BIFURCATION FOR SOME NONLINEAR ELLIPTIC VARIATIONAL INEQUALITIES (*)

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SOMMARIO. - *Si studia un problema di biforcazione relativo ad una disequazione variazionale di tipo ellittico. Vengono forniti alcuni risultati di biforcazione per gli «autovalori del problema linearizzato», anche successivi al primo.*

SUMMARY. - *A bifurcation problem concerning an elliptic variational inequality is studied. Bifurcation results are given, concerning the «eigenvalues» (not necessarily the first one) of the «linearized problem».*

Introduction

Several problems, which are typical in nonlinear functional analysis [2, 21], may occur, in view of many applications, in situations in which the usual regularity assumptions are not satisfied.

As in linear analysis and in convex analysis (see, for instance, [4, 12]), it may happen it is necessary to consider functionals which are not differentiable in the classical sense or even not continuous. Analogously, it may be necessary to restrict such functionals on sub-

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sets («constraints») which do not verify the usual regularity hypotheses.

Among several studies in this direction (see first of all the theory exposed in [6]), an attempt was developed in these years [7, 9, 10] to constitute a theoretical framework in order to treat problems of partial differential equations and associated variational inequalities.

In this way multiplicity results, which are typical in nonlinear analysis, were obtained also for some variational inequalities. The problem of geodesics with respect to an obstacle [17] is of this kind and also the problem of the eigenvalues with respect to an obstacle of the Laplace operator [5, 16] (or of an elliptic operator) that we briefly recall.

Given a bounded open set Ω in $\mathbf{R}^n$, let us consider a function $g:\Omega\times\mathbf{R}\to\mathbf{R}$, two functions $\varphi_1,\varphi_2:\Omega\to\bar{\mathbf{R}}$ with $\varphi_1\le\varphi_2$ and $\rho>0$.

The problem is to find the pairs (λ,u) such that

$$(\mathrm{P})\quad\begin{cases}\lambda\in\mathbf{R},u\in K\cap S_\rho,\\ \int_\Omega(DuD(v-u)+g(x,u)(v-u))\,dx\ge\int_\Omega\lambda u(v-u)\,dx\quad\forall v\in K\end{cases}$$

where $$K=\{v\in H^1_0(\Omega):\varphi_1\le v\le\varphi_2\},$$

$S_\rho=\{v\in L^2(\Omega):\int_\Omega v^2\,dx=\rho^2\}$.

If $\varphi_1\equiv-\infty$ and $\varphi_2\equiv+\infty$, the problem is that of nonlinear eigenvalues of Laplace operator.

The solutions u of problem (P) can be regarded as «points which are critical from below» (see definition (1.1) for the functional

$$f(u)=(1/2)\int_\Omega|Du|^2\,dx+\int_\Omega\int_0^u g(x,s)\,dsdx$$

on the «constraint» $K\cap S_\rho$.

The minimization of such functional and the existence of a solution to (P) have been considered for instance in [1, 14, 20].

In [5] the multiplicity of solutions to (P) is studied by means of the associated evolution variational inequality and an adaptation of Ljusternik-Schnirelmann techniques.

In this lecture the corresponding bifurcation problem is exposed: one has to individuate a «tangent» problem (the linearized problem in the classical case) and establish the connections between the «eigenvalues» of this one and the «branches» of solutions of (P).

§ 1.

Throughout this section H will denote a real Hilbert space whose scalar product and norm are denoted by $(\cdot|\cdot)$ and $|\cdot|$

(1.1) DEFINITION. - Let W be an open subset of H and

$$f: W \to \mathbf{R} \cup \{+\infty\}$$

a function. Set $D(f) = \{u \in W : f(u) < +\infty\}$. Let $u \in D(f)$. The function f is said to be subdifferentiable at u if there exists α in H such that

$$\liminf_{v \to u} \frac{f(v) - f(u) - (\alpha | v - u)}{|v - u|} \geq 0 .$$

For every u in $D(f)$ we denote by $\partial^- f(u)$ the (possibly empty) set of such α's and we set $D(\partial^- f) = \{u \in D(f) : \partial^- f(u) \neq \phi\}$. Since $\partial^- f(u)$ is convex and closed, for every u in $D(\partial^- f)$ we can denote by $grad^- f(u)$ the element of $\partial^- f(u)$ of minimal norm.

Finally, a point u in $D(f)$ is said to be critical from below if $0 \in \partial^- f(u)$. A real number c is said to be a critical value if there exists u in $D(f)$ such that $0 \in \partial^- f(u)$ and $f(u) = c$.

