NONLINEAR MONOTONE BOUNDARY CONDITIONS FOR PARABOLIC EQUATIONS (*)

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SOMMARIO. - *In questo lavoro viene trattato il problema* (0.1) - (0.3), *con l operatore monotono. I principali risultati sono contenuti nei teoremi concernenti l'esistenza, l'unicità, regolarità e comportamento asintotico delle soluzioni.*

SUMMARY. - *This paper is handling problem* (0.1) - (0.3) *with l a monotone operator. The main results are contained in the theorems concerning existence, uniqueness, regularity and asymptotic behaviour of the solutions.*

0. - Introduction

In this paper we shall investigate the problem:

$$(0.1) \qquad \frac{\partial u}{\partial t} + \sum_{i,j=0}^{n} (-1)^j \frac{\partial^j}{\partial x^j}\left[a_{ij}(x)\frac{\partial^i u}{\partial x^i}\right] + A(x,u) = f(t,x),$$

for $t > 0$, $0 < x < 1$,

(*) Pervenuto in Redazione il 24 aprile 1985.
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$$(0.2)\quad \begin{pmatrix} (M_{2n-1}u)\ (t,0) \\ -(M_{2n-1}u)\ (t,1) \\ \cdots\cdots \\ (M_{2n-k}u)\ (t,0) \\ -(M_{2n-k}u)\ (t,1) \\ \cdots\cdots \\ (M_n u)\ (t,0) \\ -(M_n u)\ (t,1) \end{pmatrix} \in l \left(\begin{pmatrix} u(t,0) \\ u(t,1) \\ \cdots\cdots \\ \dfrac{\partial^{k-1} u}{\partial x^{k-1}}\ (t,0) \\ \dfrac{\partial^{k-1} u}{\partial x^{k-1}}\ (t,1) \\ \cdots\cdots \\ \dfrac{\partial^{n-1} u}{\partial x^{n-1}}\ (t,0) \\ \dfrac{\partial^{n-1} u}{\partial x^{n-1}}\ (t,1) \end{pmatrix} \right),\ t > 0,$$

$$(0.3)\quad u(0,x) = u_0(x),\ 0 < x < 1,$$

where

$$M_{2n-k}u := \sum_{i=0}^{n} \sum_{j=k}^{n} (-1)^{j-k} \frac{\partial^{j-k}}{\partial x^{j-k}} \left[a_{ij}(x) \frac{\partial^i u}{\partial x^i} \right]$$

(n natural $\geqslant 1$, $k = 1, 2, \ldots, n$) and $l : D(l) \subset \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ is assumed to be maximal monotone (possibly multivalued) mapping.

Under suitable hypotheses on a_{ij} $(i, j = 1, 2, \ldots, n)$, A, f, and u_0 we shall handle the existence, uniqueness, and asymptotic behaviour as $t \to \infty$ for the solutions of problem (0.1), (0.2), (0.3). The idea is to restate this problem as an initial-value problem (IVP) for an ordinary differential equation in $L^2(0,1)$ in which the boundary condition (0.2) is «incorporated» in the definition of some associated operator which turns out to be maximal monotone. So we can derive our results by using the existence theory for evolution equations associated to monotone operators and also exploiting the particularities of our problem.

The stationary case of the problem we are talking about is also considered.

The physical models which lead to our problem come from «*heat conduction*» theory (case $n = 1$) or from «*beam theory*» (case $n = 2$).

We emphasize that, to our knowledge, boundary condition (0.2) was never considered in the literature (in this general form). This is the main novelty of the paper. Notice also that many classical boundary conditions (BC) can be derived from (0.2) by making suitable choices of l and, eventually, of a_{ij}. Let us give a few examples:

EXAMPLE 1 $l = \partial g_1$ (the subdifferential of g_1), where

$$g_1 : \mathbf{R}^{2n} \to]-\infty, \infty]$$

is defined by

$$g_1(\mathrm{col}\,[r_1, s_1, \ldots, r_n, s_n]) := \begin{cases} 0, & \text{if } r_k = b_k, s_k = c_k, (k = 1, \ldots, n) \\ +\infty, & \text{otherwise,} \end{cases}$$

with b_k, c_k $(k = 1, \ldots, n)$ fixed in $\mathbf{R}$. Then (0.2) becomes

$$\text{(0.2.1)} \quad \frac{\partial^{k-1} u}{\partial x^{k-1}}(t, 0) = b_k, \frac{\partial^{k-1} u}{\partial x^{k-1}}(t, 1) = c_k, t > 0,$$

$(k = 1, \ldots, n)$: *bilocal conditions.*

EXAMPLE 2 $l = \partial g_2$, where $g_2 : \mathbf{R}^{2n} \to] - \infty, \infty]$ is defined by

$$g_2(\mathrm{col}\,[r_1, s_1, \ldots, r_n, s_n]) := \begin{cases} 0, & \text{if } r_k = s_k \ (k = 1, \ldots, n) \\ +\infty, & \text{otherwise.} \end{cases}$$

In this case (0.2) becomes

$$\frac{\partial^{k-1} u}{\partial x^{k-1}}(t, 0) = \frac{\partial^{k-1} u}{\partial x^{k-1}}(t, 1), (M_{2n-k} u)(t, 0) = (M_{2n-k} u)(t, 1), t > 0$$

$(k = 1, \ldots, n)$. If in addition we admit that $a_{ij}(x) \equiv C_{ij} (i, j = 0, 1, \ldots, n)$ the matrix (C_{ij}) is positive semi-definite, and $C_{nn} > 0$ (that is a special case of hypothesis (H_2) below) then we obtain the *periodic conditions:*

$$\text{(0.2.2)} \quad \frac{\partial^k u}{\partial x^k}(t, 0) = \frac{\partial^k u}{\partial x^k}(t, 1), \ t > 0 \ (k = 0, 1, \ldots, 2_{n-1}).$$

EXAMPLE 3 - Take $n = 2$, $a_{11}(x) \equiv C_1 \geqslant 0$, $a_{22}(x) \equiv C_2 > 0$, and the other $a_{ij}(x) \equiv 0$. Let $g_3 : \mathbf{R}^4 \to] - \infty, \infty]$ be defined by

$$g_3(\mathrm{col}\,[r_1, s_1, r_2, s_2]) = \begin{cases} C_2(d_3 r_2 - d_4 s_2), & \text{if } r_1 = d_1, \ s_1 = d_2, \\ +\infty, & \text{otherwise,} \end{cases}$$

with d_1, d_2, d_3, d_4 some fixed real numbers and let $l = \partial g_3$. In this case, problem (0.1), (0.2), (0.3) becomes:

$$\text{(0.1.3)} \quad u_t + C_2 u_{xxxx} - C_1 u_{xx} + A(x, u) = f(t, x),$$

$$\text{(0.2.3)} \quad u(t, 0) = d_1, \ u(t, 1) = d_2, \ u_{xx}(t, 0) = d_3, \ u_{xx}(t, 1) = d_4,$$

$$\text{(0.3.3)} \quad u(0, x) = u_0(x),$$

i.e. a quasi-stationary problem of the *beam theory.*

EXAMPLE 4 - Let l be a *$(2n \times 2n)$-positive, semi-definite matrix* (not necessarily symmetric). In this case, (0.2) (where «$\in$» is replaced by «$=$») represents *linear boundary conditions.* Let us recall that an $m \times m$ positive, semi-definite matrix M is a subdifferential if and only if M is symmetric (more exactly, in this case the matrix M is just the Fréchet differential of $\xi \to (1/2)\, \xi^* M \xi, \xi \in \mathbf{R}^m$, where ξ^* denotes the transposed of ξ), i.e. when the (linear) BC are self-adjoint. Therefore, assuming that l is a general positive, semi-definite matrix we effectively go beyond the case of subdifferential in (0.2).

