APPLICATIONS OF THE CONLEY INDEX TO REACTION-DIFFUSION SYSTEMS (*)

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SOMMARIO. - *Si discute l'applicazione dell'indice di Conley all'esistenza di onde viaggianti in sistemi del tipo reazione-diffusione. La teoria generale è illustrata con talune equazioni-modello che si incontrano in ecologia matematica.*

SUMMARY. - *The application of the Conley index to the existence of travelling wave solutions of reaction-diffusion systems is discussed. We illustrate the general theory with some model equations arising in mathematical ecology.*

0. Introduction

The Conley index is a topological invariant for flows which is particularly useful in determining when an orbit exists which connects two Morse sets in a Morse decomposition of an isolated invariant set. These methods have recently been applied to reaction-diffusion systems, where a frequently encountered problem is to find an orbit which connects distinct rest points. Such questions arise in several different contexts. For example, the system can be viewed as generating a (semi-) flow on a suitable function space. A natural

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problem is to then determine when an orbit runs from one steady state to another (see e.g. [3, 6]). This is an inherently infinite dimensional problem; typically, the index methods are quite effective if there is a globally defined Liapunov function.

Another source of such problems, with which we shall principally be concerned here, is the existence of travelling wave solutions of the system

$$(1) \qquad u_t = Du_{xx} + f(u), \quad u \in \mathbf{R}^n.$$

Such solutions are functions of the single variable $\xi = x - \theta t$ and so, they satisfy the $2n$-dimensional system

$$(2) \qquad \begin{cases} u' = v \\ \\ Dv' = -\theta v - f(u) \end{cases} \qquad ' = \frac{d}{d\xi}.$$

The first step in determining the transient behavior of solutions of (1) is to specify the bounded solutions of (2); the simplest nontrivial solutions of the latter system are those which connect distinct rest points. The rest points of (2) are of the form $(u, v) = (\bar{u}, 0)$ where $\bar{u}$ is a critical point of $f(u)$. We shall focus on the case where $\bar{u}$ is a *stable* rest point of (1). In this case the two relevant critical points of (2) are saddles, each with n-dimensional stable and unstable manifolds, and the connection will occur only for distinguished values of the wave velocity θ. The manner in which θ enters into the equations evidently plays a crucial role in the geometry of the flow generated by (2). We therefore need to employ a construction called the *connection index,* which is basically the Conley index applied to the $(2n+1)$-dimensional system obtained from (2) by augmenting the flow with the additional equation $\theta' = 0$.

In the next section we shall describe the index-theoretic preliminaries and illustrate the constructions with a canonical example. In Section 2 we describe some applications to some two-component systems arising in mathematical ecology. We conclude in Section 3 with an example which illustrates how these fethods can be extended to several space dimensions.

1. Index theory

A. *The Conley Index.* Suppose there is given a flow on $\mathbf{R}^n$. Given a compact neighborhood $N \subset \mathbf{R}^n$ we denote by $S(N)$ the set of points on solution curves which remain in N for all time. The maximal invariant set $S(N)$ is *isolated* if $S(N)$ is interior to N; is then called an isolating neighborhood.

Given an isolated invariant set $S(I)$ we say that (N, N_2) is an *index pair* if N_2 is a compact subset of $N \backslash S(N)$ which is poistively invariant relative to N and if solution curves which exit N in the positive time direction enter the set N_2 before leaving N. It can be shown that if N is isolating then index pairs exists. The *Conley index* is defined to be $[N/N_2]$, the homotopy type of the space obtained by collapsing N_2 to a point. The index depends only on $S = S(N)$ and is invariant under continuation. The precise statement and the proofs of these theorems can be found in [1]. Thus we use the notation $h(S) = [N/N_2]$.

B. *Examples.* If S is a hyperbolic critical point with a k-dimensional unstable manifold, then $h(S) = \Sigma^k$, a pointed k-sphere (see e.g. Figure 1).