Now we consider a function $f: H \to \mathbf{R} \cup \{+\infty\}$ such that

(1.2) $$f(0) = 0 , \quad 0 \in \partial^- f(0) .$$

Our purpose is to study the set of the pairs (λ, u) in $\mathbf{R} \times D(\partial^- f)$ such that

(1.3) $$\lambda u \in \partial^- f(u) .$$

Of course for every λ in $\mathbf{R}$ the pair $(\lambda, 0)$ satisfies (1.3).

(1.4) DEFINITION. - A real number λ is said to be of bifurcation for $\partial^- f$ if there exists a sequence $((\lambda_h, u_h))_h$ in $\mathbf{R} \times D(\partial^- f)$ such that

$$\forall h \in \mathbf{N} : \lambda_h u_h \in \partial^- f(u_h) , \quad u_h \neq 0 ;$$

$$\lim_h (\lambda_h, u_h) = (\lambda, 0) \text{ in } \mathbf{R} \times H .$$

If the function f is sufficiently smooth, the study of bifurcation has been made by several authors (see, for instance, [2, 3, 11, 13, 15, 18, 19]). However for the applications to variational inequalities we are brought to consider discontinuous functions.

In order to give a characterization of the values λ of bifurcation, we make the following further assumptions on f:

(1.5) the function f is lower semicontinuous and there exists a continuous function $q: H \to \mathbf{R}$ such that
$f(v) \geq f(u) + (\alpha | v - u) - q(u) | v - u |^2$

whenever $v \in H$, $u \in D(\partial^- f)$, $\alpha \in \partial^- f(u)$;

(1.6) if $(u_h)_h$ is a sequence in $H \setminus \{0\}$ such that

$\sup_h f(u_h)/(|u_h|^2) < +\infty$, then $(u_h/|u_h|)$ possesses a convergent subsequence;

(1.7) there exists a function $f_0 : H \to \mathbf{R} \cup \{+\infty\}$ such that for every sequence $(\rho_h)_h$ in $]0,1]$ with $\lim_h \rho_h = 0$ we have
$f_0 = \Gamma^-(H) \lim_h f_{\rho_h}$
where $f_\rho(u) = f(\rho u)/(\rho^2)$ (see [8] for the definition of Γ-limit).

In these hypotheses it is readily proved that

$$f_0(0) = 0, \quad 0 \in \partial^- f_0(0);$$

$$\forall s > 0, \quad \forall u \in H : f_0(su) = s^2 f_0(u).$$

(1.8) DEFINITION. - A real number λ is said to be an eigenvalue of $\partial^- f_0$ if there exists w in $D(\partial^- f_0)$ such that

$$w \neq 0, \quad \lambda w \in \partial^- f_0(w).$$

(1.9) THEOREM. - *Under assumptions* (1.2), (1.5), (1.6), (1.7) *if* λ *is of bifurcation for* $\partial^- f$, *then* λ *is an eigenvalue of* $\partial^- f_0$.

The converse is, in general, not true, as the following example shows.

(1.10) EXAMPLE. - Let $H = \mathbf{R}^2$,

$$f(x, y) = x^2 + 2y^2 - x(x^2 + y^2) \exp(-x^2 - y^2)$$

if $x \geq 0$ and $y \geq 0$, $f(x, y) = +\infty$ elsewhere.

Then assumptions (1.2), (1.5), (1.6), (1.7) are satisfied and we have $f_0(x, y) = x^2 + 2y^2$ if $x \geq 0$ and $y \geq 0$, $f_0(x, y) = +\infty$ elsewhere.

On the other hand it is easy to check that $\lambda = 4$ is an eigenvalue of $\partial^- f_0$, but it is not of bifurcation for $\partial^- f$ (the reason is that $\lambda = 4$ is not «topologically essential» in a sense that will be precised later).

We have to make a further assumption which allows to state the converse of (1.9). For this purpose define $\tilde{f}_0 : H \to \mathbf{R} \cup \{+\infty\}$ by means of $\tilde{f}_0(u) = f_0(u)$ if $|u| = 1$, $\tilde{f}_0(u) = +\infty$ if $|u| \neq 1$.

(1.11) PROPOSITION. - *Let* $\lambda \in \mathbf{R}$. *Then the following facts are equivalent:*

a) λ *is an eigenvalue of* $\partial^- f_0$;

b) $(\lambda/2)$ *is a critical value of* $\tilde{f}_0$.

Now we can state our bifurcation theorem, which concerns only the eigenvalues which are «topologically essential».

