SOME RECENT RESULTS IN THE HOMOTOPY INDEX THEORY IN INFINITE DIMENSIONS (*)

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SOMMARIO. - *Si fornisce un criterio di ammissibilità nell'ambito della teoria dell'indice di omotopìa in spazi metrici e si confronta la condizione di ammissibilità con la condizione di Palais-Smale. Nel caso di problemi variazionali, si collega l'indice di omotopìa alla nozione di gruppi critici di un punto critico. Infine, si applica la teoria dell'indice di omotopìa per stabilire un «principio di perequazione» per soluzioni periodiche di sistemi del secondo ordine di tipo gradiente.*

SUMMARY. - *In this note we give a criterion for admissibility in the homotopy index theory on metric spaces and compare admissibility with the Palais-Smale condition. For variational problems, we relate the homotopy index to the concept of critical groups of a critical point. Finally, we use the homotopy index to establish an «averaging principle» for periodic solutions of second order gradient systems.*

1. Introduction

The homotopy or Conley index was defined by C. Conley [1] for (two-sided) flows on compact spaces. Thus the applicability of the original Conley's theory is restricted to problems definable by or reducible to finite-dimensional ordinary differential equations.

In the papers [9] - [11] the homotopy index theory was exten-

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ded to certain classes of (one-sided) semiflows on nonnecessarily locally compact metric spaces. This theory was further developed and applications to functional and partial differential equations were given in the works [12] - [21].

The key assumption in this extended homotopy index theory is the so-called admissibility condition.

It is the aim of this note to give a general criterion for admissibility and to examine the connection between the admissibility and the Palais-Smale conditions.

We also review some recent results relating the homotopy index to the well-known critical groups in Morse theory.

Finally we state a useful continuation principle for the homotopy index in the spirit of the coincidence degree and indicate how this principle can be applied to prove the existence of periodic solutions for second-order gradient systems.

2. Admissibility and the Palais-Smale condition

Let X be a metric space and π be a local semiflow on X. We write $x\pi t := \pi(t, x)$, i.e. $x\pi t$ is the value at time t of the solution with initial value x; ω_x denotes the endpoint of this solution.

We say that a closed set $N \subset X$ is *strongly π-admissible* if the following conditions are satisfied:

(1) *π does not explode in N,* i.e. whenever $x \in X$ and $x\pi\,[0, \omega_x) \subset N$, then $\omega_x = \infty$.

(2) N is π-admissible, i.e. whenever $\{x_n\} \subset X$ and $\{t_n\} \subset \mathbf{R}^+$ are sequence with $x_n\pi\,[0, t_n] \subset N$ for all $n \in \mathbf{N}$ and $t_n \to \infty$ for $n \to \infty$, then the sequence $\{x_n \pi t_n \mid n \in \mathbf{N}\}$ has a convergent subsequence.

If K is an isolated invariant set (relative to π) which has a strongly π-admissible isolating neighborhood, then the homotopy index $h(\pi, K)$ is defined and, by definition, equal to the homotopy type of the quotient space $(B/B^-, [B^-])$, where B is any strongly π-admissible isolating block for K and B^- is the set of all strict egress and bounce-off points of B. Instead of the pair $\langle B, B^- \rangle$, one can take, more generally, any «admissible» index or quasi-index pair $\langle N_1, N_2 \rangle$ (see [9], [10]).

Suppose the pair (π, K) is varied in an admissible continuous way, i.e. assume there is a «continuous» parametrized family $\lambda \to (\pi_\lambda, K_\lambda)$ of pairs consisting of a local semiflow π_λ on X and an isolated invariant set K_λ (relative to π_λ). λ is in a metric parameter space Λ and all sets K_λ are isolated by the same isolating neighborhood N which satisfies a «collective» admissibility assumption. Then $h(\pi_\lambda, K_\lambda)$ is constant for λ lying in connected components of Λ. This is the important homotopy invariance property of the index making it a useful instrument in global perturbation problems (see [9]).

We will now give a useful criterion of strong π-admissibility. Let X be a Banach space and A be a sectorial operator on X generating the family X^β, $\beta > 0$ of fractional power spaces. Fix $0 < \alpha < 1$ and let $f: X^\alpha \to X$ be a locally Lipschitzian operator. Then the strong solutions of the equation

$$\dot{u} = -Au + f(u) \tag{1}$$

generate a local semiflow π_f on X^α (see [3], [12]).