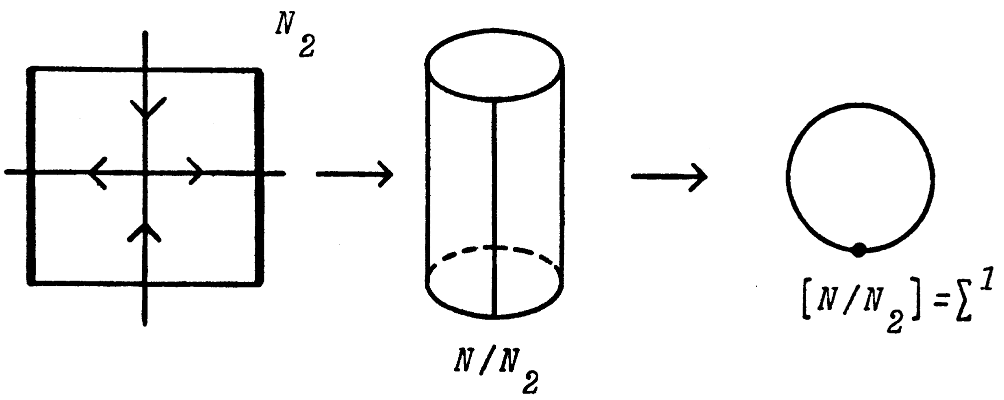

Fig. 1

Next, consider a collection of flows in the plane parametrized by θ with the aspect indicated in Figure 2.

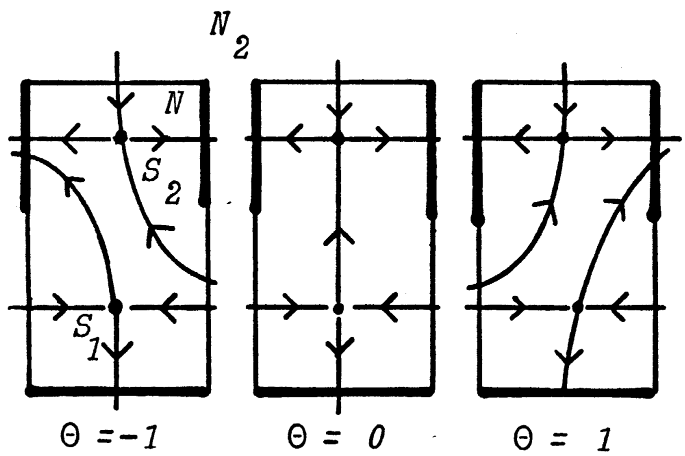

Fig. 2

In each case the exit set N_2 (indicated by the heavy lines) consists of three distinct components which continue for all θ. When $\theta \neq 0$, $S(N)$ consists of the union of the two critical points S^1 and $S^?$, and by the sum formula and the continuation theorem

$$h(S(N)) = \Sigma^1 \vee \Sigma^1$$

for all θ (see [1]). Thus the existence of the connecting solution at $\theta = 0$ is not reflected in the index. In order to define an algebraic invariant which can «see» the connection, it us useful to consider the behavior of the unstable manifold $\mathfrak{M}^u$ of the lower critical point S^1. For all $\theta \neq 0$, $\mathfrak{M}^u$ represents homology in $H_1(h(S))$. This is a free group with two generators, and $\mathfrak{M}^u$ represents *different* homology classes at $\theta = \pm 1$. The behavior of $\mathfrak{M}^u$ evidently plays a crucial role.