If $c \in \mathbf{R}$, set $\tilde{f}_0^c = \{u \in H : \tilde{f}_0(u) \leq c\}$.

(1.12) THEOREM. - *Suppose that* (1.2), (1.5), (1.6), (1.7) *hold. Let* λ *be an eigenvalue of* $\partial^- f_0$ *such that for some* $\varepsilon > 0$ $\tilde{f}_0^{(\lambda/2)-\varepsilon}$ *is not a weak deformation retract of* $\tilde{f}_0^{(\lambda/2)+\varepsilon}$ *in* $\tilde{f}_0^{(\lambda/2)+\varepsilon}$ *and* $(\lambda/2)$ *is the unique critical value of* $\tilde{f}_0$ *in* $[(\lambda/2) - \varepsilon, (\lambda/2) + \varepsilon]$.

Then λ *is of bifurcation for* $\partial^- f$.

More precisely, there exists $\rho_0 > 0$ *and two sets* $\{\lambda_\rho : 0 < \rho \leq \rho_0\} \subset \mathbf{R}$ *and* $\{u_\rho : 0 < \rho \leq \rho_0\} \subset D(\partial^- f)$ *such that*

$$\forall \rho : \lambda_\rho u_\rho \in \partial^- f(u_\rho) \, , \; |u_\rho| = \rho \, ; \; \lim_{\rho \to 0} \lambda_\rho = \lambda \, .$$

(1.13) COROLLARY. - *Suppose that* (1.2), (1.5), (1.6), (1.7) *hold and that* $f_0(u) < +\infty$ *for some* $u \neq 0$.

Then $\tilde{f}_0 \not\equiv +\infty$, *there exists* $(\lambda/2) := \min \tilde{f}_0$ *and* λ *is of bifurcation for* $\partial^- f$. *More precisely, all the thesis of theorem* (1.12) *holds.*

Bifurcation theorems for the «first eigenvalue» were already proved in [20].

There is a significant case in which theorem (1.12) is true for all the eigenvalues of $\partial^- f_0$.

(1.14) THEOREM. - *Suppose that* (1.2), (1.5), (1.6), (1.7) *hold. Suppose moreover that* $D(\partial^- f_0)$ *is a linear subspace of* H *and that* $\mathrm{grad}^- f_0 : D(\partial^- f_0) \to H$ *is a linear map. Let* λ *be an eigenvalue of* $\partial^- f_0$.

Then λ *is of bifurcation for* $\partial^- f$. *More precisely, all the thesis of theorem* (1.12) *holds.*

Under the assumptions of theorem (1.14) it is easy to prove that for every λ in $\mathbf{R}$ the set $E_\lambda = \{u \in D(\partial^- f_0) : \lambda u \in \partial^- f_0(u)\}$ is a linear subspace of H of finite dimension. A real number λ is said to be a simple eigenvalue of $\partial^- f_0$ if E_λ has dimension one.

(1.15) THEOREM. - *Under the assumption of theorem* (1.14) *let* λ *be a simple eigenvalue of* $\partial^- f_0$.

Then λ *gives rise to two branches of bifurcation for* $\partial^- f$, *that is there exists* $\rho_0 > 0$ *and*

$$\{\lambda_\rho^{(i)} : 0 < \rho \leq \rho_0 \, , \; i = 1, 2\} \subset \mathbf{R} \, ,$$

$$\{u_\rho^{(i)} : 0 < \rho \leq \rho_0 \, , \; i = 1, 2\} \subset D(\partial^- f)$$

such that

$$\forall \rho, i : \lambda_\rho^{(i)} u_\rho^{(i)} \in \partial^- f(u_\rho^{(i)}) \, , \; |u_\rho^{(i)}| = \rho \, , u_\rho^{(1)} \neq u_\rho^{(2)} ;$$

$$\lim_{\rho \to 0} \lambda_\rho^{(i)} = \lambda \, .$$

§ 2.

In this section we expose an application of the results of the previous section to a bifurcation problem for a nonlinear elliptic variational inequality.

Let Ω be a bounded open subset of $\mathbf{R}^n$, $g : \bar{\Omega} \times \mathbf{R} \to \mathbf{R}$ a function of class C^1 such that $g(x,0) = 0$, $\varphi_1 : \Omega \to [-\infty, 0]$ an upper semicontinuous function and $\varphi_2 : \Omega \to [0, +\infty]$ a lower semicontinuous function.