We now have the following

THEOREM 1 - *Suppose that there is a direct sum decomposition* $X = X_1 \oplus X_2$ *with* $A(D(A) \cap X_i) \subset X_i$, $A_i := A \mid D(A) \cap X_i$, $i = 1, 2$, *such that* $\dim X_2 < \infty$, A_i *is sectorial on* X_i, $i = 1, 2$ *and* re $\sigma(A_1) > \delta > 0$ *for some* δ.

Let N *be closed and bounded in* X^α *and* $f(N) \subset C$ *where* C *is compact in* X.

Then N *is strongly* π*-admissible.*

Proof. That π_f does not explode in N follows (under much more general hypotheses) from Theorem 3.3.4 in [3].

Let $t_n \to \infty$ as $n \to \infty$ and for $n \in \mathbf{N}$ u_n be a solution of (1) defined on $[0, t_n]$ and such that $u_n(t) \in N$ for $t \in [0, t_n]$. We have to show that the sequence $\{u_n(t_n) \mid n \in \mathbf{N}\}$ has a convergent subsequence.

Using Theorem 1.4.3 in [3] and our hypotheses we obtain an $h > 0$ and $0 < k < 1$ such that

$$\| e^{-A_1 h} u \|_\alpha \leq k \| u \|_\alpha \tag{2}$$

for all $u \in X_1^\alpha$.

Let P_i be the projector on X_i, $i = 1, 2$ in the above direct sum.

Then using our assumption on f we obtain the existence of a compact set $D \subset X$ such that for every continuous function $u: [0, h] \to N$

$$\int_0^h e^{-A_1 s} P_1 f(u(s))\, ds \in D. \tag{3}$$

Write $u_n^i(t) = P_i u_n(t)$, and let β be the Kuratowski-measure of noncompactness on X^α. Since $\dim X_2 < \infty$ and N is bounded we only have to show

$$\beta\{u^1_n(t_n) \mid n \in \mathbf{N}\} = 0 \tag{4}$$

Since $t_n \to \infty$ as $n \to \infty$ we get from the variation-of-constants formula

$$\beta\{u^1_n(t_n) \mid n \in \mathbf{N}\} = \beta\{u^1_n(t_n) \mid t_n \geq h\} =$$

$$\beta\{e^{-A_1 h} u^1_n(t_n - h) + \int_{t_n - h}^{t_n} e^{-A_1(t_n - s)} P_1 f(u_n \pi s)\, ds \mid t_n > h\}$$

$$\leq \beta\{e^{-A_1 h} u^1_n(t_n - h) \mid t_n \geq h\} + \beta(D) \leq k\beta\{u^1_n(t_n - h) \mid t_n \geq h\}$$

Repeating this argument we get for any $m \in \mathbf{N}$,

$$\beta\{u^1_n(t_n) \mid n \in \mathbf{N}\} \leq k^m \cdot \beta(N) \tag{5}$$

Since $k < 1$, we obtain (4) by letting $m \to \infty$ in (5). The theorem is proved.

REMARK - Theorem 1 is, in particular, applicable if (1) is an ordinary differential equation on X, i.e. if $A : X \to X$ is linear and bounded, $X^\alpha = X$ and the above direct sum decomposition exists.

We will now look into the relation between the admissibility assumption and the Palais-Smale condition.

Let X be a Banach space and $g : X \to X$ be a locally Lipschitzian mapping. Let π_g be the local flow on X generated by the ordinary differential equation

$$\dot{u} = g(u) \tag{6}$$

By the same symbol π_g we also denote the local semiflow consisting of solutions of (6) defined for nonnegative times.

Let N be a subset of X. We say that g satisfies a *Palais-Smale condition on* N if every sequence $\{x_n\} \subset N$ with $g(x_n) \to 0$ has a convergent subsequence. Usually X is a Hilbert space and $g = -\nabla\Phi$ for some functional $\Phi : X \to \mathbf{R}$. Then we say that Φ *satisfies a Palais-Smale condition on* N *if* g *does.*

Now we have

THEOREM 2 - *Let* $\tilde{N}$, N *be subsets of* X *with* $\tilde{N} \subset N$ *and* N *closed. Suppose that* $\operatorname{dist}(\tilde{N}, \partial N) =: c > 0$ *and* g *is Lipschitzian on* N. *If* N *is* π*-admissible, then* g *satisfies the Palais-Smale condition on* $\tilde{N}$.

