SYSTEMS OF REACTION - DIFFUSION TYPE WITH UNILATERAL BOUNDARY CONDITIONS. THE APPROACH OF M. KUČERA (*)

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SOMMARIO. - *In questo lavoro si considerano alcune condizioni al contorno di tipo universale relative a sistemi del tipo reazione-diffusione e se ne esamina l'influenza sulla stabilità della soluzione stazionaria e spazialmente omogenea. Simultaneamente viene presa in esame l'influenza delle condizioni al contorno sul punto di biforcazione del relativo sistema stazionario.*

SUMMARY. - *This paper deals with certain unilateral boundary conditions for systems of reaction-diffusion type and their influence on the stability of the stationary and spatialy homogeneous solution. The influence of unilateral boundary conditions on the bifurcation points of the corresponding stationary system is also considered.*

0. Introduction

Let us consider a reaction-diffusion system of the type

$$\text{(RD)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + f(u,v)\,, \\[2ex] \dfrac{\partial v}{\partial t} = d_2 \Delta v + g(u,v)\,, \end{cases} \qquad \text{in } \Omega$$

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with the boundary conditions

(NC) $$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega,$$

or

(BC) $$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N, u = \bar{u}, v = \bar{v} \text{ on } \Gamma_D.$$

Suppose that Ω is a bounded domain in $\mathbf{R}^n$ with boundary $\partial\Omega = \Gamma_N \cup \Gamma_D, f, g$ are real valued functions defined on $\mathbf{R}^2, d_1, d_2$ are positive parameters (diffusion coefficients) and $\bar{u}, \bar{v}$ are constants such that $f(\bar{u}, \bar{v}) = g(\bar{u}, \bar{v}) = 0$, i.e. $[\bar{u}, \bar{v}]$ is a stationary and spatialy homogeneous (constant) solution of (RD), (NC) (resp. (RD), (BC)). In some mathematical models arising in biochemistry, morphogenesis, population dynamics etc. the following situation occurs: for some fixed d_2 (we shall suppose $d_2 = 1, d_1 = d$) there is a critical value $d_0 > 0$ such that $\bar{U} = [\bar{u}, \bar{v}]$ is stable if $d > d_0$ and unstable if $d < d_0$; moreover this d_0 is the greatest bifurcation point of the corresponding stationary system

(RD_S) $$\begin{cases} d\Delta u + f(u,v) = 0, \\ \Delta v + g(u,v) = 0, \end{cases} \quad \text{in } \Omega$$

with (NC) (resp. (BC)), i.e. the branch of the spatialy nonhomogeneous stationary solutions of (RD_S), (NC) (resp. (RD_S), (BC)) bifurcates at $[d_0, \bar{U}]$ from the trivial branch

$$\{[d, U] : d \in \mathbf{R}, U = \bar{U}\}.$$

Such a situation for (RD), (NC) in the case $n = 1$ (i.e. $\Omega = (0, 1)$, $\Delta u = u_{xx}, \Delta v = v_{xx}$) is described in detail for instance in [11] (see also [10]).

The aim of the previous works [2,3,4,8,9] was to study how this situation changes if we consider some unilateral boundary conditions instead of (NC), resp. (BC). In this paper we give a survey of the results in that direction obtained by the author, Milan Kučera and Marta Míková in [2,3,4].

In Section 1 we present an abstract formulation of the problems in question. Section 2 is devoted to the modification of a global bifurcation theorem due to E.N. Dancer [1] which is the tool in the proofs of our main results. An abstract result about the greatest bifurcation point for the associated variational inequality is contained in Section 3 and destabilizing effect of unilateral conditions for linearized reaction-diffusion system is explained in Section 4. More general result concerning linearized reaction-diffusion systems is presented in Section 5. The last Section contains some final remarks and comments.

As it was already pointed out the proofs are based on the application of the Dancer's bifurcation result and they come from a modification of the argument developed by Milan Kučera in [5, 6, 7].

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1. Abstract formulation of the problems

We shall denote by $\mathbf{V}$ and $\mathbf{H}$ two Hilbert spaces equipped with the inner products $\langle .,. \rangle$ and $(.,.)$ respectively, and such that

(V, H) $\quad \mathbf{V} \hookrightarrow\hookrightarrow \mathbf{H}$

algebraically and topologically with completely continuous embedding. The corresponding norms will be denoted by $\| u \| = \langle u, u \rangle^{1/2}$ and $|v| = (v, v)^{1/2}$, respectively. Let $K \subset \mathbf{V}$ be a closed convex cone in $\mathbf{V}$ with its vertex at the origin. We shall denote by $\tilde{\mathbf{V}}$ and $\tilde{\mathbf{H}}$ the Hilbert spaces $\mathbf{V} \times \mathbf{V}$ and $\mathbf{H} \times \mathbf{H}$ with the inner products given by

$$\langle U, W \rangle_{\sim} = \langle u, w \rangle + \langle v, z \rangle, \; (U, W)_{\sim} = (u, w) + (v, z),$$

where $U = [u, v]$, $W = [w, z]$, and with the corresponding norms $\| U \|_{\sim} = \langle U, U \rangle_{\sim}^{1/2}, |V|_{\sim} = (V, V)_{\sim}^{1/2}$, respectively. The identity mappings in $\mathbf{V}(\mathbf{H})$ and $\tilde{\mathbf{V}}(\tilde{\mathbf{H}})$ will be denoted by I and $\tilde{I}$. We shall suppose that $K \neq \mathbf{V}, K^0 \neq \phi$ (the interior of K is nonempty). The symbols $\to$ and $\rightharpoonup$ will denote the strong and the weak convergence in the corresponding spaces, $\mathbf{R}$ and $\mathbf{R}^+$ will be the set of all reals and of all positive reals, respectively. Throughout the paper we suppose that

(A) A is a linear completely continuous symmetric positive (i.e. $\langle Au, u \rangle > 0$, for all $u \neq 0$) operator in $\mathbf{V}$.