For $i = 1, 2$ set $F_i = \{x \in \Omega : \varphi_i(x) = 0\}$ and denote by K the closure in $H^1_0(\Omega)$ of the set

$$\{u \in C^\infty_0(\Omega) : \forall x \in \Omega : \varphi_1(x) \leq u(x) \leq \varphi_2(x)\}$$

and by K_0 the closure in $H^1_0(\Omega)$ of the set

$$\{u \in C^\infty_0(\Omega) : \forall x \in F_1 : u(x) \geq 0 ; \ \forall x \in F_2 : u(x) \leq 0\}.$$

We want to study the pairs (λ, u) in $\mathbf{R} \times K$ such that

$$(2.1) \quad u \in L^\infty(\Omega), \quad \int_\Omega (DuD(v-u) + g(x,u)(v-u))\, dx \geq$$
$$\geq \int_\Omega \lambda u(v-u)\, dx \quad \forall v \in K.$$

Because of the assumptions we have made, for every λ in $\mathbf{R}$ the pair $(\lambda, 0)$ satisfies (2.1).

(2.2) Definition. - A real number λ is said to be of bifurcation for $-\Delta u + g(x,u)$ with respect to K, if there exists a sequence $((\lambda_h, u_h))_h$ in $\mathbf{R} \times K$ such that

$$\forall h \in \mathbf{N} : (\lambda_h, u_h) \text{ satisfies (2.1)}, \ u_h \neq 0 ;$$

$$\lim_h \lambda_h = \lambda ; \ \lim_h u_h = 0 \text{ in } L^\infty(\Omega) .$$

(2.3) Definition. - A real number λ is said to be an eigenvalue of $-\Delta u + g'_u(x,0)\, u$ with respect to K_0, if there exists u in K_0 such that

$$u \neq 0, \ \int_\Omega (DuD(v-u) + g'_u(x,0)\, u(v-u))\, dx \geq$$
$$\geq \int_\Omega \lambda u(v-u)\, dx \quad \forall v \in K_0.$$

(2.4) Theorem. - *Let $\lambda \in \mathbf{R}$. If λ is of bifurcation for $-\Delta u + g(x,u)$ with respect to K, then λ is an eigenvalue of $-\Delta u + g'_u(x,0)\, u$ with respect to K_0.*

To formulate the converse, we have to consider the functional $\mathcal{J}_0 : L^2(\Omega) \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\mathcal{J}_0(u) = \int_\Omega (1/2)(|Du|^2 + g'_u(x,0)\, u^2)\, dx$$

if $u \in K_0$ and $\int_\Omega u^2\, dx = 1$; $\mathcal{f}_0(u) = +\infty$ elsewhere.

(2.5) THEOREM. - *Let* λ *be an eigenvalue of* $-\Delta u + g'_u(x,0)\, u$ *with respect to* K_0. *Suppose that for some* $\varepsilon > 0$ $\mathcal{f}_0^{(\lambda/2)-\varepsilon}$ *is not a weak deformation retract of* $\mathcal{f}_0^{(\lambda/2)+\varepsilon}$ *in* $\mathcal{f}_0^{(\lambda/2)+\varepsilon}$ *and that* $(\lambda/2)$ *is the unique critical value of* $\mathcal{f}_0$ *in* $[(\lambda/2) - \varepsilon, (\lambda/2) + \varepsilon]$.

Then λ *is of bifurcation for* $-\Delta u + g(x,u)$ *with respect to* K.

More precisely, there exists $\rho_0 > 0$ *and* $\{\lambda_\rho : 0 < \rho \le \rho_0\} \subset \mathbf{R}$, $\{u_\rho : 0 < \rho \le \rho_0\} \subset K$ *such that* $\forall \rho : (\lambda_\rho, u_\rho)$ *satisfies* (2.1), $\int_\Omega u_\rho^2\, dx = \rho^2$; $\lim_{\rho \to 0} \lambda_\rho = \lambda$; $\lim_{\rho \to 0} u_\rho = 0$ *in* $L^\infty(\Omega)$ *and in* $H_0^1(\Omega)$.

A particular case of theorem (2.5) is the bifurcation theorem for the «first eigenvalue» (see also [20]).

(2.6) COROLLARY. - *Suppose we are not in the trivial case*

$$\varphi_1 = \varphi_2 = 0.$$

Then there exists $\lambda := 2 \min \mathcal{f}_0$, λ *is an eigenvalue of*

$$-\Delta u + g'_u(x,0)\, u$$

with respect to K_0 *(the «first eigenvalue») and* λ *is of bifurcation for* $-\Delta u + g(x,u)$ *with respect to* K. *More precisely, all the thesis of theorem* (2.5) *holds.*

Now we consider the particular case in which $F_1 = F_2$ and we set $\Omega' = \Omega \setminus F_1 = \Omega \setminus F_2$.