C. *The Connection Index.* Next consider the parametrized flow generated by

$$\begin{aligned} x' &= f(x, \theta) \qquad (x \in \mathbf{R}^n) \\ \theta' &= 0 . \end{aligned}$$

Given a set $M \subset \mathbf{R}^{n+1}$ we denote by M_θ the slice of M *with* $\theta =$ constant. Suppose that there is given a set $\bar{N} = N \times I$ with $N \subset \mathbf{R}^n$ and $I = [-\theta_1, \theta_1]$ such that $S(\bar{N}_\theta)$ is an isolated invariant set for each $\theta \in I$. Moreover, suppose that there exist subsets of $S(\bar{N})$, $\bar{S}^i = S^i \times I$ such that $\bar{S}^i_\theta = S^i$ is an isolated invariant set for each $\theta \in I$, $i = 1, 2$.

DEFINITION 1 - Suppose that $\bar{S} = S(\bar{N})$, that $\bar{S}^1$ and $\bar{S}^2$ satisfy the above conditions, and that

$$\bar{S}_\theta = S^1 \cup S^2 \ \text{ at } \ \theta = \pm \theta_1 .$$

Then $(\bar{S}, \bar{S}^1, \bar{S}^2)$ is called a *connection triple.*

In the following we give a definition of the connection index which for technical reasons, is not strictly correct; hwoever it gives the correct qualitative picture. The modifications needed to make the general index theory available in the present situation are described in [2]. These aspects of the construction are not important in computations and we shall therefore suppress them.

DEFINITION 2 - Let $\mathfrak{M}^u(\bar{S}^1_\theta)$ be the set of all points on solution curves which tend to $\bar{S}^1_\theta$ in backward time (the unstable manifold of $\bar{S}^1_\theta$). Let N_2 be such that $(\bar{N}_\theta, (N_2)_\theta)$ is an index pair for each $\theta \in I$ and define

$$\bar{N}_2 = N_2 \cup \mathfrak{M}^u(\bar{S}^1_{-\theta_1}) \cup \mathfrak{M}^u(\bar{S}^1_{\theta_1}).$$

The *connection index* is defined to be

$$\bar{h}(\bar{S}, \bar{S}^1, \bar{S}^2) = [\bar{N}/\bar{N}_2] .$$

It can be shown that $\bar{h}$ depends only on the connection triple $(\bar{S}, \bar{S}^1, \bar{S}^2)$ and that it is invariant under continuation (see [2]).

D. *An Example.* It is evident that $N \times I$, where N is the indicated rectangle in Figure 2, can be used to define a connection triple for the parametrized flow provided that $0 \notin \partial I$. We will compute $\bar{h}$ in two cases.

Case 1. Suppose that $I = [-\theta_1, \theta_1]$ for some $\theta_1 > 0$. The sets $\bar{N}$ and $\bar{N}_2$ are depicted in Figure 3a, from which it can be seen that $\bar{N}_2$ is

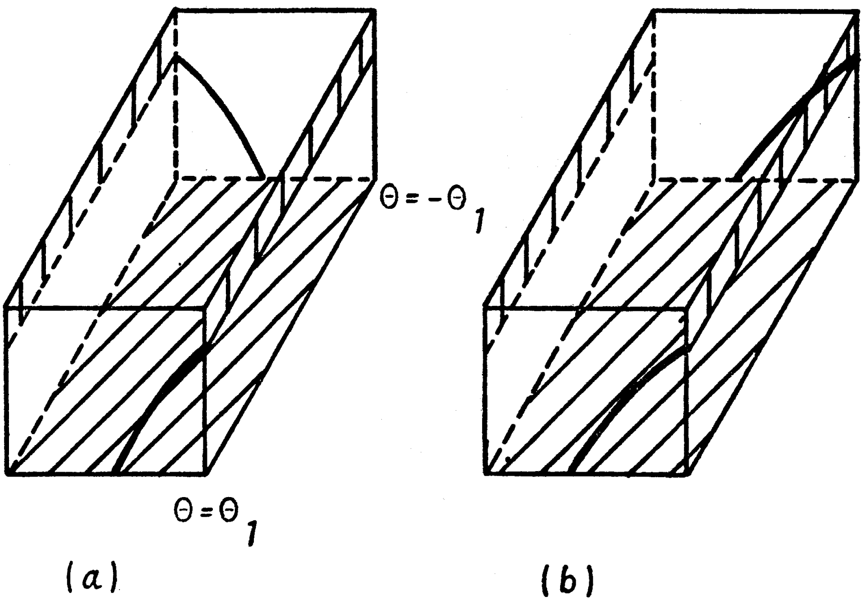

Fig. 3

contractible to a point. Hence in this case

$$\bar{h}(\bar{S}, \bar{S}^1, \bar{S}^2) = [3\ \text{cell/point}] = [\text{point}].$$