Proof. Let L be the Lipschitz constant of g on N, and $u : [0, t_0] \to N$ be any solution of (6), $t_0 > 0$. We claim that

$$\| x(t) - x(0) \| \leq \| f(x(0)) \| \cdot L^{-1} \cdot e^{Lt} \tag{7}$$

for all $t \in [0, t_0]$.

In fact

$$\| x(t) - x(0) \| \leq \int_0^t \| f(x(s)) - f(x(0)) \| \, ds + \int_0^t \| f(x(0)) \| \, ds \leq$$

$$\leq L \int_0^t \| x(s) - x(0) \| \, ds + \| f(x(0)) \| \, t \tag{8}$$

Now (8) and simple differential inequality arguments easily imply (7).

Now let $\{x_n\} \subset N$ be a sequence with $g(x_n) \to 0$ as $n \to \infty$. Let $\pi = \pi_g$. Define

$$s^-(x_n) = \sup \{ t \geq 0 \mid x_n \pi [-t, 0] \text{ is defined and } \subset N \}.$$

Set $u_n(t) = x_n \pi (-t)$ as long as defined. Then $\dot{u}_n(t) = - g(u_n(t))$ as long as defined.

From (7) we obtain

$$\| u_n(t) - u_n(0) \| \leq \| g(x_n) \| \cdot L^{-1} e^{Lt} \tag{9}$$

for $t < \infty$, $t \in [0, s^-(x_n)]$.

This implies that either $s^-(x_n) = +\infty$ or else $u_n(s^-(x_n))$ is defined and $\in \partial N$. In the latter case we obtain from (9)

$$c \leq \| u_n(s^-(x_n)) - x_n \| \leq \| g(x_n) \| \cdot L^{-1} e^{Ls^-(x_n)} \tag{10}$$

Now (10) and the fact that $g(x_n) \to 0$ as $n \to \infty$ imply that $s^-(x_n) \to \infty$ for $n \to \infty$.

Define $t_n = n$ if $s^-(x_n) = \infty$ and $t_n = s^-(x_n)$ otherwise and $z_n = u_n(-t_n)$.

Then $z_n \pi [0, t_n] \subset N$ as $t_n \to \infty$. The admissibility of N implies that $\{z_n \pi t_n \mid n \in \mathbf{N}\}$ has a convergent subsequence. Since $z_n \pi t_n = x_n$, the theorem follows.

Remark - There is no converse of Theorem 2: In fact let X be a Hilbert space with X_1 and $X_2 := X_1^{\perp}$ two infinite dimensional subspaces. For $u \in X$ let $u^i \in X_i$, $i = 1, 2$ be such that $u = u^1 + u^2$. Define $\Phi : X \to \mathbf{R}$ by $\Phi(u) = ½(\| u^1 \|^2 - \| u^2 \|^2)$.

Then $g(u) = -\nabla \Phi(u) = u^2 - u^1$.

If $\{u_n\} \subset X$ is such that $g(u_n) \to 0$, then $u_n \to 0$, so that the Palais-Smale condition holds on any set $N \subset X$. However, if N is the closed unit ball and $\{e_n\}$ is an orthonormal sequence in X_2, then $e_n \pi_g t = \exp(t) \cdot e_n \in N$ for all $t \leq 0$. Therefore if N were π-admissible, $\{e_n\}$ would contain a (strongly) convergent subsequence, a contradiction.

3. Critical groups and the homotopy index

Let X be a Hilbert space and u_0 be an isolated equilibrium of equation (6). Suppose that, locally around u_0, $g = -\nabla\Phi$, where $\Phi : X \to \mathbf{R}$ is some functional.

For $c \in \mathbf{R}$ write

$$\Phi^c = \{u \in X \mid \Phi(u) \leq c\}.$$

Let H_q, $q \in \mathbf{Z}$ by any (unreduced) homology or cohomology theory. Let B be any closed neighborhood of u_0 such that there is no other equilibrium of (6) in B.

The *critical groups* $C_q(\Phi, u_0)$ are defined as

$$C_q(\Phi, u_0) = H_q(\Phi^c \cap B, \Phi^c \cap B \setminus \{u_0\}), \quad q \in \mathbf{Z} \tag{11}$$

The excision axiom of (co)homology easily implies that $C_q(\Phi, u_0)$ are independent (up to an isomorphism) of the choice of B.