Particularly, this is fulfilled for the operator defined by

(1.1) $\quad \langle Au, \varphi \rangle = (u, \varphi)$, for all $u, \varphi \in \mathbf{V}$,

by the assumption (V, H). Let further $N_i : \tilde{\mathbf{V}} \to \mathbf{V}$ be nonlinear completely continuous mapping such that

(N) $$\lim_{\|U\|_{\sim}\to 0} \frac{N_i(U)}{\|U\|_{\sim}} = 0, \quad i = 1,2.$$

Let $d_i, b_{ij} \in \mathbf{R}$ $(i,j = 1,2)$ be given and define

$$B = \begin{pmatrix} b_{11}, b_{12} \\ b_{21}, b_{22} \end{pmatrix}, \quad D = \begin{pmatrix} d_1, 0 \\ 0, d_2 \end{pmatrix}, \quad D(d) = \begin{pmatrix} d, 0 \\ 0, 1 \end{pmatrix},$$

$\tilde{A}U = [Au, Av]$, for all $U = [u,v] \in \tilde{\mathbf{V}}$. Then we have
$B\tilde{A}U = [b_{11}Au + b_{12}Av, b_{21}Au + b_{22}Av]$, $DU = [d_1 u, d_2 v]$.
Further, introduce a cone $\tilde{K}$ in $\tilde{\mathbf{V}}$ by $\tilde{K} = \{U \in \tilde{\mathbf{V}} : [u,v], v \in K\}$ and $N : \tilde{\mathbf{V}} \to \tilde{\mathbf{V}}$ by $N(U) = [N_1(u,v), N_2(u,v)]$, $U = [u,v] \in \tilde{\mathbf{V}}$.

We shall study the bifurcation problem for the stationary inequality

$$(\tilde{\mathrm{SI}}) \quad \begin{cases} U \in \tilde{K}, \\ \langle D(d)U - B\tilde{A}U + N(U), \Phi - U\rangle_{\sim} \geqq 0, \ \forall \Phi \in \tilde{K}. \end{cases}$$

Simultaneously the system of stationary equations (written in the vector form) will be considered

$$(\tilde{\mathrm{SE}}) \quad D(d)U - B\tilde{A}U + N(U) = 0.$$

REMARK 1.1 - Consider the reaction-diffusion system (RDs) from Introduction and suppose that f, g are twice continuously differentiable and $f(\bar{u},\bar{v}) = g(\bar{u},\bar{v}) = 0$ with some constants $\bar{u} > 0$, $\bar{v} > 0$. Set

$$b_{11} = \frac{\partial f}{\partial u}(\bar{u},\bar{v}), \quad b_{12} = \frac{\partial f}{\partial v}(\bar{u},\bar{v}), \quad b_{21} = \frac{\partial g}{\partial u}(\bar{u},\bar{v}), \quad b_{22} = \frac{\partial g}{\partial v}(\bar{u},\bar{v}),$$

and let us suppose, for simplicity that $n = 1$, $\Omega = (0,1)$, $\Gamma_D = 0$, $\Gamma_N = 1$. Define the space $\mathbf{V} = \{u \in W_2^1(0,1) : u(0) = 0\}$ with inner product

$$(1.2) \quad \langle u, \varphi\rangle = \int_0^1 u_x \varphi_x \, dx, \quad \forall u, \varphi \in \mathbf{V},$$

and with the corresponding norm $\|\cdot\|$ which is equivalent in $\mathbf{V}$ to the usual norm of the Sobolev space $W_2^1(0,1)$. Further, denote by $\mathbf{H}$ the Lebesgue space $L_2(0,1)$ with the usual inner product $(.,.)$ and the corresponding norm $|\cdot|$. Introduce the operators A, N_1, N_2 by

$$\langle Au, \varphi\rangle = \int_0^1 u\varphi \, dx,$$

$$\langle N_1(u,v), \varphi\rangle = \\ = \int_0^1 [f(\bar{u}+u, \bar{v}+v) - \frac{\partial f}{\partial u}(\bar{u},\bar{v})\, u - \frac{\partial f}{\partial v}(\bar{u},\bar{v})\, v]\, \varphi \, dx,$$

$$\langle N_2(u,v),\psi\rangle =$$
$$= \int_0^1 [g(\bar{u}+u,\bar{v}+v) - \frac{\partial g}{\partial u}(\bar{u},\bar{v})\,u - \frac{\partial g}{\partial v}(\bar{u},\bar{v})\,v]\,\psi\,dx,$$

for all $u, v, \varphi, \psi \in \mathbf{V}$. It is easy to see that if u, v satisfy $(\tilde{\mathrm{SE}})$ then the couple $u+\bar{u}$, $v+\bar{v}$ is the classical solution of $(\mathrm{RD_S})$, (BC). If we choose $K = \{v \in \mathbf{V} : v(1) \geqq 0\}$, then if u, v satisfy $(\tilde{\mathrm{SI}})$, $u+\bar{u}$, $v+\bar{v}$ is the classical solution of $(\mathrm{RD_S})$ with unilateral boundary conditions

$$\begin{cases} u(0) = \bar{u}, u_x(1) = 0, \\ v(0) = \bar{v}, v(1) \geqq \bar{v}, v_x(1) \geqq 0, \ (v(1)-\bar{v})\,v_x(1) = 0. \end{cases}$$

We shall investigate also the stability of the trivial solution of the abstract inequality

$$(\tilde{\mathrm{AI}}) \quad \begin{cases} U(t) \in \tilde{K} \\ (\frac{\partial U}{\partial t}(t), \Phi - U(t))_{\sim} + \langle D(d)\,U(t) - B\tilde{A}U(t), \Phi - U(t)\rangle_{\sim} \geqq 0, \end{cases}$$

for all $\Phi \in \tilde{K}$, a.a. $t \in \mathbf{R}^+$. Simultaneously the corresponding system of equations will be considered

$$(\tilde{\mathrm{AE}}) \quad (\frac{\partial U}{\partial t}(t), \Phi)_{\sim} + \langle D(d)\,U(t) - B\tilde{A}U(t), \Phi\rangle_{\sim} = 0,$$

for all $\Phi \in \tilde{\mathbf{V}}$, a.a. $t \in \mathbf{R}^+$.