Case 2. Suppose that $0 \notin I$, the case depicted in Figure 3. The set $\bar{N}^2$ is homotopically equivalent to the disjoint union of a circle and a point. The connection index $\bar{h}$ can be seen to equal $\Sigma^1 \vee \Sigma^2$ through the sequence of identifications in Figure 4.

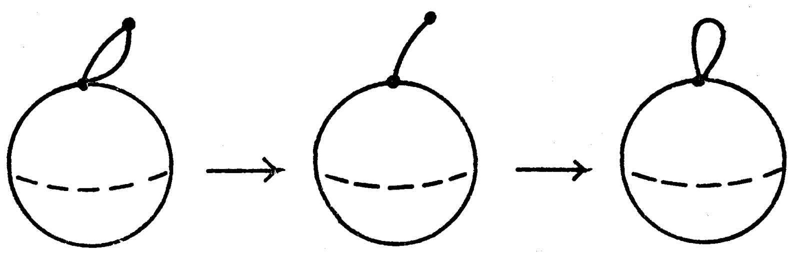

Fig. 4

From the above computation it follows that in Case 1 $\bar{S} \underset{\neq}{\supset} \bar{S}^1 \cup \bar{S}^2$ since if this were not the case we could replace the rectangle N with the union of small neighborhoods of S^1 and S^2 because $\bar{h}$ depends only on the connection triple. The problem could then be continued to Case 2, yielding a contraction.

The solution in $\bar{S} \backslash \bar{S}^1 \cup \bar{S}^2$ is easily seen to be a connecting orbit running from $\bar{S}^1$ to $\bar{S}^2$. In more general settings this usually requires some additional information about the flow in $\bar{S}$ such as the existence of a Liapunov function.

The above example frequently appears as a canonical form embedded in higher dimensional systems. In particular, such examples can be continued to a product system consisting of the example depicted in Figure 2 crossed with a linear system with a hyperbolic critical point at the origin. The connection index in this case is $\Sigma^k \wedge \bar{h}_m$ where $\bar{h}_m$ is the index of the model problem and k is the dimension of the unstable subspace of the linear components.

We finally mention a general existence theorem whose proof is similar to that of the special case just considered.

THEOREM 1 - *Suppose h^i is the Conley index of S^i, $i = 1, 2$; (this is independent of θ). If $\bar{h} = \bar{h}(\bar{S}, \bar{S}^1, \bar{S}^2)$ and*

$$h \neq (\Sigma^1 \wedge h^1) \vee h^2$$

then $\bar{S} \underset{\neq}{\supset} \bar{S}^1 \cup \bar{S}^2$.

2. Predator-prey interactions

A. We next apply these methods to a system of two reaction-diffusion equations arising in mathematical ecology modelling predator-prey interactions (see [4]). These techniques are also available for competitive interactions (see [2]), however, the predator-prey equations display more interesting phenomena due to a lact of monotonicity usually found in competitive dynamics.

