MULTIPLICITY PROBLEMS FOR ELLIPTIC EQUATIONS WITH NONLINEARITIES AT CRITICAL GROWTH (*)

by GIOVANNA CERAMI and SERGIO SOLIMINI (**)

SOMMARIO. - *Si comunicano i risultati di* [4], *in cui è dimostrato che il problema* (P), *in seguito definito, ha infinite soluzioni radiali su un dominio sferico in dimensione maggiore o uguale di* 7.

SUMMARY. - *The results in* [4] *are communicated, which show that problem* (P), *defined below, has infinitely many radial solutions on a ball in dimension bigger or equal than* 7.

We communicate with some further comments the results of [4], obtained jointly with M. Struwe, concerning the problem:

$$
\text{(P)} \qquad \begin{cases} -\Delta u = \lambda u + |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}
$$

where Ω is a bounded regular domain of $\mathbf{R}^N$, $N \geq 3$ and $p = \dfrac{2N}{N-2}$ is the critical Sobolev exponent.

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(**) Indirizzi degli Autori: G. Cerami, Dipartimento di Matematica e Appl., Via Archirafi, 34 - I 90123 Palermo — S. Solimini, International School for Advanced Studies, Strada Costiera, 11 - I 34014 Trieste.

The fact that the injection of $H_0^1(\Omega)$ in $L^p(\Omega)$ is not compact makes the study of (P) particularly difficult with respect to the case: $p < \frac{2N}{N-2}$ (subcritical case) and actually from the well-known Pohozaev identity [6] one knows that (P) has no nontrivial solution if $\lambda \leq 0$ and Ω is star-shaped. In [1] H. Brezis and L. Nirenberg have proved the following result:

THEOREM 1 - *There exist a* $\lambda^* \in [0, \lambda_1[$ *such that* (P) *has a positive solution if* $\lambda \in]\lambda^*, \lambda_1[$. *Moreover if* $N \geq 4$ *then* $\lambda^* = 0$.

The case $N \geq 4$ is the case we are concerned with and that we shall always consider in what follows. The condition $\lambda < \lambda_1$ is clearly necessary for the existence of a positive solution to (P), however one can expect a nontrivial solution for any given $\lambda > 0$. Partial answers to this question were given by the first A. with D. Fortunato and M. Struwe in [3] and subsequently A. Capozzi, D. Fortunato and G. Palmieri [2] proved the following:

THEOREM 2 - *For any* $\lambda > 0$ (P) *has a nontrivial solution.*

The results in Theorems 1-2 were substantially proved in the following way. One defines on $H_0^1(\Omega)$ the functional:

$$I_\lambda(u) = \frac{1}{2}\left[\int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2\right] - \frac{1}{p}\int_\Omega |u|^p$$

and looks for critical points of I_λ. For a given $c \in \mathbf{R}$ one considers the local Palais-Smale condition in c:

$[\mathrm{P.S.}]_c$: For any sequence $(u_n)_n$ in $H_0^1(\Omega)$ such that:

a) $I_\lambda(u_n) \to c$

b) $\nabla I_\lambda(u_n) \to 0$

there exists a converging subsequence.

The «lack of compactness» of (P) is expressed by the fact that the Palais-Smale condition does not hold «globally» namely $[\mathrm{P.S.}]_c$ is false for some levels $c \in \mathbf{R}$.

Then one shows (see [3]) that, denoting by S the Sobolev constant:

$$S = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2}{(\int_\Omega |u|^p)^{2/p}}$$ the lowest c such that $[\mathrm{P.S.}]_c$ does not hold

is $\frac{1}{N} S^{N/2}$. The above stated theorems are then proved by constructing a min-max class which works at a level $c_0 < \frac{1}{N} S^{N/2}$. In the case $\lambda < \lambda_1$, c_0 is the lowest nonzero critical level.

A similar approach does not look reasonable if one is looking for multiple (pairs of) solutions to (P), in particular for infinitely many ones since one expects in this case to find critical points at larger and larger positive levels.

The fact that (P) has infinitely many solutions in some cases has been shown by D. Fortunato and E. Jannelli [5] by considering some domains Ω with suitable symmetries and working on suitable subspaces of $H^1_0(\Omega)$. In such a way one can replace S by larger and larger values, considering functions with many sign changes. A rough explanation of the underlying idea, in the case that Ω is a ball B_R, is the following: for any integer k large enough one splits Ω in $2k$ equal sectors, puts in one of them the positive solution given by Theorem 1 and replaces it in the other sectors by odd specular reproduction.

A different approach is needed if one works with a general Ω or if one looks for particular solutions which do not allow this use of the symmetry, for instance if one looks for radial solutions on a ball.

In [4] some contributions in this direction are given, in particular the following results are proved.