The following eigenvalue problems

$$(\tilde{\mathrm{EI}}) \quad \begin{cases} U \in \tilde{K}, \\ \langle D(d)\,U - B\tilde{A}U + \lambda\tilde{A}U, \ \Phi - U\rangle_{\sim} \geqq 0, \ \forall \Phi \in \tilde{K}, \end{cases}$$

and

$$(\tilde{\mathrm{EE}}) \quad D(d)\,U - B\tilde{A}U + \lambda\tilde{A}U = 0$$

will play the key role in the investigation of $(\tilde{\mathrm{AI}})$, $(\tilde{\mathrm{AE}})$.

REMARK 1.2 - We shall not discuss the existence and the smoothness of the solutions to $(\tilde{\mathrm{AI}})$. Our aim will be to show the existence of a solution of the type $U(t) = \exp(\lambda t)\,W_0$ of $(\tilde{\mathrm{AI}})$ with $\lambda > 0$ (for a suitable parameter d only) such that $\frac{\partial U}{\partial t}(t) \in \tilde{\mathbf{H}}$, for any $t \in \mathbf{R}^+$, and $(\tilde{\mathrm{AI}})$ is fulfilled for all $t \in \mathbf{R}^+$. If we wanted to give a general correct definition of the solution on $\langle 0, T)$ we could consider for instance $u, v \in L_2(0, T; \mathbf{V})$ such that $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \in L_2(0, T; \mathbf{V}^*)$ and $(\tilde{AE})$ (or $(\tilde{AI})$) is fulfilled for a.a. $t \in (0, T)$.

REMARK 1.3 - Consider the linearized reaction-diffusion system

$$(RD_L) \quad \begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + b_{11} u + b_{12} v, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + b_{21} u + b_{22} v. \end{cases}$$

Suppose $n = 1$, $\Omega = (0, 1)$, $\Gamma_D = 0$, $\Gamma_N = 1$, $A, \mathbf{V}, \mathbf{H}, K$, $b_{ij} (i, j = 1, 2)$ are defined as in Remark 1.1. Let $d_2 := 1$, $d_1 := d$. Then $(\tilde{AE})$ is an abstract formulation of (RD_L), (BC_0), where (BC_0) is nothing but (BC) with $\bar{u} = \bar{v} = 0$ (1), and $(\tilde{AI})$ represents (RD_L) with unilateral boundary conditions

$$(1.3) \quad \begin{cases} u(0) = 0, \; u_x(1) = 0, \\ v(0) = 0, \; v(1) \geqq 0, \; v_x(1) \geqq 0, \; v(1)\, v_x(1) = 0. \end{cases}$$

Analogously $(\tilde{EE})$ and $(\tilde{EI})$ are weak formulations of

$$(RD_\lambda) \quad \begin{cases} d \Delta u + b_{11} u + b_{12} v = \lambda u, \\ \Delta v + b_{21} u + b_{22} v = \lambda v, \end{cases}$$

with boundary conditions (BC_0) and (1.3), respectively.

We shall also investigate the stability of the trivial solution of the more general linearized systems with unilateral boundary conditions:

$$(\hat{AI}) \quad \begin{cases} U(t) \in \tilde{K} \\ \left(\dfrac{\partial U}{\partial t}(t), \Phi - U(t) \right)_{\sim} + \\ \qquad\qquad + \langle DU(t) - B(d_1, d_2)\, \tilde{A} U(t), \Phi - U(t) \rangle_{\sim} \geqq 0, \end{cases}$$

for all $\Phi \in \tilde{K}$, a.a. $t \in \mathbf{R}^+$, using the properties of the corresponding homogeneous problem

$$(\hat{AE}) \quad \left(\frac{\partial U}{\partial t}(t), \Phi \right)_{\sim} + \langle DU(t) - B(d_1, d_2)\, \tilde{A} U(t), \Phi \rangle_{\sim} = 0,$$

for all $\Phi \in \tilde{\mathbf{V}}$ (see Remark 1.4 below for the definition of $B(d_1, d_2)$). The stationary eigenvalue problems

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(1) The solution $\bar{u}, \bar{v}$ is automatically transformed to zero, i.e. we write u, v instead of $u - \bar{u}$, $v - \bar{v}$, in the linearized problem.

$$(\hat{\mathrm{EI}}) \quad \begin{cases} U \in \tilde{K}, \\ \langle DU - B(d_1, d_2)\,\tilde{A}U + \lambda\tilde{A}U,\ \Phi - U\rangle \geqq 0, \quad \forall \Phi \in \tilde{K}, \end{cases}$$

$$(\hat{\mathrm{EE}}) \qquad DU - B(d_1, d_2)\,\tilde{A}U + \lambda\tilde{A}U = 0,$$

will play the key role in our considerations.