Not only classical problems, but also many other problems can be expressed in the form (0.1), (0.2), (0.3). To illustrate this we give the following simple example

EXAMPLE 5 - Take $n = 1$, $a_{11}(x) := 1$, $a_{00}(x) := 0$, $a_{10}(x) := 0$ and let $l = \partial I_K$, where K is a closed, convex subset of $\mathbf{R}^2$ and I_K is its indicator function, i.e.

$$I_K\left(\begin{bmatrix} r \\ s \end{bmatrix}\right) := \begin{cases} 0, & \text{if } \begin{bmatrix} r \\ s \end{bmatrix} \in K, \\ +\infty, & \text{elsewhere}. \end{cases}$$

Then problem (0.1), (0.2), (0.3) becomes

(0.1.5) $\quad u_t - u_{xx} + A(x,u) = f(t,x),$

(0.2.5) $\quad \begin{bmatrix} u(t,0) \\ u(t,1) \end{bmatrix} \in K, \begin{bmatrix} u_x(t,0) \\ -u_x(t,1) \end{bmatrix} \in N\left(\begin{bmatrix} u(t,0) \\ u(t,1) \end{bmatrix}\right),$

(0.3.5) $\quad u(0,x) = u_0(x).$

Here $N(\operatorname{col}[r,s])$ is the normal cone to K at the point $\operatorname{col}[r,s] \in K$, i.e.

$$N\left(\begin{bmatrix} r \\ s \end{bmatrix}\right) := \left\{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbf{R}^2; \right.$$

$$\left. w_1(r - \xi_1) + w_2(s - \xi_2) \geqslant 0, (\forall) \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \in K \right\}.$$

NOTATION, TERMINOLOGY, AND BACKGROUND MATERIAL

Let X be a Banach space and denote by $\|\cdot\|_X$ its norm. If X is Hilbert, we denote by $\langle .,. \rangle_X$ its inner product.

By $W^{k,p}(a,b)$ (k natural, $1 \leqslant p \leqslant +\infty$, and $-\infty \leqslant a < b \leqslant +\infty$) we denote the usual Sobolev spaces. For the particular cases $k=0$ and $p=2$ we shall use the well-known notations $L^p(a,b)$ and $H^k(a,b)$, respectively. In the vectorial case we use the usual notations $W^{k,p}(a,b;X)$, $L^p(a,b;X)$, $H^k(a,b;X)$.

The space of all continuous functions:

$$[a,b] \to X \; (-\infty < a < b < +\infty)$$

endowed with sup-norm will be denoted by $C([a,b];X)$. If $X = \mathbf{R}$ we simply write $C[a,b]$ instead of $C([a,b];\mathbf{R})$. Finally, $C_0^\infty(a,b)$ denotes the subspace of $C[a,b]$ of all infinitely times differentiable functions with support included in the open interval $]a,b[$.

We assume the familiarity of the reader with the monotone operator theory, convex analysis, and the existence theory for evolution equations associated to monotone operators. For notation, terminology and fundamental results in this direction we refer the reader, e.g., to V. Barbu [2], H. Brézis [3].

1. - Some Lemmas

We state first the following

LEMMA A - *Let $m \geqslant 2$ be a natural number and let $u \in W^{m,1}(0,1)$. Then, for each $j \in \{1,2,\dots,m-1\}$, there exists a constant $C_{jm} > 0$ such that*

$$\| u^{(j)} \|_{C[0,1]} \leqslant C_{jm} \| u \|_{L^1(0,1)} + \| u^{(m)} \|_{L^1(0,1)} . \tag{1.0}$$

We omit the proof of this «Poincaré Lemma» A. Notice only that *for* $m = 2$ we found the proof of Lemma A in R.A. Adams [1, p. 70].

In all which follows we denote by (H_1), (H_2), $(H_2)'$ and (H_3) the following sets of hypotheses:

(H_1) $l : D(l) \subset \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ *is a maximal monotone (possibly multivalued) mapping* (n is a fixed natural number $\geqslant 1$).

(H_2) $a_{ij} \in W^{j,\infty}(0,1)$ $(i,j = 0,1,\dots,n)$ *and, besides,*

$$\sum_{i,j=0}^{n} a_{ij}(x)\, \xi_i \xi_j \geqslant 0 , \text{ a.e. } x \in]0,1[,$$

for any $\xi := \text{col}\,[\xi_0, \xi_1, \dots, \xi_n] \in \mathbf{R}^{n+1}$. *Moreover* $a_{nn}(x) > 0$, $0 \leqslant x \leqslant 1$.

$(H_2)'$ $a_{ij} \in W^{j,\infty}(0,1)$ $(i,j = 0,1,\dots,n)$ *and, besides,*

$$\sum_{i,j=0}^{n} a_{ij}(x)\, \xi_i \xi_j \geqslant c_0\, \xi_n^2 , \text{ a.e. } x \in]0,1[,$$

for any $\xi := \text{col}\,[\xi_0, \xi_1, \dots, \xi_n] \in \mathbf{R}^{n+1}$ $(c_0 > 0)$.
Clearly, $(H_2)'$ is stronger than (H_2).

(H_3) *The function* $x \mapsto A(x,r)$ *is in* $L^2(0,1)$ *for any fixed* $r \in \mathbf{R}$. *Besides, the function* $r \mapsto A(x,r)$ *is continuous and nondecreasing from* $\mathbf{R}$ *into* $\mathbf{R}$, *for a.e.* $x \in]0,1[$.

Now, denote $X = L^2(0,1)$ and consider the operator

$$T_{2n} : D(T_{2n}) \subset X \to X$$

defined by

$$D(T_{2n}) := \{ u \in H^{2n}(0,1);\ u \text{ satisfies (1.3) below} \}, \tag{1.1}$$

$$T_{2n} u := \sum_{i,j=0}^{n} (-1)^j \,[a_{ij} u^{(i)}]^{(j)} \tag{1.2}$$

$$(1.3)\qquad \begin{bmatrix} (M_{2n-1}u)(0) \\ -(M_{2n-1}u)(1) \\ \cdots\cdots \\ (M_{2n-k}u)(0) \\ -(M_{2n-k}u)(1) \\ \cdots\cdots \\ (M_n u)(0) \\ -(M_n u)(1) \end{bmatrix} \in l \left(\begin{bmatrix} u(0) \\ u(1) \\ \cdots\cdots \\ u^{(k-1)}(0) \\ u^{(k-1)}(1) \\ \cdots\cdots \\ u^{(n-1)}(0) \\ u^{(n-1)}(1) \end{bmatrix} \right)$$

where l and a_{ij} satisfy (H_1) and (H_2), respectively, and

$$(1.4)\qquad M_{2n-k}u := \sum_{i=0}^{n}\sum_{j=k}^{n} (-1)^{j-k}[a_{ij}(x)\,u^{(i)}]^{(j-k)} \quad (k = 1, 2, \ldots, n).$$

Remark that in the special case $n = 1$ the operator T_2 is connected with the *«heat conduction»*, whilst in the case $n = 2$ the operator T_4 is connected with the *elastic beam theory*.