The concept of critical groups is due to Rothe [8]. It provides an extension of the classical Morse-index to degenerate equilibria. We will now see how critical groups relate to the homotopy index. Since g is gradient around u_0, it follows that $K = \{u_0\}$ is an isolated invariant set relative to π_g. If K admits a strongly π_g-admissible isolating neighborhood, then the homotopy index $h(\pi_g, \{u_0\})$ is defined and equal to the homotopy type of pointed spaces $(N_1 / N_2, [N_2])$, $\langle N_1, N_2 \rangle$ being any «admissible» (quasi-)index pair for $\{u_0\}$. Consequently the groups $H_q(N_1 / N_2, \{[N_2]\})$, $q \in \mathbf{Z}$ are independent of the choice of $\langle N_1, N_2 \rangle$ and, therefore, the (co)homology groups $H_q(h(\pi_g, \{u_0\}))$ are well-defined (up to an isomorphism, of course).

Now the following result holds:

THEOREM 3 - *Let u_0 be as above and assume that $K = \{u_0\}$ admits a strongly π_g-admissible isolating neighborhood.*

Then

$$H_q(h(\pi_g, \{u_0\})) \cong C_q(\Phi, u_0), \tag{12}$$

for all $q \in \mathbf{Z}$.

In other words, the critical groups of u_0 are nothing else but the (co)homology groups of the homotopy index of $K = \{u_0\}$.

For the proof of Theorem 1, we refer the reader to [19]. The critical groups are defined even if $\{u_0\}$ does not admit a strongly π_g-admissible neighborhood. However, under some additional hypotheses, they are all trivial in this case:

THEOREM 4 - *Suppose u_0 is as above. Moreover, assume that Φ is of class C^2 in a neighborhood of u_0 with $L := \Phi''(u_0) = -g'(u_0)$ being a Fredholm operator.*

Then either $\{u_0\}$ admits a strongly π_g-admissible isolating neighborhood or else all critical groups $C_q(\Phi, u_0)$ are zero.

This theorem is proved in [19]. The proof uses, among other arguments, a generalized Morse-Lemma due to Mawhin and Willem [7].

4. A continuation principle

The homotopy invariance property of the homotopy index mentioned above enables us to formulate a useful continuation principle similar, in spirit, to the coincidence degree of Mawhin [5].

THEOREM 5 - *Let A and f be as in equation* (1). *Assume the following hypotheses:*

(1) *Equation* (1) *is gradient-like with respect to some function $V : X^\alpha \to \mathbf{R}$.*

(2) *A has compact resolvent. Moreover, there is a $\delta > 0$ such that $\sigma(A) = \{0\} \cup \sigma'$, re $\sigma' > \delta$.*
Let $X_1 = \ker A$ and $P : X \to X$ be the projector onto X_1 associated with this spectral decomposition.

(3) *There is a set $\Gamma \subset X^\alpha$ open in X^α bounded in X and $f(\Gamma)$ is bounded in X. We write N for the closure of Γ in X.*

(4) *For every $\lambda \in (0, 1)$, if $u : \mathbf{R} \to X^\alpha$ is a (strong) solution of*

$$\dot{u} = -Au + \lambda(I - P)f(u) + Pf(u) \tag{13_λ}$$

with $u(\mathbf{R}) \subset N$, then $u(\mathbf{R}) \subset \Gamma$.

(5) *Let $\Gamma_1 = \Gamma \cap X_1$, and $\hat{\pi}$ be the local flow on X_1 generated by the ODE*

$$\dot{u}_1 = Pf(u_1) \tag{14}$$

Let $\hat{K}$ be the largest invariant set in the closure of Γ_1 relative to X_1. Then $K \subset \Gamma_1$ and the Conley index $h(\hat{\pi}, \hat{K}) \neq 0$.

Under the above hypotheses, there exists a solution $u_0 \in N$ of the equation

$$-Au + f(u) = 0 \tag{15}$$

REMARKS - Hypothesis (1) means, essentially, that (1) has variational structure.

The proof of Theorem 5 is an application of the homotopy invariance of our (extended) homotopy index. For details, see [17].

Theorem 5 is applicable to periodic boundary value problems:

Let $f : \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^m$ be a continuous mapping. Assume that for some T and all $(x, u) \in \mathbf{R} \times \mathbf{R}^m$, $f(x+T, u) = f(x, u)$.