THEOREM 3 - *Let* $N \geq 6$ *and* $\lambda \in]0, \lambda_1[$. *Then* (P) *has at least two (pairs of) nontrivial solutions.*

THEOREM 4 - *Let* $\Omega = B_R$, $N \geq 7$ and $\lambda \in]0, \lambda_1[$. *Then* (P) *has infinitely many radial solutions.*

The two results are closely related. One considers the class U given by:

$$U = \{u \in H^1_0 \mid u^{\pm} \neq 0 \,,\ (\nabla I_\lambda(u^{\pm}), u^{\pm}) = 0 \text{ for both the } \pm \text{ signs}\}.$$

It is easily seen that to look for a minimum of I_λ on U is equivalent to looking for a two-dimensional min-max surface. Therefore, by using a suitable version of the Rabinowitz saddle point Theorem, see [7], [4], one gets a sequence (u_n) such that:

(1) $d(u_n, U) \to 0$

(2) $I_\lambda(u_n) \to c_1 := \inf_U I_\lambda$

(3) $\nabla I_\lambda(u_n) \to 0$.

Moreover in the same way one sees that if u is any point of minimum of I_λ on U, then u is a critical point for I_λ. Collecting these informations one sees that if $[\text{P.S.}]_c$ holds for $c = c_1$ then one immediately gets a solution to (P) in U. Unfortunately there is no reason to expect this. However, using the result by M. Struwe in [8] one also has the following information:

(i) The lowest level $c > \frac{1}{N} S^{N/2}$ such that $[\text{P.S.}]_c$ does not hold is

$$c_0 + \frac{1}{N} S^{N/2};$$

(ii) if $(u_n)_n$ satisfies (a) and (b) in $[\text{P.S.}]_c$, for $c = \frac{1}{N} S^{N/2}$, and $(u_n)_n$ has no converging subsequences then one has in $H_0^1(\Omega)$:

$$u_n^+ \to 0 \quad \text{or} \quad u_n^- \to 0, \tag{4}$$

(ii) follows from the fact that $(u_n)_n$ must be approximated by a sequence of rescaled one instanton solutions cut off at infinity, and those have a constant sign. The assumption $N \geq 6$ produces the estimates:

$$c_1 < c_0 + \frac{1}{N} S^{N/2}. \tag{5}$$

Moreover $u \in U$ implies:

$$\frac{\int_\Omega |u^\pm|^p}{\int_\Omega |\nabla u^\pm|^2 - \lambda \int_\Omega (u^\pm)^2} = 1 \quad \text{and therefore:}$$

$$\left(\int_\Omega |u^\pm|^p\right)^{\frac{p-2}{p}} = \frac{\int_\Omega |\nabla u^\pm|^2 - \lambda \int_\Omega (u^\pm)^2}{\left(\int_\Omega |u^\pm|^p\right)^{2/p}} \geq S_\lambda := \inf_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^2 - \lambda \int_\Omega v^2}{\left(\int_\Omega |v|^p\right)^{2/p}} > 0$$

since $\lambda < \lambda_1$.

From this last estimate, since the L^p norm is a monotone norm, one sees that (4) cannot hold if $(u_n)_n$ verifies (1)-(2)-(3). Therefore from (5), (i) and (ii) the fact that $(u_n)_n$ has a converging subsequence follows and Theorem 3 is proved.

Finally we shortly indicate how one can derive Theorem 4 from Theorem 3. First we need a radial version of the first result.

Assume Ω is a ball or an annulus and let E denote the radial functions in $H_0^1(\Omega)$. By using the same variational approach than in Theorem 1 on E one proves:

THEOREM 3' - *Let $N \geqslant 7$ and $\lambda \in [0, \lambda_1[$. Then* (P) *has two (pairs of) nontrivial radial solutions. Moreover if u is any point of minimum for I_λ in $U \cap E$, then u is a solution to* (P).

The fact that we ask one dimension more than in Theorem 3 is due to the bigger difficulty in deriving (5) without the possibility of testing nonradial functions of $H^1_0(\Omega)$.

Now one assumes $\Omega = B(0, R)$ and defines for any $k \in \mathbf{N}$:

$$U_k = \{u \in E \mid \exists\ 0 \equiv r_k < r_{k-1} < r_{k-2} < \ldots < r_0 = R \text{ such that } u(r_i) = 0,\ 0 \leq i < k,\ (-1)^i u(x) \geq 0 \text{ in } S_i,\ u \neq 0 \text{ in } S_i \text{ and } \int_{S_i}(|\nabla u|^2 - \lambda u^2 - |u|^p) = 0 \text{ for } i = 1, \ldots, k\}$$

where $S_i = \{x \in B(0, r) \mid r_i \leq |x| \leq r_{i-1}\}$ for $i = 1, \ldots, k$.

From Theorem 3' we deduce:

PROPOSITION 5 - *If u is a minimum for I_λ on U_k, then u is a radial solution to* (P).

The proof of Proposition 5 is immediate. If $k = 2$ it is just the last part of Theorem 3' since to find a minimum in $U \cap E$ is equivalent to finding a minimum in U_2. For $k > 2$ one can use the preceding case by restricting oneself to the subdomains $\Omega_i := \overset{0}{\widehat{S_i \cup S_{i+1}}}$ for $i = 1, \ldots, k-1$. Of course the restriction of u on Ω_i is in the class U of Ω_i and minimizes I_λ on it. Since $(\Omega_i)_{1<i<k-1}$ is an open covering of Ω and u solves (P) on Ω_i, then u is a solution to (P) on Ω.

