SOLUTION OF A BVP CONSTRAINED IN AN INFINITELY DEEP POTENTIAL WELL (*)

by VITTORIO COTI ZELATI (**)

SOMMARIO. - *Si dimostra l'esistenza di una soluzione per il problema al contorno*

$$-\ddot{x} = \nabla U(x), x(0) = x(a) = 0,$$

dove $x:[0,a]\to \mathbf{R}^N, U:\Omega\subset\mathbf{R}^N\to\mathbf{R}$, *U convessa e* $U(x)\to+\infty$ *quando* $x\to\partial\Omega$. *Il metodo usato si basa sul Principio di Azione Duale di Clarke e Ekeland.*

SUMMARY. - *We prove existence of a solution for the boundary value problem*

$$-\ddot{x} = \nabla U(x), x(0) = x(a) = 0,$$

where $x:[0,a]\to \mathbf{R}^N$, $U:\Omega\subset\mathbf{R}^N\to\mathbf{R}$, *U convex and* $U(x)\to+\infty$ *as* $x\to\partial\Omega$. *The method employed is based on the use of the Dual Action Principle of Clarke and Ekeland.*

Let Ω be a bounded, open set in $\mathbf{R}^N$ and $U\in C^1(\Omega;\mathbf{R})$ be such that $U(x)\to+\infty$ as $x\to\partial\Omega$. Denote by ∇U the gradient of U.

Here we will discuss the problem of existence of solutions of the boundary value problem

$$(1)\quad \begin{cases} -\ddot{x} = \nabla U(x) & \forall t\in\,]0,a[\\ x(0) = x(a) = 0 . \end{cases}$$

(*) Comunicazione fatta al «Meeting on *Variational Methods in Differential Problems*» (Trieste, 26-28 settembre 1985).

(**) Indirizzo dell'Autore: International School for Advanced Studies - Strada Costiera, 11 - I 34014 Trieste.

The problem will be studied from a variational point of view through the use of the Dual Action Principle. To do this we will have to require U (and Ω) to be convex. The advantage of using this tecnique is the fact that the functional (whose critical points will be the solutions of (1)) will be defined on a whole Banach space E, while a more direct approach forces one to work in a subset of a Banach space, thus requiring a modification of the usual tools employed in the search of the critical points.

This work is strictly related to [1], where the periodic bvp is considered for equation (1), and where minimality of the period is also proved. For other results on periodic bvp's with infinitely deep potential well, see also [3], [6], where existence is proved working in an open subset of a Banach space.

Let $\Omega \subset \mathbf{R}^N$ be a open, bounded, convex set such that $0 \in \Omega$. Let $\Gamma = \partial\Omega$, $\Gamma_\varepsilon = \{x \in \Omega : d(x, \Gamma) < \varepsilon\}$ where

$$\mathrm{dist}(x, \Gamma) = \inf\{|x - y| : y \in \Gamma\}$$

and $|\cdot|$ is the euclidean norm in $\mathbf{R}^N$ corresponding to the scalar product $(\cdot|\cdot)$.

Let $U : \Omega \to \mathbf{R}$ be given. We say that U satisfies assumption (A) if:

(A1) $U \in C^1(\Omega; \mathbf{R})$, U is stricly convex;

(A2) $U(0) = 0 = \min U$;

(A3) $U(x) \to +\infty$ as $x \to \Gamma$ (uniformly);

(A4) $\exists \varepsilon > 0$ and $\theta \in]0, 1/2[$ such that

$$U(x) \leqq \theta(x \mid \nabla U(x)).$$

We will prove:

THEOREM - *Suppose U satisfies assumption* (A) *and*

$$\exists \varepsilon, \delta > 0 : \quad U(x) \leqq \frac{1}{(1+\varepsilon)\, a^2} |x|^2 \quad \forall |x| \leqq \delta ;$$

then problem (1) *has at least one solution.*

Proof. The proof will be carried out in two steps:

Step 1: use of the Dual Action Principle [4, 5] to transform (1) in a critical point problem for a functional f in a Banach space E.

Step 2: application of the Mountain Pass theorem [2] to f.