REMARK 1.4 - Suppose again $n = 1$, $\Omega = (0,1)$, and consider (RD_L) from Remark 1.3, $b_{ij}(i,j = 1,2)$, $A, K, \mathbf{H}$ defined as in Remark 1.1. If we require (NC) instead of (BC_0) then (1.2) does not define any inner product on $\mathbf{V} = W^1_2(0,1)$ and so it is necessary to use the usual inner product

$$\langle u, \varphi\rangle = \int_0^1 [u_x\varphi_x + \tau u\varphi]\, dx,$$

with some $\tau > 0$. Then ($\hat{\mathrm{AE}}$) becomes an abstract formulation of (RD_L), (NC) if we put

$$B(d_1, d_2) = \begin{pmatrix} b_{11} + \tau d_1, b_{12} \\ b_{21}, b_{22} + \tau d_2 \end{pmatrix}.$$

Analogously ($\hat{\mathrm{AI}}$) is an abstract formulation of (RD_L) with unilateral boundary conditions

$$\begin{cases} u_x(0) = u_x(1) = v_x(0) = 0 \\ v(1) \geqq 0,\ v_x(1) \geqq 0,\ v(1)\, v_x(1) = 0. \end{cases}$$

DEFINITION 1.1 - Let $d_1, d_2 > 0$ (resp. $d > 0$) be given. If $\lambda > 0$ is such that there exists a nontrivial solution U of ($\hat{\mathrm{EI}}$) or of ($\hat{\mathrm{EE}}$) (resp. of ($\tilde{\mathrm{EI}}$) or of ($\tilde{\mathrm{EE}}$) then λ and U are said to be an *eigenvalue* and an *eigenvector* of ($\hat{\mathrm{EI}}$) or of ($\hat{\mathrm{EE}}$) (resp. of ($\tilde{\mathrm{EI}}$) or of ($\hat{\mathrm{EE}}$)) with the parameters d_1, d_2 (resp. with parameter d). The sets of all solutions of ($\hat{\mathrm{EI}}$) and ($\hat{\mathrm{EE}}$) (resp. ($\tilde{\mathrm{EI}}$) and ($\tilde{\mathrm{EE}}$)) will be denoted by $E_I(d_1, d_2, \lambda)$ and $E_B(d_1, d_2, \lambda)$ (resp. $E_I(d, \lambda)$ and $E_B(d, \lambda)$). We say that an *eigenvalue* λ of ($\hat{\mathrm{EE}}$) (resp. of ($\tilde{\mathrm{EE}}$)) is simple if dim $E_B(d_1, d_2, \lambda) = 1$ (resp. $E_B(d, \lambda) = 1$).

DEFINITION 1.2 - A couple $[d_1, d_2] \in \mathbf{R}^+ \times \mathbf{R}^+$ (resp. a point $d \in \mathbf{R}^+$) is called a *critical couple* of ($\hat{\mathrm{EI}}$) or ($\hat{\mathrm{EE}}$) (resp. a *critical* point of ($\tilde{\mathrm{EI}}$) or ($\tilde{\mathrm{EE}}$)) if $\lambda = 0$ is an eigenvalue of ($\hat{\mathrm{EI}}$) or ($\hat{\mathrm{EE}}$) (resp. ($\tilde{\mathrm{EI}}$) or ($\tilde{\mathrm{EE}}$)). A *critical couple* $[d_1, d_2]$ of ($\hat{\mathrm{EE}}$) (resp. a *critical point* d of ($\tilde{\mathrm{EE}}$)) is *simple* if $\lambda = 0$ is a simple eigenvalue of ($\hat{\mathrm{EE}}$) (resp. of ($\tilde{\mathrm{EE}}$)).

DEFINITION 1.3 - A point $d_0 > 0$ is called a *bifurcation* point of $(\tilde{SI})$ or of $(\tilde{SE})$ if any neighbourhood of $[d_0, 0]$ in $\mathbf{R} \times \mathbf{V}$ contains a solution $[d, U]$ of $(\tilde{SI})$ or $(\tilde{SE})$, respectively, satisfying $\|U\|_{\sim} \neq 0$.

2. Modification of the global bifurcation result due to E.N. Dancer

Consider a general bifurcation equation of the type

(BE) $\quad x - L(\mu)\,x + G(\mu, x) = 0$

in a real Hilbert space X with the inner product $\langle .,. \rangle_x$ and with the corresponding norm $|||\cdot|||$. Let us suppose that

(L) $\quad \left\{\right.$ for any $\mu \in \mathbf{R}$, $L(\mu)$ is a linear completely continuous operator on X; the mapping $\mu \to L(\mu)$ *of* $\mathbf{R}$ into the space $\mathfrak{L}(X, X)$ of linear continuous operators on X is continuous;

(LG) $\quad \left\{\right.$ the mapping $M : \mathbf{R} \times X \to X$ defined by $M(\mu, x) = L(\mu)\,x + G(\mu, x)$ is completely continuous;

(G) $\quad \displaystyle\lim_{|||x||| \to 0} \frac{G(\mu, x)}{|||x|||} = 0$ uniformly on bounded subsets of $\mathbf{R}$.

Denote by C the closure (in $\mathbf{R} \times X$) of the set of all nontrivial solutions of (BE), i.e. $C = \overline{\{[\mu, v] \in \mathbf{R} \times X : |||x||| \neq 0, \text{ (BE) is fulfilled}\}}$. Suppose that μ_0 is a simple critical point of

(BE$_L$) $\quad x - L(\mu)\,x = 0,$

i.e. there is a nontrivial solution x_0 of (BE$_L$) with $\mu = \mu_0$ and $\dim \bigcup_{k=1}^{\infty} \ker(I - L(\mu_0))^k = 1$. Let C_0 be the component of C containing $[\mu_0, 0]$. Further suppose that

(Ind) $\quad \mathrm{ind}(I - L(\mu_0 + \varepsilon)) \neq \mathrm{ind}(I - L(\mu_0 - \varepsilon)),$

for all $\varepsilon \in (0, \varepsilon_0)$, with some $\varepsilon_0 > 0$ (by «ind» we mean the Leray-Schauder index with respect to the origin).

THEOREM 2.1 - *(cf.* [1, *Theorem* 2]*). Let* u_0 *be a simple critical point of* (BE$_L$) *and let* (L), (LG), (G), (Ind) *be fulfilled. Then there are* C_0^+, C_0^- *connected subsets of* C_0 *containing the point* $[\mu_0, 0]$ *with the following property: either both* C_0^+ *and* C_0^- *are unbounded or* $C_0^+ \cap C_0^- \neq \{[\mu_0, 0]\}$.