In this way the proof of Theorem 4 is complete if we construct a minimum in every U_k. To this aim one first looks for the optimal position of the nodal values $0 < r_{k-1} < \ldots r_1 < r_0 = R$ by taking a minimizing sequence $(u_n)_n$ on U_k and considering the corresponding nodal points $r_{i,n}$. Then one takes as r_i a limit point for the sequence $(r_{i,n})_n$. The fact that $r_{i+1} < r_i < r_{i-1}$ for $1 \leq i \leq k-1$ is then a consequence of the assumption $N \geq 7$. By easy computations one can observe that if one defines u on S_i as a radial solution at minimum level for I_λ, for $i = 1, \ldots, k$, then u minimizes I_λ on U_k.

We now make a few remarks about the difference with respect to the subcritical case. Let

$$\bar{c}_k = \inf_{u \in U_k} I_\lambda(u)$$

and let $u_k \in U_k, I_\lambda(u_k) = \bar{c}_k$. Then one has

$$\forall k \in \mathbf{N} : \bar{c}_{k+1} < \bar{c}_k + \frac{1}{N} S^{N/2}$$

and

$$\lim_k (\bar{c}_{k+1} - \bar{c}_k) = \frac{1}{N} S^{N/2}$$

(while in the subcritical case:

$$\lim_k (\bar{c}_{k+1} - \bar{c}_k) = +\infty) .$$

As a consequence one sees that the nodal lines of u_k can only accumulate at the centre of the ball Ω as $k \to +\infty$.

More precisely, if we fix $r \in]0;R]$ then $\exists \varepsilon > 0$ such that for every $k \in \mathbf{N}$ there exists at most one $r' \in [r-\varepsilon, \inf(R, r+\varepsilon)]$ such that $u_k(x) = 0$ if $|x| = r'$. In the subcritical case the number of the values r' increases to $+\infty$ as $k \to +\infty$.

This fact also allows us to pass to the limit with respect to k and to find a radial solution u_∞ of the equation

$$(6) \qquad -\Delta u = \lambda u + |u|^{p-2} u$$

on the open set $B_R \backslash \{0\}$ which has infinitely many oscillations in 0. Such a solution does not exist in the subcritical case. In fact if $R = r_0 > r_1 > \ldots > r_n > \ldots$ denotes an infinite sequence of positive real numbers such that $u(x) = 0$ if $|x| = r_i$, S_i is defined as above and $a_i = r_i \int_{|x|=r_i} |\nabla u_\infty|^2$ then the Pohozaev identity implies

$$(7) \qquad a_{i-1} - a_i \geq \left(\frac{N}{p} - \frac{N-2}{2} \right) (\textstyle\int_{S_i} |u_\infty|^p + \lambda \int_{S_i} u_\infty^2) \geq 0$$

for $i > 1, p < \dfrac{2N}{N-2}$. Therefore $(a_i)_i$ is a decreasing sequence of positive real numbers and then: $\lim_i (a_{i-1} - a_i) = 0$. On the other hand if $p < \dfrac{2N}{N-2}$ then $\varepsilon = \min(1, \dfrac{N}{p} - \dfrac{N-2}{2}) > 0$ and from (6) - (7) one has

$$a_{i-1} - a_i \geq \varepsilon \textstyle\int_{S_i} (|u_\infty|^p + \lambda u_\infty^2) = \varepsilon \int_{S_i} |\nabla u_\infty|^2$$

and therefore: $\lim_i \int_{S_i} |\nabla u_\infty|^2 = 0$, which is a contradiction since u_∞ is a nontrivial solution to (P) for $\Omega = S_i$ and $|S_i| \to 0$.

REFERENCES

[1] BREZIS H., NIRENBERG L., *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents.* Comm. Pure. Appl. Math. XXXVI (1983), pp. 437-477.

[2] CAPOZZI A., FORTUNATO D., PALMIERI G., *An existence result for non-*

linear elliptic problems involving critical Sobolev exponent, to appear in Ann. Ins. H. Poincaré: Analyse Nonlinéaire.

[3] CERAMI G., FORTUNATO D., STRUWE M., *Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents.* Ann. Inst. H. Poincaré: Analyse Nonlinéaire, 1 (1984), pp. 341-350.

[4] CERAMI G., SOLIMINI S., STRUWE M., *Some existence results for superlinear elliptic boundary value problems involving critical exponents.* Bonn University preprint n. 734 (1985), to appear in J. Functional Analysis.

[5] FORTUNATO D., JANNELLI E., *Infinitely many solutions for some nonlinear elliptic problems in symmetrical domains,* to appear in Proc. Roy. Soc. Edinburgh.

[6] POHOZAEV S. I., *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$. Soviet Math. Doklady 6 (1965), pp. 1408-1411.

[7] SOLIMINI S., *On the solvability of some elliptic partial differential equations with the linear part at resonance,* to appear in J. Math. Analysis Appl.

[8] STRUWE M., *A global compactness result for elliptic boundary value problems involving limiting nonlinearities.* Math. Z. 187 (1984), pp. 511-517.