SOLUTIONS OF MINIMAL PERIOD FOR HAMILTONIAN SYSTEMS WITH QUADRATIC GROWTH AT THE ORIGIN AND SUPERQUADRATIC AT INFINITY (*)

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SOMMARIO. - *Vengono presentate alcune tecniche basate sulla teoria dell'indice di Morse e su un'opportuna versione del principio di dualità di Clarke ed Ekeland per dare alcuni risultati sull'esistenza di soluzioni di periodo minimo prefissato di sistemi Hamiltoniani del tipo*

$$\dot{x}_i = \omega_i y_i + \frac{\partial}{\partial x_i}\hat{H}(x, y), \quad -\dot{y}_i = \omega_i x_i + \frac{\partial}{\partial y_i}\hat{H}(x, y) \quad (i = 1, \ldots, N),$$

dove $0 < \omega_1 \leqslant \ldots \leqslant \omega_N$ *e* $\hat{H} \in C^2(\mathbf{R}^{2N}; \mathbf{R})$ *è strettamente convessa ed ha un comportamento superquadratico.*

SUMMARY. - *Some techniques based on the Morse index theory and a suitable version of the duality principle by Clarke and Ekeland are presented here in order to give some results about the existence of periodic solutions with prescribed minimal period to Hamiltonian systems of the type*

$$\dot{x}_i = \omega_i y_i + \frac{\partial}{\partial x_i}\hat{H}(x, y), \quad -\dot{y}_i = \omega_i x_i + \frac{\partial}{\partial y_i}\hat{H}(x, y) \quad (i = 1, \ldots, N),$$

where $0 < \omega_1 \leqslant \ldots \leqslant \omega_N$ *and* $\hat{H} \in C^2(\mathbf{R}^{2N}; \mathbf{R})$ *is strictly convex and has a superquadratic behaviour.*

Introduction

Recently many authors have dealt with the problem of periodic solutions of prescribed minimal period for Hamiltonian systems (see, e.g., [1], [3], [6], etc...).

A very remarkable result has been obtained by Ekeland and

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Hofer in [5], for the case where the Hamiltonian function is convex and has a «superquadratic» behaviour at the origin and at infinity. In this framework, Ekeland and Hofer show the existence of a periodic solution of minimal period T for every positive T. The proof relies on the appropriate use of the Morse index theory (see [4]) and the complete analysis of the neighbourhood of a critical point of Mountain Pass type (see [7]).

In this paper, we are able to extend their results, showing a more general theorem for convex Hamiltonian systems where the Hamiltonian function has a quadratic growth at the origin. We use the same general arguments introduced by Ekeland and Hofer, but in our case the proofs present many differences, mainly in the computation of the Morse index of a solution.

In the first section we present our results. The proofs of the theorems are outlined in sections 2,3.

1. Main results

Let the following Hamiltonian system be given

(H) $J\dot{z} = H'(z)$
with $J(x,y) = (y,-x) \quad \forall (x,y) \in \mathbf{R}^N \times \mathbf{R}^N$.
We consider the function H of the form
$H(z) = ½\langle Qz, z\rangle + \hat{H}(z)$
verifying the following conditions:

(H_1) Q is the $2N \times 2N$ matrix

$$Q = \begin{pmatrix} Q_0 & 0 \\ 0 & Q_0 \end{pmatrix}$$

where Q_0 is the diagonal $N \times N$ matrix

$$Q_0 = \begin{pmatrix} \omega_1 & & \\ & \ddots & \\ & & \omega_N \end{pmatrix}$$

with positive eigenvalues, $0 < \omega_1 \leqslant \ldots \leqslant \omega_N$;

(H_2) $\hat{H} \in C^2(\mathbf{R}^{2N}; \mathbf{R})$, $\hat{H}(0) = 0$, $\hat{H}$ is *strictly convex;*

(H_3) $a_1|z|^\beta \leqslant \hat{H}(z) \leqslant a_2|z|^\beta \quad \forall z \in \mathbf{R}^{2N}, \quad a_1, a_2 > 0, \quad \beta > 2$;

(H_4) $\beta\hat{H}(z) \leqslant \langle \hat{H}'(z), z\rangle \quad \forall z \in \mathbf{R}^{2N}$.