LEMMA 1 - *Suppose l satisfies* (H_1) *and* a_{ij} $(i, j = 0, 1, \ldots, n)$ *satisfy* (H_2). *Then operator* $T_{2n}: D(T_{2n}) \subset X \to X$ $(X = L^2(0,1))$ *given by* (1.1), (1.2) *is maximal monotone.*

Proof. T_{2n} is *monotone:* For any $u, v \in D(T_{2n})$ we have by repeated integration by parts:

$$(1.5)\qquad \langle T_{2n}u - T_{2n}v, u - v\rangle_X =$$

$$= \sum_{i,j=0}^{n} (-1)^j \int_0^1 [a_{ij}(u-v)^{(i)}]^{(j)}\,(u-v)\,dx =$$

$$= \sum_{i,j=0}^{n} \int_0^1 a_{ij}\,[u^{(i)} - v^{(i)}]\cdot[u^{(j)} - v^{(j)}]\,dx +$$

$$+ \sum_{k=1}^{n} \{[u^{(k-1)}(x) - v^{(k-1)}(x)]\,[(M_{2n-k}u)(x) - (M_{2n-k}v)(x)]\}\,_{x=1}^{x=0} \geqslant 0,$$

because of (H_1) and (H_2).

To show that T_{2n} is *maximal* monotone, it is sufficient to prove that for any fixed $p \in X$ the problem made up by (1.6), (1.3), where

$$(1.6)\qquad \sum_{i,j=0}^{n} (-1)^j [a_{ij}u^{(i)}]^{(j)} + u = p,$$

has (at least) a solution.

Remark that if u_1 is a solution of the problem made up by equation (1.6) and boundary conditions

$$(1.7)\qquad u(0) = u(1) = u'(0) = u'(1) = \ldots = u^{(n-1)}(0) = u^{(n-1)}(1) = 0,$$

whilst u_2 is a solution of problem

(1.6.H) $\sum_{i,j=0}^{n} (-1)^i [a_{ij} u^{(i)}]^{(j)} + u = 0,$

(1.8) $w_0 + \mathrm{col}\,[(M_{2n-1} u)(0),\ -(M_{2n-1} u)(1), \ldots, (M_n u)(0),$

$-(M_n u)(1)] \in l(\mathrm{col}\,[u(0),\ u(1), \ldots, u^{(n-1)}(0),\ u^{(n-1)}(1)]),$

where

$w_0 := \mathrm{col}\,[(M_{2n-1} u_1)(0),\ -(M_{2n-1} u_1)(1), \ldots, (M_n u_1)(0),\ -(M_n u_1)(1)]$ then $u = u_1 + u_2$ is a solution of problem (1.6), (1.3).

Since $a_{nn}(x) > 0$ $(0 \leqslant x \leqslant 1)$, it is obvious that equation (1.6.H) has a fundamental system of solutions

$$y_1(x), y_2(x), \ldots, y_{2n}(x)$$

i.e. the general integral of (1.6.H) is

(1.9) $u(x) = \sum_{j=1}^{2n} c_j \cdot y_j(x)\,,\ c_j \in \mathbf{R}\ (j = 1, 2, \ldots, 2n).$

As it is known, problem (1.6), (1.7) has a unique solution (which can be found by «variation of constants»). So all we have to prove is that problem (1.6.H), (1.8) has (at least) a solution. Denote

(1.10) $w = w(u) = \mathrm{col}\,[w_1, w_2, \ldots, w_{2j-1}, w_{2j}, \ldots, w_{2n-1}, w_{2n}] :=$
$:= \mathrm{col}\,[u(0), u(1), \ldots, u^{(j-1)}(0), u^{(j-1)}(1), \ldots, u^{(n-1)}(0),$
$u^{(n-1)}(1)]\,,$

and $c := \mathrm{col}\,[c_1, c_2, \ldots, c_{2n}]$. We have by (1.9)

(1.11) $w = Dc,$

where $D := [d_1, \ldots, d_s, \ldots, d_{2n}]$ is the $2n \times 2n$-matrix whose $s-th$ column is

$$d_s := \mathrm{col}\,[y_s(0), y_s(1), \ldots, y_s^{(n-1)}(0), y_s^{(n-1)}(1)]\ \ (s = 1, \ldots, 2n).$$

Remark that D is an invertible matrix, no matter how the fundamental system was chosen. (Indeed, problem (1.6.H), (1.7) has the unique solution $u = 0$. I.e., imposing to (1.9) to satisfy (1.7) we have the implication $w = Dc = 0 \Rightarrow c = 0$, i.e. $\det D \neq 0$).

On the other hand, we obtain from (1.9) and (1.11)

(1.12) $\mathrm{col}\,[(M_{2n-1} u)(0),\ -(M_{2n-1} u)(1), \ldots, (M_n u)(0),$

$-(M_n u)(1)] = Bc = BD^{-1} w,$

where $B := [b_1, .., b_s, .., b_{2n}]$ is the matrix whose $s-th$ column is $b_s := \mathrm{col}\,[(M_{2n-1} y_s)(0),\ -(M_{2n-1} y_s)(1), .., (M_n y_s)(0),\ -(M_n y_s)(1)]$. Thus, boundary condition (1.8) reduces, by (1.10) and (1.12), to

(1.13) $w_0 \in -BD^{-1} w + l(w).$

Since l is maximal monotone, it suffices to prove that the matrix $-BD^{-1}$ is positive definite to conclude that there is a unique solution $\hat{w}$ of (1.13), i.e. $u(x)$ given by (1.9) with $c:=D^{-1}\hat{w}$ satisfies both (1.6.H) and (1.8). Thus all we have to prove is that

$$(1.14) \qquad -w^* BD^{-1} w > 0\,, \text{ for any } w \in \mathbf{R}^{2n},\ w \neq 0\,,$$

where w^* is the transposed of w.

To prove now (1.14) remark that for any choice of $c_1, c_2, \ldots, c_{2n}$, $u(x)$ given by (1.9) satisfies (after repeated integration by parts and essentially the same computation as in (1.5))

$$(1.15) \qquad 0 = \langle T_{2n} u + u, u\rangle_X = \| u \|_X^2 + \sum_{i,j=0}^{n} (-1)^j \int_0^1 [a_{ij} u^{(i)}]^{(j)} u dx =$$

$$= \| u \|_X^2 + \sum_{k=1}^{n} \{u^{(k-1)}(x)\ [M_{2n-k} u](x)\}_{x=1}^{x=0} + \sum_{i,j=0}^{n} \int_0^1 a_{ij} u^{(i)} u^{(j)} dx\,.$$

Thus, by (1.15), (1.10), (1.11) and (1.12) we obtain

$$-w^* BD^{-1} w = \| u \|_X^2 + \sum_{i,j=0}^{n} \int_0^1 a_{ij} u^{(i)} u^{(j)} dx \geqslant \| u \|_X^2 > 0\,,$$

for any $u = \sum_{j=1}^{n} c_j y_j(x) \neq 0$, that is for any $c \neq 0$, i.e. for any $w \in \mathbf{R}^{2n}$, $w \neq 0$. Thus (1.14) holds and the maximality of T_{2n} is proved. In particular, the proof of maximality of T_{2n} shows that *$D(T_{2n})$ is a non void set*. Lemma 1 is proved.

REMARK 1.1 - In addition to Lemma 1 we have: *$D(T_{2n})$ is dense in* $X = L^2(0,1)$. Indeed, take $\hat{u} \in D(T_{2n})$ and note that

$$(1.16) \qquad \hat{u} + C_0^\infty(0,1) := \{\hat{u} + v\,;\, v \in C_0^\infty(0,1)\} \subset D(T_{2n})\,.$$

Since $C_0^\infty(0,1)$ is dense in X, the set $\hat{u} + C_0^\infty(0,1)$ is also dense in X and thus, by (1.16), $D(T_{2n})$ is dense in X.

