $u = \sup \{u_j \mid j \in \omega\}$. Clearly, u is a denumerable ordinal and from (4) it follows that for the rows r_v and r_u of M , we have:

$$(5) \quad r_v = r_u \text{ for } v \in \omega_1 \text{ with } v > u.$$

But then (5) implies that M has only countably many distinct rows which is a contradiction since by our assumption M has uncountably many (in fact ω_1) distinct rows. Thus, our assumption is false and the Theorem is proved.

REFERENCES

- [1] COOKE, R. G., *Infinite matrices and sequence spaces*, Dover Pub. N. Y. 1965.
- [2] HALMOS, P. R., *Introduction to Hilbert space*, Chelsea Pub. N. Y. 1957.