Then we can the state the following

THEOREM 1 - *Let H satisfy* $(H_1), \dots, (H_4)$. *Then, for any positive $T < 2\pi / \omega_N$, there exists a solution z_T of* (H) *having minimal period T. Moreover, one has:*

$$\| z_T \|_\infty \to \infty \quad as \quad T \to 0^+, \quad \| z_T \|_\infty \to 0 \quad as \quad T \to (2\pi / \omega_N)^-.$$

REMARK 1 - In general, it is impossible, without adding some other hypotheses, to find solutions of minimal period greater than $2\pi / \omega_1$.

For example, let $H(z) = ½ |z|^2 + |z|^4$ (in this case $\omega_1 = \dots = \omega_N = 1$): then it is easy to show that any solution has minimal period $T < 2\pi$ (For more general examples see [7]).

An open question is the existence of solutions of minimal period between $2\pi / \omega_N$ and $2\pi / \omega_1$. In this direction, we are able to prove the following partial result:

THEOREM 2 - *Let H satisfy* $(H_1), \dots, (H_4)$ *and let $j \in \{1, \dots, N\}$ be such that*

$(+)$ $\quad \omega_j / \omega_i \notin \mathcal{Q} \quad \forall i \in \{1, \dots, N\}$ *with* $\omega_i \neq \omega_j$.
Then, there exists an $\varepsilon_j > 0$ such that, for any T belonging to $(2\pi / \omega_j - \varepsilon_j, 2\pi / \omega_j)$, there is a solution z_T of minimal period T. Moreover one has

$$\| z_T \|_\infty \to 0 \quad as \quad T \to (2\pi / \omega_j)^-.$$

2. An outline of the proof of Theorem 1

The proof is broken in many steps.

Step 1. We state an appropriate dual principle for this case, which allows to find solutions of Mountain Pass type.

Let the operator $\mathcal{L}_T : H^{1,\alpha}(\mathbf{R} / T\mathbf{Z}; \mathbf{R}^{2N}) \to L^\alpha(0, T; \mathbf{R}^{2N})$ be defined as

$$\mathcal{L}_T(v) = J\dot{v} - Qv$$

and let us consider the Legendre transform of $\hat{H}$, defined as

$$\hat{G}(v) = \sup \{ \langle v, w \rangle - \hat{H}(w) : w \in \mathbf{R}^{2N} \}.$$

Then, setting

$$F_T(v) = \int_0^T \hat{G}(v) - ½ \int_0^T \langle \mathcal{L}_T^{-1} v, v \rangle \qquad \forall v \in L^\alpha(0, T; \mathbf{R}^{2N}),$$

it is easy to show the following

Duality principle. $F'_T(\bar{u}) = 0$ *if and only if* $\bar{z} = \mathcal{L}_T^{-1}\bar{u}$ *is a T-periodic solution of* (H).

Moreover one can show that F_T satisfies the following properties:

(a) F_T verifies the Palais-Smale condition,

(b) $F_T(v) > 0$ for $v \neq 0, \|v\|_\alpha < \rho$,

(c) There exists $\bar{e} \in L^\alpha$ such that $F_T(\bar{e}) < 0$.

Then, applying the well known theorem by Ambrosetti and Rabinowitz (see [2]), one gets the following

PROPOSITION 1 - *Let H verify* $(H_1), \ldots, (H_4)$. *Then, for every positive* $T \neq 2k\pi/\omega_j$ $(k \in \mathbf{N}, j = 1, \ldots, N)$, *there exists a T-periodic solution* z_T *of* (H) *of Mountain Pass type. Moreover one has*

$$\|z_T\|_\infty \to 0 \text{ as } T \to (2k\pi/\omega_j)^-, \ \|z_T\|_\infty \to \infty \text{ as } T \to 0^+,$$

REMARK 2 - An analogous theorem was already stated by Rabinowitz [9] in a more general case (without convexity hypothesis), but without energy estimates.