REMARK 2.1 - Let us remark that [1, Theorem 2] deals with more special case $L(\mu) = \mu L$ but, using hypothesis (Ind), the proof of Theorem 2.1 can be performed in the same way as that of [1, Theorem 2].

REMARK 2.2 - Both C_0^+ and C_0^- are «starting» from $[\mu_0, 0]$ in the direction x_0 and $-x_0$, respectively.

3. Bifurcation points of variational inequality

Throughout Sections 3 and 4 we shall suppose that $b_{ij}(i,j=1,2)$ satisfy

(SIGN) $\quad b_{11} > 0,\ b_{12} < 0,\ b_{21} > 0,\ b_{22} < 0,\ b_{11} + b_{22} < 0.$

We can now formulate the result concerning the bifurcation points of the variational inequality $(\tilde{\mathrm{SI}})$.

THEOREM 3.1 - *(cf. [2, Theorem 2.1]). Let $d_0 > 0$ be the greatest critical point of* $(\tilde{\mathrm{EE}})$. *Suppose that d_0 is simple, $E_B(d_0, 0) \cap \tilde{K}^0 \neq \phi$ and* (A), (N), (SIGN) *are fulfilled. Then there exists a bifurcation point d_I of the inequality* $(\tilde{\mathrm{SI}})$ *satisfying $d_I > d_0$. More precisely, there is $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ there exist $[d(\delta), U(\delta)]$ satisfying* $(\tilde{\mathrm{SI}})$, *$U(\delta) \in \partial\tilde{K}$, $\| U(\delta)\|^2 = \delta$, $d(\delta) > d_0$ and all the limit points d_I (there is at least one) of $d(\delta)$ for $\delta \to 0_+$ are greater than d_0; $d(\delta)$, $U(\delta)$ do not satisfy* $(\tilde{\mathrm{SE}})$.

REMARK 3.1 - According to the assertion of Theorem 3.1 there are spatialy nonconstant stationary solutions of $(\tilde{\mathrm{EI}})$ bifurcating from the point $[d_I, 0]$ lying in the domain of stability of the trivial solution of $(\tilde{\mathrm{AE}})$ (because d_0 is the greatest critical point of $(\tilde{\mathrm{EE}})$).

Main ideas of the proof of Theorem 3.1. Using the projection $\tilde{P}$ onto the convex cone $\tilde{K}$ in $\tilde{\mathbf{V}}$ we can write $(\tilde{\mathrm{SI}})$ in the operator form

$$D(d)\,U - \tilde{P}(B\tilde{A}U - N(U)) = 0 \tag{3.1}$$

(see [2, 12] for details).

For each $\delta > 0$ fixed we shall denote by Z_δ the closure (in $\mathbf{R} \times \tilde{\mathbf{V}} \times \mathbf{R}$) of the set of all $[d, U, \tau] \in \mathbf{R}^+ \times \tilde{\mathbf{V}} \times (0, 1\rangle$ such that

(a) $\quad \| U \|_{\sim}^2 = \delta\tau,$

(b) $\quad D(d)\,U - B\tilde{A}U + \tau(\tilde{I} - \tilde{P})\,B\tilde{A}U + \tau R(U) = 0,$

where $R(U) = \tilde{P}B\tilde{A}U - \tilde{P}(B\tilde{A}U - N(U))$, for $U = [u, v]$. Put $\mu = 1/d$, $\tau = \varepsilon/(1+\varepsilon)$, $X = \tilde{\mathbf{V}} \times \mathbf{R}$ (with points $[U, \varepsilon]$) and define the operators $L(\mu) : X \to X$, $G_\delta : \mathbf{R} \times X \to X$ by

$$(\delta) \qquad \begin{cases} L(\mu)\, x = L(\mu)\,[U, \varepsilon] = [D(\mu)\, B\tilde{A}U, 0], \\ G_\delta(\mu, x) = G_\delta(\mu, U, \varepsilon) = [\dfrac{\varepsilon}{1+\varepsilon}(\tilde{I} - \tilde{P})\, D(\mu)\, B\tilde{A}U + \\ \qquad + \dfrac{\varepsilon}{1+\varepsilon} D(\mu)\, R(U), - \dfrac{1+\varepsilon}{\delta} \| U \|^2_{\sim}]. \end{cases}$$

It is possible to show that these operators satisfy (L), (LG), (G) (for any $\delta > 0$). The assumption (SIGN) also implies that $\mu_0 = 1/d_0$ is algebraically simple critical point of (BE_L) (see [3, Remark 2.2]) and that (Ind) is fulfilled (see [3, Lemma 2.1]). Hence the system of equations (a), (b) can be thought as an abstract bifurcation equation (BE) with operators L, G defined by (δ). It follows from Theorem 2.1 that for each $\delta > 0$ there exist closed connected subsets $Z^+_{\delta,0}$ and $Z^-_{\delta,0}$ of Z_δ starting from $[d_0, 0, 0]$ in the direction $W_0 \in \tilde{K}$ and $-W_0 \in \tilde{K}^0 \cap E_B(d_0, 0)$, respectively, and either

(i) $Z^+_{\delta,0}$ contains a point of the type $[d(\delta), U(\delta), 1]$

or

(ii) $Z^+_{\delta,0}$ is unbounded in d

or

(iii) $Z^+_{\delta,0}, Z^-_{\delta,0}$ meet each other at a point different from $[d_0, 0, 0]$.

It is possible to show that the cases (ii) and (iii) cannot occur for δ small enough. The boundedness of $Z^+_{\delta,0}$ follows from the elementary considerations about the equation (b) (see [2, Lemma 4.1, Remark 4.3]) which excludes the case (ii). In order to exclude also the case (iii) we need the following.