LEMMA 2 - *Suppose a_{ij} $(i,j = 0,1,\ldots,n)$ satisfy (H_2) and, besides, the matrix (a_{ij}) is symmetric, a.e. $x \in]0,1[$. Suppose also l is the subdifferential of a function $g: \mathbf{R}^{2n} \to]-\infty, +\infty]$ which is proper, convex and lower semicontinuous* (LSC). *Denote, as usual,*
(1.17) *$D(g) := \{w \in \mathbf{R}^{2n}\,;\, g(w) < +\infty\}$ (the effective domain of g), take $w(u)$ as in* (1.10) *and define $\varphi: X \to]-\infty, \infty]$, $X := L^2(0,1)$ by*

$$(1.18) \qquad \varphi(u) := \begin{cases} \tfrac{1}{2} \sum_{i,j=0}^{n} \int_0^1 a_{ij} u^{(i)} u^{(j)} dx + g(w(u)), & \text{if } u \in H^n(0,1) \\ & \text{and } w(u) \in D(g), \\ +\infty, & \text{otherwise}. \end{cases}$$

Then φ is proper, convex and LSC *and operator T_{2n} defined by* (1.1), (1.2) *is just the subdifferential of* φ.

Proof. Obviously $D(T_{2n}) \subset D(\varphi)$, i.e. φ is proper. It is also clear that φ is convex. Let $\partial\varphi$ denote its subdifferential. It is easy to see by a straightforward, standard computation that $D(T_{2n}) \subset D(\partial\varphi)$ and

$$T_{2n}\, u \in \partial\varphi(u), \text{ for any } u \in D(T_{2n}).$$

Since by Lemma 1 T_{2n} is maximal monotone it follows that $T_{2n} = \partial\varphi$. Hence T_{2n} is cyclically maximal monotone and so, according to a well-known result (see, e.g., V. Barbu [2, p. 59]), T_{2n} is the subdifferential of a proper, convex and LSC function which is uniquely determined up to an additive constant. Therefore φ is LSC. Q.E.D.

REMARK 1.2 - Let (H_1) and (H_2) be satisfied. If we assume further that $a_{ij} \equiv 0$, for $i \neq j$, then (H_2) becomes equivalent to $(H_2)'$ and, denoting $a_j : = a_{jj} (j = 0, 1, \ldots, n)$, this hypothesis can be rewritten as: $(H_2)^0 : a_j \in W^{j,\infty}(0,1)$; $a_n(x) > 0$, $a_r(x) \geqslant 0$ $(r = 0, 1, \ldots, n-1)$.

Operator T_{2n} becomes

$$T^0_{2n}\, u = \sum_{j=0}^{n} (-1)^j \, [a_j\, u^{(j)}]^{(j)},$$

which is the general form of a formally self-adjoint linear differential operator of order $2n$, whilst M_{2n-k} in (1.3) become

$$M^0_{2n-k}\, u = \sum_{j=k}^{n} (-1)^{j-k} \, [a_j\, u^{(j)}]^{(j-k)} \quad (k = 1, 2, \ldots, n).$$

If in addition $l = \partial g$ then $T^0_{2n} = \partial\varphi_0$, where

$$\varphi_0(u) : = \begin{cases} \frac{1}{2} \sum_{j=0}^{n} \int_0^1 a_j \, | u^{(j)} |^2 \, dx + g(w(u)), & \text{if } u \in H^n(0,1) \text{ and } w(u) \in D(g) \\ + \infty, & \text{otherwise}. \end{cases}$$

2. - A perturbation result

The purpose of this section is to establish the following perturbation result:

LEMMA 3 - *Suppose* $l, a_{ij} (i, j = 0, 1, \ldots, n)$, *and* A *satisfy* (H_1), $(H_2)'$, *and* (H_3) *respectively. Then, operator* $Q_{2n} : D(Q_{2n}) \subset X \to X$, $X : = L^2(0,1)$, *defined by*

(2.1) $\quad D(Q_{2n}) : = D(T_{2n})$, *(see* (1.1) *and* (1.2)*)*

(2.2) $\quad Q_{2n}\, u : = T_{2n}\, u + A(\cdot\,, u(\cdot))$

is maximal monotone.

Proof. Remark first that, because of (H_3), we have that $A(.,q(\cdot)) \in X$ for any $q \in D(Q_{2n})$ and consequently $Q_{2n}\, q \in X$ for any $q \in D(Q_{2n})$.

Now, it is easy to see that Q_{2n} is *monotone*. To prove its *maximality*, consider the «approximate» equation

(2.3) $$\begin{cases} u + T_{2n}\, u + A_\lambda(x,u) = p(x), \\ u \in D(Q_{2n}) := D(T_{2n}), \end{cases}$$

for any fixed $p \in X$ and $\lambda > 0$. Here $A_\lambda(x,.)$ denotes the Yosida approximate of $A(x,.)$, a.e. $x \in]0,1[$.

Obviously, for any $q \in X$ the function

$$x \mapsto A_\lambda(x, q(x)) := A(x, [I + \lambda A(x,.)]^{-1} q(x))$$

is measurable in $]0,1[$ and since

$$|A_\lambda(x,q(x))| \leqslant |A_\lambda(x,q(x)) - A_\lambda(x,0)| +$$

$$+ |A_\lambda(x,0)| \leqslant \frac{1}{\lambda}|q(x)| + |A(x,0)|,$$

it follows that $x \mapsto A_\lambda(x,q(x))$ is in X for any fixed $\lambda > 0$. Moreover, the canonic extension $q \mapsto A_\lambda(.,q(\cdot))$ is maximal monotone in X (for it is everywhere defined, monotone and continuous in X). Since the domain of the extension is all the space, T_{2n} + *extension* is maximal monotone (cf. Rockafellar's Theorem; see, e.g., V. Barbu [2, p. 46]). Thus (2.3) has a unique solution u_λ for any fixed $\lambda > 0$, i.e.

(2.4) $$u_\lambda + T_{2n}\, u_\lambda + A_\lambda(x,u_\lambda) = p, \quad u_\lambda \in D(T_{2n}).$$

We intend to go to the limit in (2.4) as $\lambda \to 0$ to conclude that Q_{2n} is *maximal* monotone. To this purpose, some a priori estimates are needed. Take first $u_0 \in D(T_{2n})$ *and denote*

(2.5) $$u_0 + T_{2n}\, u_0 + A_\lambda(x,u_0) := p_\lambda \quad (\lambda > 0).$$

From $|A_\lambda(x,u_0(x))| \leqslant |A(x,u_0(x))|$, from $\inf u_0(x) \leqslant u_0(x) \leqslant \sup u_0(x)$ and from (H_3) we find that $\{A_\lambda(.,u_0(\cdot));\lambda > 0\}$ is bounded in X. Consequently

(2.6) $\{p_\lambda;\lambda > 0\}$ is bounded in X.