Step 1 (Dual Action Principle)

Let U^* be the Legendre transform of U, defined as

$$U^*(y) = \sup_{x \in \Omega} \{(x \mid y) - U(x)\}. \tag{2}$$

The properties of U^* are collected in the following lemma:

LEMMA 1 - *$U^* \in C^1(\mathbf{R}^N;\mathbf{R})$ and convex. Moreover $\exists c_1, c_2, c_3 > 0$ such that $\forall y \in \mathbf{R}^N$, $|y|$ large, one has:*

(3) $c_1|y| \leqq U^*(y) \leqq c_2|y|$;

(4) $|\nabla U^*(y)| \leqq c_3$;

(5) $U^*(y) \geqq (1-\theta)(y|\nabla U^*(y))$.

Proof. See [1, lemma 2.1].

Let $E = L^1(0,a;\mathbf{R}^N)$ with norm $\|u\|_1 = \int_0^a |u|\,dt$. Define $L: E \to E$ setting

$$Lu = v \text{ if and only if } -\ddot{v} = u,\ v(0) = v(a) = 0.$$

L is a linear selfadjoint operator from E into E; moreover, since $L(E) = W_0^{2,1}(0,a;\mathbf{R}^N)$, L is compact. It is easy to see that

(6) $\|Lu\|_{L^\infty} \leqq (a/4)\|u\|_1$.

Define $f \in C^1(E;\mathbf{R})$ setting

$$f(u) = \int_0^a [U^*(u) - \tfrac{1}{2}(u|Lu)]\,dt.$$

It is well defined by (3) and (6). If $u \in E$ is such that $f'(u) = 0$, then $-Lu + \nabla U^*(u) = 0$. Setting $x = \nabla U^*(u)$, one has $x = Lu$, hence $-\ddot{x} = u$ and $x(0) = x(a) = 0$. From $x = \nabla U^*(u)$ follows, from the properties of the Legendre transform, $u = \nabla U(x)$, and we finally get

$$\begin{cases} -\ddot{x} = \nabla U(x) \\ x(0) = x(a) = 0. \end{cases}$$

So we have proved that the critical points of f are solutions of (1). We now go to:

Step 2 (The Mountain Pass Theorem).

To apply the Mountain Pass Theorem we begin showing that f satisfies the Palais Smale (PS) condition, namely: $\forall\{u_n\} \subset E$ such that $f(u_n)$ is bounded and $f'(u_n) \to 0$ has a converging subsequence. The proof of the (PS) condition works as in [1]; we repeat it here for completeness. One has

$$f'(u_n)[u_n] = \int_0^a [(u_n|\nabla U^*(u_n)) - (u_n|Lu_n)]\,dt.$$

From (5) and Holder's inequality we get:

$$\int_0^a (u_n|Lu_n) \leqq \frac{1}{1-\theta}\int_0^a U^*(u_n) + \|f'(u_n)\|_{L^\infty}\|u_n\|_1 + c_4.$$

From this inequality and $f(u_n) \leqq c_5$, we get

$$\int_0^a U^*(u_n) \leqq c_5 + \tfrac{1}{2} \int_0^a (u_n \mid Lu_n) \leqq \frac{1}{2(1-\theta)} \int_0^a U^*(u_n) +$$

$$+ \tfrac{1}{2} \|f'(u_n)\|_{L^\infty} \|u_n\|_1 + c_6 .$$

Since $1 - \frac{1}{2(1-\theta)} > 0$ we get, using (3) ,

(7) $\quad \|u_n\|_1 \leqq c_7 + c_8 \|f'(u_n)\|_{L^\infty} \|u_n\|_1 .$

Since $\|f'(u_n)\|_{L^\infty} \to 0$, we get $\|u_n\|_1 \leqq$ const., and, up to subsequence, $Lu_n \to \bar{v}$ in C^0. Setting

$$z_n = Lu_n - f'(u_n) ,$$

it follows $z_n \to \bar{v}$ in L^∞, with $\bar{v}$ continuous function. Since $z_n(t) = \nabla U^*(u_n(t)) \in \Omega$ a.e. ($u_n \in L^1$ implies $|u_n(t)| < +\infty$ a.e.) we have that $\bar{v}(t) \in \bar{\Omega}$ $\forall t \in [0, a]$. We now claim that:

(8) $\quad \bar{v}(t) \in \Omega \quad \forall t \in [0, a]$.