Step 2: The Morse index of a critical point of Mountain Pass type. Let $\bar{u}_T$ a given critical point of F_T and let us consider the quadratic form

$$Q_T(v) = \langle F''_T(\bar{u}_T)\, v, v\rangle =$$
$$= \int_0^T \langle \hat{G}''(\bar{u}_T)\, v, v\rangle - \int_0^T \langle \mathcal{L}_T^{-1} v, v\rangle \qquad \forall v \in L^2(0, T; \mathbf{R}^{2N}).$$

Let E^- the maximal subspace of L^2 on which Q_T is negative definite. Since $\dim E^-$ is finite, it makes sense to define, as the *Morse index* of $\bar{u}_T$, the following nonnegative integer number

$$m(\bar{u}_T) = \dim E^-.$$

PROPOSITION 2 - *Let* $\bar{u}_T$ *be a critical point of* F_T *of Mountain Pass tyje. Then one has*

(1) $m(\bar{u}_T) \leqslant 1$.

Moreover, if $m(\bar{u}_T) = 1$, *then there exists an open neighbourhood W of* $\bar{u}_T$ *such that the set*

$$W(\bar{u}_T) = \{u \in W : F_T(u) < F_T(\bar{u}_T)\}$$

has exactly two path components, P_1 *and* P_2 *say, and, if* λ_1 *denotes the (unique) negative eigenvalue of* Q_T *and* v_1 *an associated eigenvector, then, for all* $\eta > 0$, *with* η *sufficiently small, one has*

(2) $\bar{u}_T + \eta v_1 \in P_1, \quad \bar{u}_T - \eta v_2 \in P_2$.

The proof of Proposition 2 is obtained, firstly, by a finite dimensional reduction (the so called «broken geodesics» method), secondly, by a generalized version of the Morse Lemma (see Hofer [7]).

Step 3: The Morse index of a solution and the order of its isotropy group.

Let a T-periodic solution z_T of (H) be given. The *order of its isotropy group* for the S^1-action, denoted by $\mathcal{O}(z_T)$, is defined as the greatest integer number k such that z_T *is* T/k-periodic.

Given z_T and given any $s \in (0, T)$, one says that $z_T(s)$ is *conjugate* to $z_T(0)$ if the linear problem

$$(3) \qquad \begin{cases} J\dot{y} = H''(z_T(t))\, y \\ y(0) = y(s) \end{cases}$$

has a non-zero solution; the *multiplicity* of $z_T(s)$ is the defined as the dimension of the space of solutions to (3).

PROPOSITION 3 - *For any T-periodic solution z_T of* (H), *there exists only a finite number of points $z_T(s_i)$ conjugate to*

$$z_T(0) \; (i = 1, \ldots, r).$$

Denoting by m_i the relative multiplicity, the following relation holds:

$$(4) \qquad m(z_T) = \sum_{i=1}^{r} m_i - 2 \sum_{i=1}^{N} [T\omega_j / 2\pi]$$

(where [ ] denotes the integer part).

By taking into account that the points

$$z_T(T/k), \ldots, z_T((k-1)\, T/k),$$

with $k = \mathcal{O}(z_T)$, are obviously conjugate to $z_T(0)$, then one easily gets the following

COROLLARY. *If $T < 2\pi / \omega_N$, then one has*

$$(5) \qquad m(z_T) \geqslant \mathcal{O}(z_T) - 1.$$

Step. 3. Conclusion. Let z_T a T-periodic solution of (H) of Mountain Pass type, and let $T < 2\pi / \omega_N$, then, by (1) and (5), it follows

$$(6) \qquad \mathcal{O}(z_T) \leqslant 2.$$

Now we have only to exclude the case $m(z_T) = 1$ and $\mathcal{O}(z_T) = 2$. By contradiction, let us suppose that $m(z_T) = 1$ and that $T/2$ is the minimal period of z_T.