Now, from (2.4) and (2.5) we obtain by using the monotonicity of $A_\lambda(x,.)$:

(2.7) $$\|u_\lambda - u_0\|_X^2 +$$

$$+ \sum_{i,j=0}^{n} (-1)^j \int_0^1 [a_{ij}(u_\lambda^{(i)} - u_0^{(i)})]^{(j)}(u_\lambda - u_0)\, dx \leqslant$$

$$\leqslant \tfrac{1}{2}\|p_\lambda - p\|_X^2 + \tfrac{1}{2}\|u_\lambda - u_0\|_X^2 .$$

After some integrations by parts in (2.7) we get by virtue of (2.6), (H_1), (H_2)' and on account of $u_\lambda, u_0 \in D(T_{2n})$:

$$\|u_\lambda - u_0\|_X^2 + \int_0^1 \sum_{i,j=0}^n a_{ij}\,(u_\lambda^{(i)} - u_0^{(i)})\,(u_\lambda^{(j)} - u_0^{(j)})\,dx \leqslant \text{Const.},$$

or, on account of (H₂)':

$$\|u_\lambda - u_0\|_X^2 + c_0\,\|u_\lambda^{(n)} - u_0^{(n)}\|_X^2 \leqslant \text{Const.}\ \ (c_0 > 0).$$

Hence, we have

(2.8) $\{u_\lambda\,;\lambda > 0\}$ *is bounded in* X and

(2.9) $\{u_\lambda^{(n)}\,;\lambda > 0\}$ *is bounded in* X.

By (2.8) and (2.9) we have, according to Lemma A

(2.10) $\{u_\lambda^{(j)}\,;\lambda > 0\}$ is bounded in $C[0,1]$, $(j = 0,1,\ldots,n-1)$.

From (2.10), (H₃) and $|A_\lambda(x,r)| \leqslant |A(x,r)|$ we immediately obtain

(2.11) $\{A_\lambda(.,u_\lambda(\cdot))\,;\lambda > 0\}$ *is bounded in* X.

Therefore, by virtue of (2.4) and (2.11) we have:

(2.12) $\{\sum_{i,j=0}^n (-1)^j\,(a_{ij}\,u_\lambda^{(i)})^{(j)}\,;\lambda > 0\}$ *is bounded in* X.

Denote now

$$(2.13)\qquad w_\lambda(x) := \sum_{i=0}^n (-1)^n\,a_{in}(x)\,u_\lambda^{(i)}(x) + $$
$$+ \sum_{i=0}^n (-1)^{n-1} \int_0^x a_{i\,n-1}(s)\,u_\lambda^{(i)}(s)\,ds + $$
$$+ \sum_{i=0}^n \sum_{j=0}^{n-2} (-1)^j \int_0^x ds_{n-j-1} \int_0^{s_{n-j-1}} ds_{n-j-2} \ldots \int_0^{s_1} a_{ij}(s)\,u_\lambda^{(i)}(s)\,ds.$$

Note that by (2.8), (2.9), (2.10) we have

(2.14) $\{w_\lambda\,;\lambda > 0\}$ *is bounded in* X.

Also, it is obvious that

$$w_\lambda^{(n)} = \sum_{i,j=0}^n (-1)^j (a_{ij}\,u_\lambda^{(i)})^{(j)}.$$

Hence, from (2.12) we have

(2.15) $\{w_\lambda^{(n)}\,;\lambda > 0\}$ *is bounded in* X.

Moreover, from Lemma A it follows, by virtue of (2.14) and (2.15),

(2.16) $\{w_\lambda^{(j)}\,;\lambda > 0\}$ is bounded in $C[0,1]$ $(j = 0,1,\ldots,n-1)$.

Now, differentiating in (2.13) and using (2.9), (2.10) and (2.16) we see that $\{u_\lambda^{(n+1)}\,;\lambda > 0\}$ is bounded in X. After differentiating two times in (2.13) we find $\{u_\lambda^{(n+2)}\,;\lambda > 0\}$ is bounded in X. Thus, by repeated use of this argument we get

(2.17) $\{u_\lambda^{(j)}\,;\lambda > 0\}$ *is bounded in* X $(j = n+1,\ldots,2n)$.

and $\{u_\lambda^{(j)}\ ;\lambda > 0\}$ is bounded in $C[0,1]$ $(j = 0,1,..,2n-1)$
$\{u_\lambda^{(2n)}\ ;\lambda > 0\}$ is bounded in $L^2(0,1)$.

Therefore, by Arzelà's Criterion it follows that there exists $u \in H^{2n}(0,1)$ such that (eventually on a subsequence)

(2.18) $\quad u_\lambda^{(j)} \to u^{(j)}$ in $C[0,1]$, as $\lambda \to 0$ $(j = 0,1,..,2n-1)$

(2.19) $\quad u_\lambda^{(2n)} \to u^{(2n)}$ weakly in $L^2(0,1)$, as $\lambda \to 0$.

In addition (see (2.4) and (2.11))

(2.20) $\quad A_\lambda(.,u_\lambda(\cdot)) \to h := p - u - T_{2n}u$, weakly in $L^2(0,1)$.

Let us prove now that $h = A(.,u(\cdot))$. To this purpose remark first that for a.e. $x \in]0,1[$ and any $\lambda > 0$

$$|[I + \lambda A(x,.)]^{-1} u_\lambda(x) - u(x)| \leqslant |u_\lambda(x) - u(x)| +$$
$$+ |[I + \lambda A(x,.)]^{-1} u(x) - u(x)|.$$

So by (2.18) we have for a.e. $x \in]0,1[$

$$[I + \lambda A(x,.)]^{-1} u_\lambda(x) \to u(x) \text{ as } \lambda \to 0,$$

eventually on a subsequence. Therefore, for a.e. $x \in]0,1[$

(2.21) $\quad A_\lambda(x,u_\lambda(x)) = A(x,[I + \lambda A(x,.)]^{-1} u_\lambda(x)) \to A(x,u(x))$,

as $\lambda \to 0$. From (2.11) and (2.21) we get by virtue of Lebesgue's Dominated Convergence Theorem (eventually on a subsequence)

(2.22) $\quad A_\lambda(.,u_\lambda(\cdot)) \to A(.,u(\cdot))$, strongly in $L^2(0,1)$, as $\lambda \to 0$.

Now, on account of (2.18), (2.19), (2.22) we conclude by taking the limit in (2.4) that

$$u + Q_{2n}u = p$$

and, in addition, as l is closed, u verifies (1.3). Therefore Q_{2n} is *maximal* monotone, as desired, and Lemma 3 is proved.

REMARK 2.1 - If all the assumptions of Lemma 3 are satisfied and $(a_{ij}(x))$ is a symmetric matrix, a.e. $x \in]0,1[$ whilst $l = \partial g$, where $g : \mathbf{R}^{2n} \to]-\infty, +\infty]$ is proper, convex and LSC then operator Q_{2n} defined by (2.1), (2.2) is the subdifferential of the function

$$\psi : X \to]-\infty, +\infty]$$

defined by

(2.23) $$\psi(u) := \begin{cases} \varphi(u) + \int_0^1 dx \int_0^{u(x)} A(x,s)\, ds, & \text{if } u \in D(\varphi) \\ +\infty, & \text{otherwise}. \end{cases}$$

The function φ and its $D(\varphi)$ were defined in Lemma 2.

Actually, by (2.9), (2.10) and (2.17) we have from Lemma A

Indeed, ψ is proper, convex and $D(\psi) = D(\varphi)$. Moreover, it is easy to see that $D(Q_{2n}) = D(T_{2n}) \subset D(\partial\psi)$ and

$$Q_{2n}u = T_{2n}u + A(.,u(\cdot)) \subset \partial\psi(u), \text{ for any } u \in D(T_{2n}).$$

As Q_{2n} is maximal monotone (Lemma 3) it follows that $Q_{2n} = \partial\psi$ as asserted and ψ is LSC.