LEMMA 3.1 - *For each $\delta \in (0, \delta_0)$ (with $\delta_0 > 0$ fixed and sufficiently small) the following assertion are true for all $[d, U, \tau] \in Z^+_{\delta,0}$:*

(3.2) *if $[d, U, \tau] \neq [d_0, 0, 0]$ then $B\tilde{A}U \notin \tilde{K}$,*

(3.3) *if $[d, U, \tau] \neq [d_0, 0, 0]$ then $d_0 < d < \bar{d}$, for some $\bar{d} > 0$ independent of δ.*

The proof of (3.2), (3.3) is based on the following principles:

(3.4) for an arbitrary $\delta > 0$, the values d are locally increasing along $Z^+_{\delta,0}$ near $d = d_0$, $\| U \|_{\sim} = 0$, $\tau = 0$ (see [2, Lemma 2.2]);

(3.5) for any $\delta > 0$ small enough, $B\tilde{A}U$ cannot intersect $\partial\tilde{K}$ with $\tau > 0$ as long as $d \geqq d_0$ for $[d, U, \tau] \in Z^+_{\delta,0}$; at the same time $Z^+_{\delta,0}$ cannot intersect the line $d = d_0$ as long as $B\tilde{A}U \notin \tilde{K}$ (see [2, Lemmas 2.1, 4.2, 4.3]);

(3.6) the sets Z^+_δ are uniformly bounded with respect to $\delta \in (0, \delta_0)$ with δ_0 sufficiently small.

The assertion of Lemma 3.1 together with the fact that $B\tilde{A}U \in \tilde{K}^0$ for all $[d, U, \tau] \in Z^-_{\delta,0}$ excludes (iii).

REMARK 3.2 - The equation (b) represents the homotopy joining the equation $(\tilde{\text{E}}\text{E})$ with $\lambda = 0$ (for $\tau = 0$) and the variational inequality (3.1), i.e. $(\tilde{\text{S}}\text{I})$ (for $\tau = 1$). We obtain from (i), (a) and (b) that $d(\delta), U(\delta)$ satisfy $(\tilde{\text{S}}\text{I})$, $\| U(\delta) \|^2_{\sim} = \delta$ and $D(d(\delta))\, U(\delta) \in \tilde{K}$, i.e. $U(\delta) \in K$.

It is possible to show that $U(\delta) \in \partial\tilde{K}$, for all $\delta \in (0, \delta_0)$, with $\delta_0 > 0$ small. In the opposite case there are $d_n = d(\delta_n)$, $U_n = \overline{U(\delta_n)}$, satisfying $(\tilde{\text{S}}\text{I})$ and $\| U_n \|^2_{\sim} = \delta_n \to 0$, $U_n \in \tilde{K}^0$. It follows from (b) (putting $\tau = 1$) that $\tilde{P}(B\tilde{A}U_n - N(U)) \in \tilde{K}^0$ i.e. $BAU_n - N(U_n) \in \tilde{K}^0$. Using (3.3) we can suppose that $d_n \to d \geqq d_0$, $W_n = U_n / \| U_n \|_{\sim} \rightharpoonup W$. It follows from (b) that $W_n \to W$ and $D(d)\, W - B\tilde{A}W = 0$. Hence d is a critical point of $(\tilde{\text{E}}\text{E})$, i.e. $d = d_0$ because d_0 is assumed to be the greatest one. Simultaneously we have $B\tilde{A}W_n - N(U_n) / \| U_n \| \in \tilde{K}^0$ and $B\tilde{A}W_n \notin \tilde{K}$ by (3.2). Using assumption (N) we obtain from here that $D(d_0)\, W = B\tilde{A}W \in \partial\tilde{K}$, i.e. $W \in \partial\tilde{K}$. This contradicts the assumption that d_0 is simple and $E_B(d_0, 0) \cap \tilde{K}^0 \neq \phi$.

The existence of a limit point d_I of $d(\delta)$ for $\delta \to 0_+$ follows from (3.3). Suppose that $d(\delta_n) \to d_0$ for some $\delta_n \to 0_+$. We obtain (by the limiting process) from (b) the existence of $W \in \partial\tilde{K}$, $\| W \|_{\sim} = 1$ such that $D(d_0)\, W - \tilde{P}B\tilde{A}U = 0$ which contradicts the simplicity of d_0 and the assumption $E_B(d_0, 0) \cap \tilde{K}^0 \neq \phi$. Hence$d_I > d_0$.

Suppose that $d(\delta_n)$, $U(\delta_n)$ satisfy also $(\tilde{\text{S}}\text{E})$ for some $\delta_n \to 0$,

$d(\delta_n) > d_0, U(\delta_n) \in \partial\tilde{K}$. This implies the existence of a bifurcation point of ($\tilde{S}$E) which is greater than d_0. This contradiction proves the last assertion of Theorem 3.1.

4. Destabilizing effect of unilateral boundary conditions for linearized systems

In this Section we shall deal with the influence of unilateral boundary conditions on the stability of the trivial solution of the linearized system ($\tilde{A}$I).

THEOREM 4.1 - *(cf.* [3, *Theorem* 1.2]*). Assume the same as in Theorem* 3.1 *and* (V,H). *Then there are $d_I > d_0$, $\lambda > 0$ and $W_I \in \partial\tilde{K} \setminus \{0\}$ such that the function $U(t) = \exp(\lambda t) W_I$ satisfies* ($\tilde{A}$I).

REMARK 4.1 - We have $U(0) = W_I$ and $\| U(t) \|_{\sim} \to +\infty$, for $t \to +\infty$. It implies the unstability of the trivial solution of ($\tilde{A}$I) in an arbitrary reasonable sense. On the other hand the trivial solution of ($\tilde{A}$E) is stable for any $d > d_0$ under our assumptions. Hence unilateral conditions of the type considered have a destabilizing effect.