The equations assume the form

$$(3) \qquad u_{i_t} = d_i u_{i_{xx}} + u_i f_i(u), \quad i = 1, 2,$$

where $u = (u_1, u_2)$, $d_i > 0$ is a positive constant, and f_i is the per-capita growth rate of u_i, $i = 1, 2$. We assume that

$$\frac{\partial f_1}{\partial u_2} < 0 , \frac{\partial f_2}{\partial u_1} > 0$$

so that u_1 is the density of the prey and u_2 is the density of the predator. We further assume that the null-clines of f_1 and f_2 have the aspect depicted in Figure 5 — the arrows indicate the reaction flow for spatially homogeneous

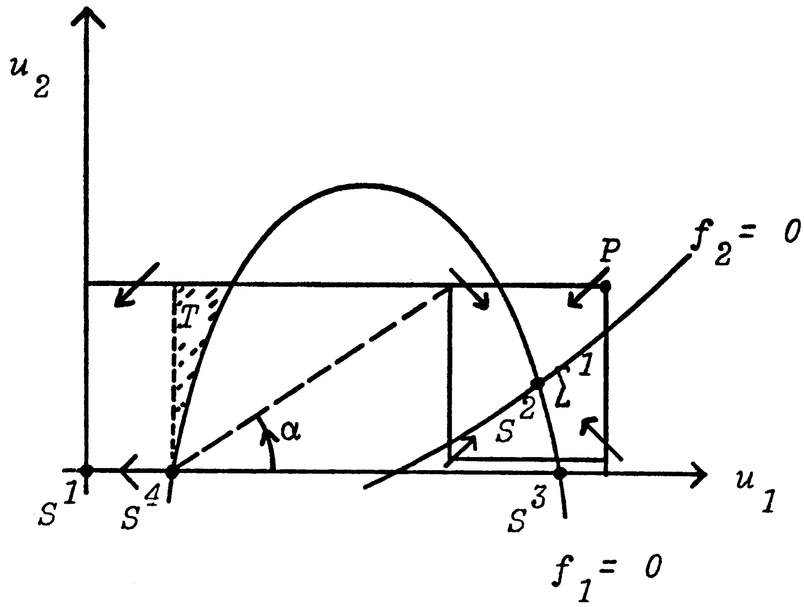

Fig. 5

solutions. The system admits four critical points. The rest points S^1 and S^2 are stable, while S^3 and S^4 are unstable.

Travelling wave solutions of (3) satisfy a four-dimensional system

$$(4) \qquad \begin{aligned} u'_i &= v_i , \quad i = 1,2 \\ d_i v'_i &= - \theta v_i - u_i f_i(u) . \end{aligned}$$

With a slight abuse of notation we denote the four critical points of (4) also by S^i. Thus we seek a solution $(u(\xi), v(\xi))$ of (4) which satisfies

$$(5) \qquad (u,v)\ (-\infty) = S^1 , \quad (u,v)\ (+\infty) = S^2 .$$

To this end we impose the following hypotheses.

(H_1) There exists a family of contracting rectangles Σ_τ, $\tau \in (0,1]$ centered about the point S^2 in Figure 4.

(H_2) Let θ_1 be the velocity of the travelling wave connecting S^1 to S^3 along which $u_2 \equiv v_2 \equiv 0$, and let $c_1 < 0$ be the maximal velocity of all connections from S^3 to S^2 along which the u-components remain non-negative (see [4] for further discussion). Assume $c_1 < \theta_1$.

(H_3) Let $f(u) = (u_1 f_1(u), u_2 f_2(u))$ and let φ be defined by

$$\varphi = \min_{u \in T} \arg(-f(u))$$

where T is the shaded region in Figure 4. (It is easily seen that $0 < \varphi \leq \pi/2$). Assume that

$$d_1 > d_2 \frac{\tan \alpha}{\tan \varphi}.$$

We remark that (H_1) implies the stability of S^2 relative to (3). (H_2) ensures that the critical point at S^3 does not split the connection into a stached family of waves. Finally, (H_3) is used to force the u-components of the connection to remain monotone increasing while $u(\xi) \notin \Sigma_1$. While this condition is somewhat artificial, some such criterion is needed to ensure that the connection is not interrupted by a family of periodic wave trains. Further discussion of the mathematical aspects of (H_1-H_3) together with an ecological interpretation is provided in [4].