3. - The stationary problem

In keeping with the notation above consider the following BVP:

$$(3.1) \qquad Q_{2n}u = p(x) \ (0 < x < 1), \ u \text{ satisfies } (1.3),$$

where $p \in L^2(0,1)$. Obviously, if (H_1), $(H_2)'$, (H_3) are fulfiled and, in addition, A satisfies some appropriate condition which assures the coerciveness of Q_{2n} then problem (3.1) has at least a solution. For example, assume that

$$(3.2) \qquad (A(x,r_1) - A(x,r_2))\,(r_1 - r_2) \geqslant k_0(r_1 - r_2)^2, \text{ a.e. } x \in]0,1[,$$

for any $r_1, r_2 \in \mathbf{R}$, where k_0 is some positive constant. Then, there exists a unique $u \in H^{2n}(0,1)$ which satisfies (3.1) (u is unique because in this case Q_{2n} is strongly monotone !). Remark also that Q_{2n} could be non-surjective (i.e. problem (3.1) might have no solution for some p) if no additional assumption is made on A, as the following simple counterexample shows.

Counterexample - Take in (3.1) $: n = 1$, $a_{11} := 1$, $a_{00} := 0$, $a_{01} := 0$, $a_{10} := 0$, $A(.,.) := 0$, $p := 0$ and let $l : D(l) = \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $l(\text{col}[r_1, r_2]) = \text{col}[0, -1]$, for any $\text{col}[r_1, r_2] \in \mathbf{R}^2$. Then, problem (3.1) becomes

$$u'' = 0; \ u'(0) = 0, \ u'(1) = 1$$

which has no solution.

Let us also mention here an existence result for the case of a linear equation:

$$(3.3) \qquad \begin{cases} \sum_{i,j=0}^{n} (-1)^j [a_{ij}u^{(i)}]^{(j)} = p(x), \ 0 < x < 1, \ (p \in L^2(0,1)) \\ u \text{ satisfies } (1.3). \end{cases}$$

Proposition 3.1 - *Assume l satisfies* (H_1) *and a_{ij} satisfy: $a_{nn}(x) > 0$, $(0 \leqslant x \leqslant 1)$, and $a_{ij} \in W^{j,\infty}(0,1)$ $(i,j = 0,1,..,n)$, while*

$$(3.4) \qquad \sum_{i,j=0}^{n} a_{ij}(x)\,\xi_i\xi_j \geqslant \alpha_0\xi_0^2, \text{ a.e. } x \in]0,1[,$$

for any $\xi = \text{col}[\xi_0, \xi_1, .., \xi_n] \in \mathbf{R}^{n+1}$ $(\alpha_0 > 0)$. Then (3.3) has a unique solution $u \in H^{2n}(0,1)$.

Proof. By Lemma 1 operator T_{2n} is maximal monotone. Moreover, by (3.4) T_{2n} is surjective.

REMARK 3.1 - The existence in the BVP (3.1) can be treated by a «variational» approach, i.e. by introducing the functional

$$a(u,v) := \sum_{i,j=0}^{n} \int_0^1 a_{ij} u^{(i)} v^{(j)} dx + \int_0^1 A(x,u)\, v dx + \langle l(w(u)), w(v) \rangle_{\mathbf{R}^{2n}}$$

for $u, v \in H^n(0,1)$, with $w(u) \in D(l)$. Under assumptions (H_1), (H_2)', (H_3) one can prove that for any $p \in (H^n(0,1))'$ (the dual of H^n) there exists a $u \in H^n(0,1)$ satisfying

$$a(u,v) \ni p(v), \text{ for any } v \in H^n$$

which is called a variational solution of (3.1). The problem of «variational solutions» and regularity results is discussed in a more general frame in [7]. Let us only remark that such results like Lemma 1 and Proposition 3.1 cannot be obtained via such theorems which use a (sufficient) condition of coercivity «on» $a(u,v)$ (with $A := 0$). In any case, the proof of Lemma 1 given here is preferable for it has the advantage to be direct and elementary.

REMARK 3.2 - In the particular case $l = \partial g$, $a_{ij} = a_{ji}$, $p \in L^2(0,1)$ u is a solution of (3.1) if and only if u is a minimum point of

$$v \mapsto \psi(v) - \int_0^1 pv dx, \quad v \in L^2(0,1), \tag{3.5}$$

where ψ is given by (2.23). Therefore, in this case, the existence for (3.1) (in particular the maximality of Q_{2n}) can also be obtained by studying the functional (3.5).

4. - Existence, uniqueness and regularity of solutions to Problem (0.1), (0.2), (0.3)

THEOREM 4.1 - *Assume that l, a_{ij} $(i,j = 0,1,\dots,n)$ and A satisfy* (H_1), (H_2)' *and* (H_3) *respectively (or $A := 0$ and l, a_{ij} satisfy* (H_1) *and* (H_2)*). Let $f \in W^{1,1}(0,t_1; L^2(0,1))$ ($t_1 > 0$ fixed), and $u_0 \in D(T_{2n})$ defined before by* (1.1)*. Then, there is a unique function*

$$u \in W^{1,\infty}(0,t_1; L^2(0,1)) \cap L^\infty(0,t_1; H^{2n}(0,1))$$

such that u satisfies equation (0.1) *(for a.e. $(t,x) \in]0,t_1[\times]0,1[$), boundary condition* (0.2) *(for every $t \in [0,t_1]$) and initial condition* (0.3)*. Moreover*

$$\frac{\partial^j u}{\partial x^j} \in L^\infty(]0,t_1[\times]0,1[) \quad (j = 0,1,\dots,2n-1). \tag{4.1}$$

Proof. Consider in $X := L^2(0,1)$ the following Cauchy problem

(4.2) $$\frac{du}{dt} + Q_{2n} u = f(t,.), \ 0 < t < t_1$$

(4.3) $$u(0) = u_0 ,$$

where Q_{2n} is the operator defined by (2.1), (2.2). From Lemma 3 (respectively Lemma 1 for $A := 0$) we know that Q_{2n} is maximal monotone in X. Thus, from the general existence theory for evolution equations associated to monotone operators (see, e.g. V. Barbu [2, Ch. III, § 2]) we know that there is a unique function $u \in W^{1,\infty}(0, t_1; X)$ which satisfies (4.2) (for a.e. $t \in]0, t_1[$) and IC (4.3). Moreover, $u(t)$ is everywhere differentiable from the right in $[0, t_1[$, $u(t) \in D(Q_{2n})$ for every $t \in [0, t_1[$ and $u(t)$ satisfies for every $t \in [0, t_1[$ equation (4.2), where du/dt is replaced by d^+u/dt. In fact, we may consider (4.2) on $[0, t_1 + \varepsilon]$, $\varepsilon > 0$ (by extending correspondingly f) and so we have $u(t_1) \in D(Q_{2n})$. Since

(4.4) $$Q_{2n} u(t) = f(t,.) - \frac{d^+u}{dt}(t)$$

we have

(4.5) $$\sup \{ \| Q_{2n} u(t) \|_X ; 0 \leqslant t \leqslant t_1 \} < +\infty .$$

Denote $D_x^k := \partial^k / \partial x^k$. Now, a straightforward computation gives

(4.6) $$\langle Q_{2n} u(t,.) - Q_{2n} u_0 , u(t,.) - u_0 \rangle_X \geqslant$$

$$\geqslant \sum_{i,j=0}^{n} \int_0^1 a_{ij}(x)\, D_x^i [u(t,x) - u_0(x)]\, D_x^j [u(t,x) - u_0(x)]\, dx \geqslant$$

$$\geqslant c_0 \| D_x^n u(t,.) - u_0 \|_X^2 .$$

By (4.5) and (4.6) it follows that

(4.7) $$\sup \{ \| D_x^n u(t,.) \|_X ; \ 0 \leqslant t \leqslant t_1 \} < +\infty .$$