Main ideas of the proof of Theorem 4.1. It is sufficient to show that W_I satisfies ($\tilde{E}$I) (see e.g. [9], cf. [3, Theorem 1.1]). The assumption (SIGN) and elementary investigation of the equation ($\tilde{E}$E) yield the existence of $\rho > 0$ such that: for any $d_1 \in (d_0 - \rho, d_0)$ there is simple eigenvalue $\lambda(d_1) > 0$ of ($\tilde{E}$E) (with $d = d_1$) and

$$E_B(d_1, \lambda(d_1)) \cap \tilde{K}^0 \neq \phi .$$

Moreover, replacing B by $B_{\lambda(d_1)} = B - \lambda(d_1) E$ (E is the unit matrix), d_1 satisfies the assumptions of Theorem 3.1 (see [3, Lemmas 2.1, 2.2, 2.3]). We can repeat the same procedure as in the proof of Theorem 3.1 replacing d_0 by d_1, B by $B_{\lambda(d_1)}$, setting $N = 0$, $\delta = 1$ and using the fact that $B_{\lambda(d_1)}$ satisfies (SIGN) provided $\rho > 0$ is sufficiently small. We obtain that for each $d_1 \in (d_0 - \rho, d_0)$ there exist $d_{1,I} > d_1$ and $\lambda = \lambda(d_1)$ such that ($\tilde{E}$I) with $d = d_{1,I}$ and $\lambda = \lambda(d_1)$ is fulfilled and $E_I(d_{1,I}, \lambda) \subset \partial\tilde{K}$.

It remains to show that $d_{1,I} > d_0$ if $\rho > 0$ is small enough. If this were not true we should have sequences $\{d_n\}, \{d_{n,I}\}$ such that

$$d_n < d_{n,I} \leqq d_0 \, , d_n \to d_0 \, ,$$

with corresponding eigenvalues $\lambda_n = \lambda(d_n)$ and eigenvectors

$$W_n \in \partial\tilde{K} \cap E_I(d_{n,I}, \lambda_n), \ \| W_n \|_{\sim} = 1 \, .$$

Using the compactness of A and a suitable limiting process in $(\tilde{EI})$ (with $d = d_{n,I}, U = W_n, \lambda = \lambda_n)$ we derive

$$\langle D(d_0) W - B\tilde{A}W, \Phi - W \rangle_{\sim} \geqq 0 \, ,$$

for all $\Phi \in \tilde{K}$, with some $\| W \|_{\sim} = 1$, $W \in E_I(d_0, 0) \cap \partial\tilde{K}$. But

$$E_B(d_0, 0) \cap \tilde{K} = E_I(d_0, 0)$$

(this holds in general under the assumption $E_B(d_0, 0) \cap \tilde{K}^0 \neq \phi$, see [2, Lemma 2.1]). Hence $W \in \partial\tilde{K} \cap E_B(d_0, 0)$ which contradicts the simplicity of d_0 and the assumption $E_B(d_0, 0) \cap \tilde{K}^0 \neq \phi$.

5. More general result concerning linearized systems

In this Section we shall deal with the destabilizing effect of unilateral boundary conditions for more general linearized problem $(\hat{AE})$. The main tool is the same as that in proving the results in Sections 3, 4 (i.e. Theorem 2.1). The difference is in it's application. Roughly speaking the situation in the previous Sections was the following: we had only one diffusion coefficient d (we supposed $d_1 = d, d_2 = 1$) which played the role of «moving» parameter μ (see Theorem 2.1, $\mu = 1/d$) and the proofs were based on the existence of certain branches in d. To prove the results of this Section (we shall consider two diffusion coefficients $d_1, d_2 > 0$) we use again Theorem 2.1. But the corresponding eigenvalue λ of $(\hat{EE})$ will play the role of the «moving» parameter μ.

We need to add a further condition on $b_{ij} (i, j = 1, 2)$:

(5.1) $\quad b_{11} b_{22} - b_{12} b_{21} > 0 \, .$

It is possible to show that the assumptions (A), (SIGN), (5.1) imply the existence of a continuous function $d_2 = \xi(d_1)$, defined for all $d_1 \in (0, c)$, $\lim_{d_1 \to 0_+} \xi(d_1) = 0$, $\lim_{d_1 \to c_-} \xi(d_1) = +\infty$, ξ is increasing on $(0, c)$ and such that

(5.2) if $d_2 = \xi(d_1)$ then $[d_1, d_2]$ is a critical couple of $(\hat{\mathrm{EE}})$;

(5.3) if either $d_2 < \xi(d_1)$, $d_1 \in (0, c)$ or $[d_1, d_2] \in \mathbf{R}^+ \times \mathbf{R}^+$, $d_1 > c$ then $[d_1, d_2]$ is not a critical couple of $(\hat{\mathrm{EE}})$ (c is some constant).

REMARK 5.1 - The illustration of (5.2), (5.3) is the following: the graph of ξ (we shall denote it by G_ξ) is the set of the «greatest critical couples with respect to d_1». Simultaneously G_ξ divides the first quadrant into two domains: the first one $G_\xi^- = \{[d_1, d_2] \in \mathbf{R}^+ \times \mathbf{R}^+ : d_2 > \xi(d_1), d_1 \in (0, c)\}$ and the second one G_ξ^+ is the complement of $G_\xi \cup G_\xi^-$. The trivial solution of $(\hat{\mathrm{AE}})$ is stable whenever $[d_1, d_2] \in G_\xi^+$ and unstable if $[d_1, d_2] \in G_\xi^-$ (see [4]).

THEOREM 5.1 - *Let the assumptions* (A), (V,H), (SIGN), (5.1) *be fulfilled,* $[d_1^0, d_2^0] \in G_\xi$ *is a simple critical couple. Suppose that* $E_B(d_1^0, d_2^0, 0) \cap K^0 \neq \phi$. *Then there is a neighbourhood* $\mathcal{B}(d_1^0, d_2^0)$ *of the point* $[d_1^0, d_2^0]$ *such that for any* $[d_1, d_2] \in \mathcal{B}(d_1^0, d_2^0)$ *there are* $\lambda_I = \lambda_I(d_1, d_2) > 0$ *and* $W_I = W_I(d_1, d_2) \in \partial\tilde{K} \setminus \{0\}$ *such that the abstract function* $U(t) = \exp(\lambda_I t) W_I$ *satisfies* $(\hat{\mathrm{AI}})$.