Let λ_1 and v_1 be as in Proposition 2. By the variational characterization of v_1, one can show that

$$v_1(t + T/2) = -v_1(t).$$

Taking into account Proposition 2, we can show that the sets

$W(\bar{u}_T)$, (where $\bar{u}_T = \mathfrak{L}_T z_T$)„ P_1 and P_2 can be assumed to be S^1-invariant (i.e., if $v \in P_i$, then $v(t+\theta) \in P_i$ for any $\theta \in \mathbf{R}/T\mathbf{Z}$).

Then, if we consider the orbit

$$c(\theta) = \bar{u}_T(t+\theta) + \eta v_1(t+\theta),$$

one has

$$c(0) = \bar{u}_T + \eta v_1 \in P_1 \quad c(T/2) = \bar{u}_T - \eta v_2 \in P_2$$

which is an absurde, as it contradicts (2).

3. An outline of the proof of Theorem 1

Let z_T a T-periodic solution of (H) of Mountain Pass type: we show that, under the assumptions of Theorem 2, T/n cannot be a period of z_T, for any $n \in \mathbf{N}$, $n \geqslant 2$, that is T must be the minimal period of z_T.

One considers two possible cases: $T/n < \frac{1}{2}\, 2\pi/\omega_N$ and $T/n \geqslant \frac{1}{2}\, 2\pi/\omega_N$.

Case $T/n < \frac{1}{2}\, 2\pi/\omega_N$ - One shows that the quadratic form Q_T, defined at Step 2 of Section 2, is positive definite on $L^2(0,s)$ for any $s \in (0, T/n]$, so T/n is not a period of z_T. This fact is easily deduced from the estimates

$$(7) \qquad \int_0^s \langle \mathfrak{L}_S^{-1} v, v\rangle \leqslant (1/\omega_N) \int_0^s |v|^2 \quad \forall v \in L^2(0,s)$$

and, putting $u_T = \mathfrak{L}_T z_T$,

$$(8) \qquad \langle \hat{G}''(u_T)\, w, w\rangle \geqslant c(T)\, |w|^2 \quad \forall w \in \mathbf{R}^{2N}$$

where

$$(9) \qquad \lim_{T \to (2\pi/\omega_j)^-} c(T) = +\infty .$$

Case $T/n \geqslant \frac{1}{2}\, 2\pi/\omega_N$ - Firstly one defines $h_1, h_2 \in \mathbf{N}$ and $i_1, i_2 \in \{1, \ldots, N\}$ as follows:

$$2h_1\pi/\omega_{i_1} = \max\{2h\pi/\omega_i : h \in \mathbf{N} \cup \{0\},\ i \in \{1,\ldots,N\},\ 2h\pi/\omega_i < 2\pi/n\omega_j\}$$
$$2h_2\pi/\omega_{i_2} = \min\{2h\pi/\omega_i : h \in \mathbf{N} \cup \{0\},\ i \in \{1,\ldots,N\},\ 2h\pi/\omega_i > 2\pi/n\omega_j\}.$$

Then (+) implies

$$(10) \qquad 2\pi/n\omega_j \in (2h_1\pi/\omega_{i_1}, 2h_2\pi/\omega_{i_2}) .$$

At this point, it is possible to show that, if one chooses $\varepsilon > 0$ small enough, the form Q_S related to z_T for $T \in (2\pi/\omega_j - \varepsilon, 2\pi/\omega_j)$

is definite positive if $s \in \{2h_1\pi/\omega_{i_1}, 2\pi/n\omega_j\}$, so T/n still cannot be a period of z_T. This fact is a consequence of the following estimate on the maximum eigenvalue λ_S of $\mathcal{L}_S^{-1}$ (deduced from (10)),

$$\lambda_S \leqslant \omega_{i_2} / \omega_1 (nh_2\omega_j - \omega_{i_2})$$

and, again, of the relations (8), (9).

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