Since $u \in C([0, t_1]; X)$ we get from (4.7) by virtue of Lemma A

(4.8) $$\sup \{ \| D_x^j u(t,.) \|_X ; \ 0 \leqslant t \leqslant t_1 \} < \infty \quad (j = 0, 1, .., n).$$

On the other hand we have

(4.9) $$D_x^j u(t,x) = \int_0^1 [y D_y^{j+1} u(t,y) + D_y^j u(t,y)]\, dy - \\ - \int_x^1 D_y^{j+1} u(t,y)\, dy .$$

Therefore, according to (4.8) we have

(4.10) $$D_x^j u \in L^\infty(]0, t_1[\times]0,1[) \quad (j = 0, 1, .., n-1).$$

Now, from (4.8), (4.9) (with $j = 0$) and (H_3) we get

(4.11) $$\sup \{ \| A(., u(t,.)) \|_X ; \ 0 \leqslant t \leqslant t_1 \} < +\infty .$$

Henc, by (4.5) and (4.11) we have

$$(4.12) \qquad \sup\{\|\sum_{i,j=0}^{n}(-1)^i D_x^i[a_{ij}(\cdot)D_x^i u(t,.)]\|_X ; 0 \leqslant t \leqslant t_1\} < +\infty ,$$

which is similar to (2.12) (where the parameter λ is replaced by t). So, by the same argument as in the proof of Lemma 3, (4.8) can be extended for $j = 0, 1, \ldots, 2n$ and, correspondingly, (4.10) holds for $j = 0, 1, \ldots, 2n-1$. Summarising, we see that u is a solution of problem (0.1), (0.2), (0.3) in the same sense as in Theorem 4.1. Q.E.D.

THEOREM 4.2 - *Assume a_{ij} $(i, j = 0, 1, \ldots, n)$ and A satisfy* (H$_2$)' *and* (H$_3$) *(or $A := 0$ and a_{ij} satisfy* (H$_2$)*). Assume further that matrix (a_{ij}) is symmetric and $l = \partial g$, where $g : \mathbf{R}^{2n} \to]-\infty, +\infty]$ is proper, convex and* LSC. *Let $f \in L^2(0, t_1; L^2(0,1))$ and $u_0 \in L^2(0,1)$. Then, there exists a unique $u \in C([0,t_1]; L^2(0,1)) \cap W^{1,2}(\delta, t_1; L^2(0,1))$ (for every $\delta \in]0, t_1[$) such that u satisfies* (0.1) *(for a.e. $(t,x) \in]0, t_1[\times]0,1[$), BC* (0.2) *(for a.e. $t \in]0, t_1[$), and IC* (0.3) *(for a.e. $x \in]0,1[$). Moreover $\sqrt{t}\, u_t \in L^2(]0,t_1[\times]0,1[)$. If in addition $u_0 \in D(\varphi)$ (see* (1.18) *in Lemma 2) then $u_t \in L^2(]0,t_1[\times]0,1[)$.*

Proof. Consider again Cauchy problem (4.2), (4.3), where Q_{2n} is defined by (2.1), (2.2). According to *Remark 1.1*, $D(Q_{2n}) = D(T_{2n})$ is dense in $L^2(0,1)$. Moreover, by Lemma 2 and Remark 2.1, Q_{2n} (T_{2n} in case $A := 0$) is the subdifferential of ψ (respectively φ). Thus we can apply a general known result (see, e.g., V. Barbu [2, p. 189]) to problem (4.2), (4.3) and all the conclusions follow. Q.E.D.

REMARK 4.2 - For an abstract IVP of type (4.2), (4.3) we have the well-known concepts of *strong* and *weak solution* (see, e.g., H. Brézis [3, p. 64]). We shall say that u is a *strong (weak) solution* for problem (0.1), (0.2), (0.3) if u is a strong (respectively weak) solution of (4.2), (4.3) with Q_{2n} defined by (2.1), (2.2). For instance, the solution given by Theorem 4.1 is, according to this terminology, a strong solution of problem (0.1), (0.2), (0.3).

5. - Asymptotic behaviour of solutions

For the sake of brevity, we confine ourselves to give here only two results of asymptotic behaviour related to our problem.

Let us first give the following

PROPOSITION 5.1 - *Assume* (H$_1$), (H^2)' *and* (H$_3$) *are satisfied. Then, for any $\lambda > 0$, the operator $(I + \lambda Q_{2n})^{-1}$ (the resolvent of Q_{2n} defined by* (2.1), (2.2)*) maps bounded subsets of $X = L^2(0,1)$ into bounded subsets of $H^{2n}(0,1)$.*

Proof. Let $\lambda > 0$ be fixed and let Y be a bounded subset of X. Let

(5.1) $$u_p := (I + \lambda Q_{2n})^{-1} p\,,\ p \in Y\,,$$

i. e.

(5.2) $$Q_{2n} u_p = \frac{1}{\lambda}(p - u_p),\ p \in Y\,.$$

As $(I + \lambda Q_{2n})^{-1}$ is a contraction on X, it follows from (5.1) that the set $\{u_p\,; p \in Y\}$ is bounded in X. Therefore we get from (5.2)

(5.3) $$\{Q_{2n} u_p\,; p \in Y\} \text{ is bounded in } X\,,$$

which is similar to (4.5) (where the parameter t is replaced by p). Thus, we can again use the argument in the proof of Theorem 4.1 (or that of Lemma 3) to conclude that all the sets $\{u_p^{(j)}\,; p \in Y\}$ $(j = 0, 1, .., 2n)$ are bounded in X, i.e. $\{u_p\,; p \in Y\}$ is bounded in $H^{2n}(0,1)$.

THEOREM 5.1 - *Assume* a_{ij} $(i, j = 0, 1, .., n)$ *and* A *satisfy* $(H_2)'$, (H_3), *the matrix* (a_{ij}) *is symmetric (a.e.* $x \in]0,1[$*) and* $l = \partial g$, *where* $g : \mathbf{R}^{2n} \to]-\infty, \infty]$ *is proper, convex and* LSC. *Assume also*
(H_4) *There is at least a* $q \in H^{2n}(0,1)$ *which satisfies the problem*

(5.4) $$Q_{2n} q = 0,\ (0 < x < 1)\ ;\ q \text{ verifies } (1.3).$$

Let $f \in L^1(0, \infty; L^2(0,1))$, $u_0 \in L^2(0,1)$ *and let* $u(t,.)$ *be the corresponding (weak) solution of problem* (0.1), (0.2), (0.3) *on* $[0, +\infty[$. *Then, there is a solution* $\hat{q}$ *of* (5.4) *such that*

(5.5) $$u(t,.) \to \hat{q}\,, \text{ as } t \to \infty\,, \text{ strongly in } L^2(0,1)\,.$$

If in addition $f \in W^{1,1}(0, \infty; L^2(0,1))$ *then* (5.5) *holds in the weak topology of* $H^{2n}(0,1)$, *so, in particular*

(5.6) $$D_x^j u(t,.) \to \hat{q}\,, \text{ as } t \to \infty\,, \text{ in } C[0,1]\ (j = 0, 1, .., 2n-1)\,.$$

Proof. Let us first prove that for $f \in W^{1,1}(0, \infty; X)$, $X = L^2(0,1)$, and for any $\varepsilon > 0$ the set

(5.7) $$\{u(t,.)\ ;\ t \geqslant \varepsilon\} \text{ is bounded in } H^{2n}(0,1)\,.$$

Indeed, by Theorem 4.2, for every $\varepsilon > 0$ there is a $\delta \in]0, \varepsilon[$ such that $u(\delta,.) \in D(T_{2n})$. So, by Theorem 4.1, $u(t,.)$, $t \geqslant \varepsilon$ is a strong solution of (0.1), (0.2) (see also Remark 4.2). Now, we recall the well-known estimate (see, e.g., H. Brézis [3, p. 68])

$$\left\|\frac{d^+ u}{dt}(t,.)\right\|_X \leqslant \|Q_{2n} u(\varepsilon,.) - f(\varepsilon,.)\|_X + \int_\varepsilon^t \left\|\frac{d}{ds} f(s,.)\right\|_X ds\,,\ t \geqslant \varepsilon\,.$$

Therefore the set

(5.8) $\{ \frac{d^+ u}{dt}(t,.) ; t \geqslant \varepsilon \}$ is bounded in X.