Main ideas of the proof of Theorem 5.1 (see [4] for the details). Similarly as in the proof of Theorem 4.1 it is sufficient to show that for any $[d_1, d_2] \in \mathcal{B}(d_1^0, d_2^0)$ the eigenvalue problem $(\hat{\mathrm{EI}})$ has positive eigenvalue $\lambda_I = \lambda_I(d_1, d_2) > 0$. Let us define the set $Z(d_1, d_2)$ as a closure of all $[\lambda, U, \varepsilon]$ in $\mathbf{R} \times \tilde{\mathbf{V}} \times \mathbf{R}$ such that

(5.4) $$\| U \|_{\sim}^2 = \varepsilon / (1 + \varepsilon)$$

(5.5) $$DU - (B(d_1, d_2)\tilde{A}U - \lambda \tilde{A}U) + \varepsilon\beta(U) = 0,$$

with some fixed $[d_1, d_2] \in \mathbf{R}^+ \times \mathbf{R}^+$, where β is usual penalty operator corresponding to $\tilde{K}$ (see e.g. [5,6,7]). Put $X = \tilde{\mathbf{V}} \times \mathbf{R}$ (with the points $x = [U, \varepsilon]$) and define the operators $L(\lambda) : X \to X$, $G : \mathbf{R} \times X \to X$ by

$$L(\lambda)\, x = L(\lambda)\, [U, \varepsilon] = [D^{-1} B(d_1, d_2)\, \tilde{A}U - \lambda D^{-1} \tilde{A}U, 0],$$

$$G(\lambda, x) = G(\lambda, U, \varepsilon) = [\varepsilon D^{-1} \beta(U), -(1+\varepsilon)\| U \|^2],$$

where D^{-1} is the inverse of D. These operators satisfy the assumptions (L), (LG), (G) and so it is possible to prove the existence of a neighbourhood $\mathcal{B}(d_1^0, d_2^0)$ such that for any $[d_1, d_2] \in \mathcal{B}(d_1^0, d_2^0)$ there is a $\lambda = \lambda(d_1, d_2)$ which is a simple critical point of (BE_L) (with operators L, G defined above). Moreover (Ind) is fulfilled with $\mu = \lambda$. Furthermore the neighbourhood $\mathcal{B}(d_1^0, d_2^0)$ can be taken so small that $E_B(d_1, d_2, \lambda(d_1, d_2)) \cap \tilde{K}^0 \neq \phi$. Application of Theorem 2.1 (for each fixed $[d_1, d_2] \in \mathcal{B}(d_1^0, d_2^0)$) yields the existence of connected sets $Z^+(d_1, d_2)$, $Z^-(d_1, d_2)$ (starting in the directions

$$W(d_1, d_2) \in E_B(d_1, d_2, \lambda(d_1, d_2)) \cap (-\tilde{K})$$

and

$$-W(d_1, d_2) \in E_B(d_1, d_2, \lambda(d_1, d_2)) \cap \tilde{K}^0$$

both containing the point $[\lambda(d_1, d_2), 0, 0]$ and such that either

(5.6) both $Z^+(d_1, d_2)$, $Z^-(d_1, d_2)$ are unbounded

or

(5.7) $Z^+(d_1, d_2) \cap Z^-(d_1, d_2) \neq [\lambda(d_1, d_2), 0, 0]$.

The case (5.7) can be excluded in a similar way as (iii) in Section 3. The proof is based on the fact that

$$U \notin \tilde{K} \text{ if } [\lambda, U, \varepsilon] \in Z^+(d_1, d_2)$$

and

$$U \in \tilde{K}^0 \text{ if } [\lambda, U, \varepsilon] \in Z^-(d_1, d_2).$$

Simultaneously it can be proved that $Z^+(d_1, d_2)$ is bounded in the first component λ and $\lambda > \lambda(d_1, d_2)$ whenever $[\lambda, U, \varepsilon] \in Z^+(d_1, d_2)$, $[\lambda, U, \varepsilon] \neq [\lambda(d_1, d_2), 0, 0]$. Hence $Z^+(d_1, d_2)$ (and also $Z^-(d_1, d_2)$) is unbounded in ε and (5.4) implies that ε must be positive. Passing to the limit for $\varepsilon \to \infty$ in (5.5) we obtain (using the properties of the penalty operator β) the existence of $\lambda_I = \lambda_I(d_1, d_2) > \lambda(d_1, d_2)$ and $W_I = W_I(d_1, d_2)$ which satisfy ($\widehat{EI}$).

Using the limiting procedure similar to that in the proof of $d_{1,I} > d_0$ in Theorem 4.1 it is possible to show that for any $[d_1, d_2] \in \mathcal{B}(d_1^0, d_2^0)$ it is $\lambda_I(d_1, d_2) > 0$ if the neighbourhood is sufficiently small.

6. Final remarks

REMARK 6.1 - The assertion of Theorem 5.1 can be proved also without the assumption that $[d_1, d_2]$ is a simple critical couple (see [4]).

REMARK 6.2 - Unilateral boundary conditions considered in this paper concern always the second function v and they have destabilizing effect with respect to the spatialy homogeneous solution. On the other hand it is possible to study an opposite effect (some kind of stabilization) putting certain unilateral conditions on the first function u (see [8]).

REMARK 6.3 - Using an abstract approach explained in Section 5 it is also possible to deal with (RD_L), (BC_0). In this case we put $\tau = 0$ and $B(d_1, d_2) = B$ for any $[d_1, d_2] \in \mathbf{R}^+ \times \mathbf{R}^+$.

REMARK 6.4 - If we wanted to use the abstract approach from Sections 3, 4 in the case of Neumann boundary conditions (NC), we should consider the matrix $B(d, 1)$ instead of B. The problem which arizes here is the «moving parameter d» in the coefficients of $B(d, 1)$ (see [2, 3]).

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