THEOREM 2 - *Under* (H_1-H_3) *there exists a solution* $(u,v)(\xi)$ *of* (4) *which satisfies* (5).

B. *Sketch of the Proof.* We outline the main points of the proof of Theorem 2. In order to apply the methods of Section 1 we first need a suitable candidate for an isolating region which plays a role analogous to that of N in Section 1.D. To this end we start with a region N_0 defined by

$$N_0 = \{(u,v) : u \in R,\ 0 \leq v_i \leq L,\ i = 1,2\},$$

where R is the rectangle in Figure 4 with vertices at S^1 at the lower left and P at the upper right, and L is a large positive constant depending only on d_1, d_2, and R.

Notice that the u-components of solutions in $S(N_0)$ are monotone increasing. It follows that such solutions are either critical or connecting orbits. If $d_1 = d_2$ and S^2 is a spiral for the reaction flow, then S^2 must also be a spiral as a solution of (4). Hence the region N_0 is too restrictive. Let

$$N_1 = \{(u,v) : u \in \Sigma_1,\ |v_i| \leq L,\ i = 1,2\},$$

and define $N_* = N_0 \cup N_1$. It is easily seen with the aid of (H_1) that non-constant solutions in $S(N_*)$ must again connect distinct critical points.

The final difficulty is that S^1, S^3, and S^4 lie in ∂N_* so that N_* is not isolating. Let B_i be a small neighborhood in $\mathbf{R}^4$ of S^i, $i = 1,3,4$. The final neighborhood is defined to be

$$N = N_* \cup B_1 \setminus (B_3 \cup B_4).$$

The main estimate is to show that N is an isolating neighborhood.

To this end, the flow through each point in ∂N must be shown to (eventually) leave N in at least one time direction. The details can be found in [4].

Finally, it must be shown that $S(N) = S^1 \cup S^2$ when $|\theta|$ is sufficiently large. This follows from the observation that for large $|\theta|$, solutions of (4), after suitable rescaling and changes of variables, closely approximate solutions of the reaction flow,

$$\tilde{u}'_i = \tilde{u}_i f_i(\tilde{u})$$

crossed with a hyperbolic critical point. It then follows that $(\bar{S}, \bar{S}^1, \bar{S}^2)$ is a connection triple, where

$$\bar{S} = S(N \times [-\theta_1, \theta_1]$$

$$\bar{S}_i = S^i \times [-\theta_1, \theta_1], \quad i = 1,2.$$

In order to compute $\bar{h}$ the problem is continued to the example discussed in Section 1.C, D. The homotopy begins by deforming the set $u_2 f_2(u) = 0$ in the manner indicated in Figure 6. After some

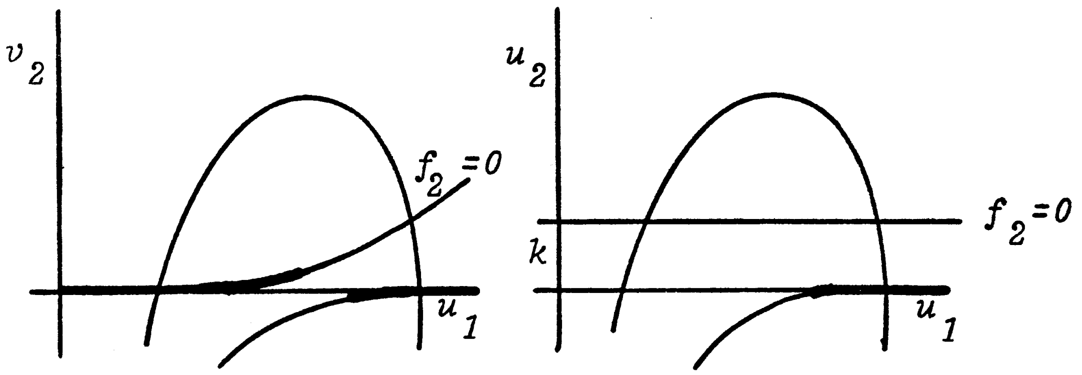

Fig. 6

additional homotopies the problem can be continued to

$$\begin{aligned} u'_2 &= v_2 & u'_2 &= v_2 \\ v'_1 &= -\theta v_1 - u_1 f_1(u_1, k) & v'_2 &= u_2; \end{aligned}$$