Since $\bar{v}(0) = 0 \in \Omega$ and $\bar{v}$ is continuous, exists $\bar{t} > 0$ such that $\bar{v}(t) \in \Omega$ $\forall t \in [0, \bar{t}]$. Since $z_n(t) \to \bar{v}(t)$ in L^∞, then $z_n(t) \in \Omega'$ a.e. for $t \in [0, \bar{t}]$ for n large, where Ω' is a compact subset of Ω, and hence $u_n(t) = \nabla U(z_n(t)) \in L^\infty(0, \bar{t})$. Since ∇U is bounded in Ω' and continuous, we have

$$u_n(t) = \nabla U(z_n(t)) \to \bar{u}(t) = \nabla U(\bar{v}(t)) \text{ in } L^\infty(0, \bar{t}).$$

It follows from this fact and the definition of L that, weakly

$$-\ddot{\bar{v}}(t) = \nabla U(\bar{v}(t)) \text{ in }]0, \bar{t}[$$

and from the usual regularity theorems we deduce that $\bar{v}$ is a classical solution of $-\ddot{x} = \nabla U(x)$ $\forall t \in]0, \bar{t}[$. In particular the conservation of energy holds, i.e.

(9) $\quad 1/2 |\dot{\bar{v}}(t)|^2 + U(\bar{v}(t)) = \bar{c} \equiv 1/2 |\dot{\bar{v}}(\bar{t}/2)|^2 + U(\bar{v}(\bar{t}/2))$

$\forall t \in]0, \bar{t}[$.

Suppose exists $t' > \bar{t}$ such that $\bar{v}(t) \in \Omega$ $\forall t \in [\bar{t}, t'[$, $\bar{v}(t') \in \partial\Omega$. Since (9) holds, repeating the preeceding argument in $[0, t' - \varepsilon[$, ε small, one has that $U(\bar{v}(t)) \leqq \bar{c}$ $\forall t \in]0, t' - \varepsilon[$; from the continuity of $\bar{v}$ and of U in the region $\{x \in \Omega : U(x) \leqq \bar{c}\}$ one has $U(\bar{v}(t')) \leqq \bar{c}$ and $\bar{v}(t') \in \Omega$, contradiction which proves claim (8).

By the claim it follows that $\bar{v}(t) \in \Omega$ $\forall t \in [0, a]$; since $z_n \to \bar{v}$

in L^∞, we have, as before, that $u_n \to \nabla U(\bar{v})$ in $L^\infty(0,a)$ and, in particular, in L^1. This proves (PS).

The behaviour of f at $u = 0$ and at infinity is described in the following lemma:

LEMMA 2 - 1^0) *Exist* $r, b > 0$ *such that* $f(u) \geqq b \quad \forall \| u \|_1 = r$;

2^0) *Exist* $\bar{u} \in E, \| \bar{u} \|_1 > r$ *such that* $f(\bar{u}) < 0$.

Proof. See [1, lemma 2.5].

We can now apply the Mountain Pass Theorem to find the desidered solution.

REFERENCES

[1] A. AMBROSETTI, V. COTI ZELATI, *Solutions with minimal period for Hamiltonian systems in a potential well*, to appear on Ann. I.H.P. «Analyse non lineare».

[2] A. AMBROSETTI, P. RABINOWITZ, *Dual variational methods in critical point theory and applications*. J. Funct. Anal. *14* (1973), pp. 349-381.

[3] V. BENCI, *Normal modes of a Lagrangian system constrained in a potential well*. Ann. I.H.P. «Analyse non lineare», *1* (1984, pp. 379-400.

[4] F. CLARKE, *Periodic solutions of Hamiltonian inclusinos*. J. Diff. Eq. *40* (1981), pp. 1-6.

[5] F. CLARKE, I. EKELAND, *Hamiltonian trajectories having prescribed minimal period*. Comm. Pure and Appl. Math. *33* (1980), pp. 103-116.

[6]C. GRECO, *Remarks on periodic solutions of second order Hamiltonian systems in a unbounded potential well*, in «Nonlinear Oscillations for conservative systems» (A. Ambrosetti ed.), Venice 1985.