On the other hand, as $f \in L^1(0, \infty ; L^2(0,1))$, it follows by virtue of (H_4):

(5.9) $\{u(t,.); t \geqslant 0\}$ is bounded in X,

because for any q verifying (5.4) we have

$$\| u(t,.) - q \|_X \leqslant \| u_0 - q \|_X + \int_0^t \| f(s,.) \|_X ds, \ t \geqslant 0 .$$

Now, by (4.2) we have

(5.10) $u(t,.) = (I + Q_{2n})^{-1}(u(t,.) - \frac{d^+ u}{dt}(t,.) + f(t,.)), \ t \geqslant \varepsilon$.

Now, since $\{f(t,.) ; t \geqslant 0\}$ is bounded in X, we obtain by (5.8) ~ (5.10) and Propositiion 5.1 the desired (5.7). So all we have to prove is (5.5). Since Q_{2n} is a subdifferential (see Lemma 2 and Remark 2.1), by virtue of a result of R. E. Bruck [4] we have for $f := 0$

(5.11) $u(t,.) \to q_1$, as $t \to \infty$, weakly in X,

where q_1 satisfies (5.4). Taking into account (5.7) and (5.11) we deduce that, for $f := 0$, $u(t,.) \to q_1$ as $t \to \infty$, weakly in $H^{2n}(0,1)$, *i.e.*, in particular, strongly in X. Hence (5.5) is proved for $f := 0$. Then, according to G. Morosanu [6, Theorem 2.1] it follows that (5.5) holds actually for f arbitrary in $L^1(0, \infty ; X)$. This ends the proof.

THEOREM 5.2 - *Assume* (H_1), (H_2)', (H_3) *and* (H_4) *are fulfiled. Moreover, assume that for a.e.* $x \in]0, [$ *the function*

(5.12) $r \mapsto A(x,r)$ *is strictly increasing on* $\mathbf{R}$.

Let $f \in L^1(0, \infty ; X)$, $u_0 \in L^2(0,1)$ *and let* $u(t,.)$, $t \geqslant 0$ *be the corresponding (weak) solution of problem* (0.1), (0.2), (0.3). *Then problem* (5.4) *has a unique solution, say* $\hat{q}$, *such that*

(5.13) $u(t,.) \to \hat{q}$, *as* $t \to \infty$, *strongly in* $L^2(0,1)$.

If in addition $f \in W^{1,1}(0, \infty ; L^2(0,1))$ *and* $u_0 \in D(T_{2n})$ *(see* (1.1)*) then* (5.13) *holds in the weak topology of* $H^{2n}(0,1)$.

Proof. As in the proof of the preceding theorem we find for $f \in W^{1,1}(0, \infty ; X)$, $X = L^2(0,1)$, and $u_0 \in D(T_{2n})$

(5.14) $\{u(t,.) ; t \geqslant 0\}$ is bounded in $H^{2n}(0,1)$.

Let us now show that Q_{2n} is strictly monotone. To this purpose let $q, \hat{q} \in H^{2n}(0,1)$ verify (5.4). Then

(5.15) $\langle T_{2n} q - T_{2n} \hat{q}, q - \hat{q} \rangle_X + \langle A(.,q(\cdot)) - A(.,\hat{q}(\cdot)), q - \hat{q} \rangle_X = 0.$

Since both terms in (5.15) are nonnegative we have

(5.16) $\quad [A(x, q(x)) - A(x, \hat{q}(x))] \cdot [q(x) - \hat{q}(x)] = 0$, a.e. $x \in]0, 1[$.

From (5.12) and (5.16) we have $q = \hat{q}$, i.e. indeed Q_{2n} is strictly monotone. Consequently (5.4) has a unique solution. To prove (5.13) we may assume without loss of generality that $f := 0$ (see again G. Morosanu [6, Theorem 2.1]).

Now, since Q_{2n} is strictly monotone we have

(5.17) $\quad \langle Q_{2n} q - Q_{2n} \hat{q}, q - \hat{q} \rangle_X = 0$ implies $q = \hat{q}$,

where $\hat{q}$ is the unique solution of (5.4). Note also that by (5.14) the set

(5.18) $\quad \{u(t, .) \; ; \; t \geqslant 0\}$ is relatively compact in X,

provided that $u_0 \in D(Q_{2n})$. According to A. Haraux [5, Theorem 29, p. 218 and Corollary 31, p. 225], (5.13) follows by (5.17) and (5.18). The second part of our theorem follows by (5.14). Q.E.D.

6. - Extensions

6.1. The above theory still works for the equation

(0.1)' $\quad \alpha(x) u_t + Q_{2n} u = f(t, x)$,

(to which we associate (0.2) and (0.3)) provided the following additional assumption holds:

$$\alpha \in L^\infty(0, 1) \text{ and } \alpha(x) \geqslant k_0 > 0, \text{ a.e. } x \in]0, 1[.$$

Indeed, let us consider equation (0.1)' divided by $\alpha(x)$ in the space $X_\alpha = L^2(0, 1; \alpha(x) dx)$ (the α-weighted L^2 space, which coincides algebrically and topologically with $X = L^2(0, 1)$) and define the operator $Q'_{2n} : D(T_{2n}) \subset X_\alpha \to X_\alpha$ by $Q'_{2n} u := [\alpha(\cdot)]^{-1} Q_{2n} u$, for any $u \in D(T_{2n})$. Then it is sufficient to observe that Q_{2n} is maximal monotone in X if and only if Q'_{2n} is maximal monotone in X_α.

6.2. We may investigate in a similar manner the same problem in the case $x \in]0, \infty[$ or $x \in \mathbf{R}$, with appropriate modifications of BC (0.2). So, in the case $x \in]0, \infty[$ we replace (0.2) by

(0.2)' $\quad \mathrm{col}\,[(M_{2n-1} u)(t, 0), (M_{2n-2} u)(t, 0), \ldots, (M_n u)(t, 0)] \in$
$\in l(\mathrm{col}\,[u(t, 0), D^1_x u(t, 0), \ldots, D^{n-1}_x u(t, 0)]), \; t > 0$
$u(t, .) \in L^2(0, \infty), \; t > 0$,

where $l : D(l) \subset \mathbf{R}^n \to \mathbf{R}^n$ is assumed to be maximal monotone. In the case $x \in \mathbf{R}$ we replace (0.2) by the asymptotic condition

(0.2)" $\quad u(t, .) \in L^2(\mathbf{R}), \; t > 0$.

As the basic spaces we take here $X = L^2(0, \infty)$ and $X = L^2(\mathbf{R})$ respectively. We leave to the reader to reformulate the hypotheses and all the results above for these cases.

ACKNOWLEDGEMENT

The results of this paper were communicated the 7th of December 1984 in the seminar of Prof. G. Mancini from the University of Trieste. The first author is very grateful to Professors G. Mancini and E. Mitidieri who kindly invited him to visit this institution.

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