of course, the isolating region N must be modified as the equations are deformed. The u_1, v_1 components have a phase plane similar to the standard example in Figure 2 while the u_2, v_2 equations are linear with a saddle point at the origin. It follows from Case 1 of Section 1.D that $\bar{h} = \Sigma^1 \wedge [\text{point}] = [\text{point}]$. Since $h(S^i) = \Sigma^2$, $i = 1,2$

it follows that $\bar{h} \neq \Sigma^2 \vee \Sigma^3$, so that by Theorem 1, $\bar{S}$ contains a connection running from S^1 to S_2 for some $|\theta| < \theta_1$.

3. Multidimensional fronts

Consider the scalar equation

$$\begin{aligned} u_t &= u_{xx} + u_{yy} + u(1-u)(u-\alpha) \\ u|_{\partial\Omega} &= 0 \\ \Omega &= \{(x,y) : x \in \mathbf{R}^1, 0 < y < L\}. \end{aligned}$$

We seek solutions of the form $u = u(\xi, y)$ where $\xi = x - \theta t$. Such solutions satisfy an elliptic equation of the form

$$-\theta u_\xi = u_{xx} + u_{\xi\xi} + f(u), \quad u = 0 \text{ on } \partial\Omega. \tag{6}$$

We also require solutions to tend to limits at $|x| = \infty$; the limiting states are therefore solutions of

$$\begin{aligned} 0 &= u_{yy} + u(1-u)(u-\alpha) \\ u(0) &= u(L) = 0. \end{aligned} \tag{7}$$

If $\alpha \in (0, 1/2)$ there exists L_0 such that (7) has exactly three solutions, $0 < u_\alpha(y) < u_1(y)$, for $L > L_0$ (see [6]).

THEOREM 3 - *Suppose that* $\alpha \in (0, 1/2)$ *and that* $L > L_0$; *there exists a solution* u, θ *of* (6) *such that*

$$\lim_{\xi \to -\infty} u = 0, \qquad \lim_{\xi \to +\infty} u = u_1(y).$$

The solution is monotone increasing in ξ.

The methods of the previous section cannot be applied directly to this problem since the system

$$\begin{aligned} u_\xi &= v \\ v_\xi &= -\theta v - [u_{yy} + f(u)] \end{aligned}$$

is elliptic. The initial value problem is ill-posed and the equations do not generate a flow. This problem is circumvented by discretizing the finite variable y into a net, $y_i = ih$, $0 < i < n$, $nh = L$, and introducing an approximate of 2n differential-difference equations

$$\begin{aligned} u'_i &= v_i \\ v'_i &= -\theta v_i - [(u_{i+1} - 2u_i + u_{i-1}) h^{-2} + f(u_i)], \; 1 \le i \le n. \end{aligned} \tag{8}$$

The end state equation, (7), is also discretized; for sufficiently small h, it admits exactly three solutions which approximate the continuous solutions.

An isolating neighborhood $N(h)$ is constructed in a manner analogous to that of N in the previous section. Connecting orbits are obtained for each $h > 0$ by deforming the boundary conditions to the Neumann problem. At the end of the homotopy we find that (8) continues to the product of the standard example of Section 2.D. with a linear hyperbolic system. The construction of $N(h)$ is such that the set of approximate connecting solutions is in a suitable sense compact. An exact connection solution of the continuous problem is obtained by passing to a subsequence as h tends to zero. (The details of the proof can be found in [5]).

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