Moreover, let f be locally Lipschitzian in u, uniformly in x.

We are looking for T-periodic solutions of

$$u''(x) + f(x, u(x)) = 0 \tag{16}$$

Let $X = L^2([0, T], \mathbf{R}^m)$ and

$$D = \{u \in H^2([0, T], \mathbf{R}^m) \mid u(0) = u(t), u'(0) = u'(T)\}.$$

Define $A : D \to X$ by $Au = -u''$. Then A is sectorial on X. Moreover

$$X^{1/2} = \{u \in H^1([0, T], \mathbf{R}^m) \mid u(0) = u(T), u'(0) = u'(T)\}.$$

Define the Nemitski operator $\hat{f} : X^{1/2} \to X$ by $\hat{f}(u)(x) = f(x, u(x))$. Dropping the hat « $\wedge$ » we then see that equation (16) has the form of equation (15). Moreover, if $f(x, u) = \dfrac{\partial F}{\partial u}(x, u)$ for some function F, i.e. if f is gradient, then the resulting equation (1) is gradient-like.

Now an application of Theorem 5 to the system (16) yields the following extension of the averaging method:

THEOREM 6 - *Assume that $f : \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^m$ is gradient and satisfies the above assumptions. Let $\mathcal{C}^+$ (respt. $\mathcal{C}^-$) be a finite (possibly empty) set of C^1-functions $V : \mathbf{R}^m \to \mathbf{R}$ such that the set*

$$G = \{u \in \mathbf{R}^m \mid V(u) < 0 \text{ for all } V \in \mathcal{C}^+ \cup \mathcal{C}^-\}$$

is nonempty, connected and such that for every $u \in \partial G$ and $V \in \mathcal{C}^+$ (resp. $V \in \mathcal{C}^-$) with $V(u) = 0$,

$$\langle \frac{\partial V}{\partial u}(x, u), f(x, u) \rangle < 0 \text{ (resp. } \langle \frac{\partial V}{\partial u}(x, u), f(x, u) \rangle > 0 \text{ for all } x \in \mathbf{R}.$$

Let $B^- = \{u \in \partial G \mid V(u) = 0 \text{ for some } V \in \mathcal{C}^-\}$.

Suppose that B^- is disconnected.

Then there exists an $M > 0$ and an $\varepsilon_0 < 0$ such that whenever $0 < \varepsilon < \varepsilon_0$, then there exists a T-periodic solution $u = u_\varepsilon$ of

$$u''(x) + \varepsilon f(x, u(x)) = 0 \tag{17}$$

satisfying

$$\bar{u} = \frac{1}{T}\int_0^T u(x)\,dx \in G$$

and $\|u(x) - \bar{u}\| \leq \varepsilon M$ *for all* $x \in \mathbf{R}$.

The proof of Theorem 6 is a verification of the hypotheses of Theorem 5. In particular, hypothesis (5) of that theorem is a consequence of our connectedness assumptions on G and B^-. For details, see [17].

Now let $g(u) = \frac{1}{T}\int_0^T f(x, u(x))\,dx$ for $u \in \mathbf{R}^m$. Our assumption on G implies that the degree $d(g, G, 0)$ is defined. If $d(g, G, 0) \neq 0$, then Theorem 6 is valid even without the gradient assumption on f ([4]).

However, in the gradient case, the homotopy index yields better results. In fact, one can easily construct examples satisfying the assumptions of Theorem 6 with $d(g, G, 0) = 0$ (see [17]).

On the other hand, the following general fact is true: Suppose $g : \mathbf{R}^m \to \mathbf{R}^m$ is locally Lipschitzian and gradient, G is open and bounded and $N = ClG$ is an isolating neighborhood with respect to the flow π generated by

$$\dot{u} = g(u) \tag{18}$$

Then

$$d(g, G, 0) = (-1)^m \sum_{q=0}^{\infty} (-1)^q \beta_q(h(\pi, K)) \tag{19}$$

where K is the largest π-invariant set in ClG and

$$\beta_q(h(\pi, K)) = \text{rank } H_q(h(\pi, K)).$$

In particular, (19) implies that whenever $h(\pi, K) = \bar{0}$, then also $d(g, G, 0) = 0$. This means that, for gradient systems, the continuation principle in Theorem 5 yields better results than the Leray-Schauder degree.

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