In such a case the eigenvalues of $-\Delta u + g'_u(x,0)\, u$ with respect to K_0 coincide with the classical eigenvalues of

$$-\Delta u + g'_u(x,0)\, u$$

in the open set Ω'.

(2.7) THEOREM. - *Suppose that* $F_1 = F_2$ *and let* λ *be an eigenvalue of* $-\Delta u + g'_u(x,0)\, u$ *in the open set* Ω'.

Then λ *is of bifurcation for* $-\Delta u + g(x,u)$ *with respect to* K. *More precisely, all the thesis of theorem* (2.5) *holds.*

(2.8) THEOREM. - *Suppose that* $F_1 = F_2$ *and let* λ *be a simple eigenvalue of* $-\Delta u + g'_u(x,0)\, u$ *in the open set* Ω'.

Then λ *gives rise to two branches of bifurcation for*

$$-\Delta u + g(x,u)$$

with respect to K, *namely there exists* $\rho_0 > 0$ *and*

$$\{\lambda_\rho^{(i)} : 0 < \rho \le \rho_0,\ i = 1,2\} \subset \mathbf{R},\ \{u_\rho^{(i)} : 0 < \rho \le \rho_0,\ i = 1,2\} \subset K$$

such that

$$\forall \rho, i: (\lambda_\rho^{(i)}, u_\rho^{(i)}) \text{ satisfies } (2.1), \int_\Omega |u_\rho^{(i)}|^2 dx = \rho^2,$$

$$u_\rho^{(1)} \neq u_\rho^{(2)};$$

$$\lim_{\rho \to 0} \lambda_\rho^{(i)} = \lambda; \lim_{\rho \to 0} u_\rho^{(i)} = 0 \text{ in } L^\infty(\Omega) \text{ and in } H_0^1(\Omega).$$

REFERENCES

[1] V. BENCI, A. M. MICHELETTI, *Su un problema di autovalori per disequazioni variazionali,* Ann. Mat. Pura Appl. (4) *107* (1975), 359-371.

[2] M. BERGER, *Nonlinearity and functional analysis. Lectures of nonlinear problems in mathematical analysis.* Pure and Applied Mathematics, *74,* Academic Press, New York - London, 1977.

[3] R. BÖHME, *Die Lösung der Verzweigungsgleichungen für nichtlineare Eigenwertprobleme,* Math. Z. *127* (1972), 105-126.

[4] H. BREZIS, *Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert,* North-Holland Mathematics Studies, n. 5, Notas de Matemàtica (50), Amsterdam-London, 1973.

[5] G. CHOBANOV, A. MARINO, D. SCOLOZZI, in preparation.

[6] F. H. CLARKE, *Optimization and non-smooth analysis,* John Wiley, New York, 1983.

[7] E. DE GIORGI, M. DEGIOVANNI, A. MARINO, M. TOSQUES, *Evolution equations for a class of nonlinear operators,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) *75* (1983), 1-8.

[8] E. DE GIORGI, T. FRANZONI, *Su un tipo di convergenza variazionale,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) *58* (1975), 842-850.

[9] E. DE GIORGI, A. MARINO, M. TOSQUES, *Problemi di evoluzione in spazi metrici e curve di massima pendenza,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) *68* (1980), 180-187.

[10] M. DEGIOVANNI, A. MARINO, M. TOSQUES, *Evolution equations with lack of convexity,* Nonlinear Anal. *9* (1985), 1401-1443.

[11] E. R. FADELL, P. H. RABINOWITZ, *Bifurcation for odd potential operators and an alternative topological index,* J. Funct. Anal. *26* (1977), 48-67.

[12] D. KINDERLEHRER, G. STAMPACCHIA, *An introduction to variational inequalities and their applications,* Pure and Applied Mathematics, *88,* Academic Press, New York - London - Toronto, Ont., 1980.

[13] M. A. KRASNOSELSKII, *Topological methods in the theory of nonlinear integral equations,* Gosudarstv. Izdat. Tehn. - Teor. Lit., Moscow, 1956. The Macmillan Co., New York, 1964.

[14] M. KUČERA, J. NEČAS, J. SOUČEK, *The eigenvalue problem for variational inequalities and a new version of the Ljusternik - Schnirelmann theory,* Nonlinear analysis (collection of papers in honor of Erich H. Rothe), pp. 125-143, Academic Press, New York, 1978.

[15] A. MARINO, *La biforcazione nel caso variazionale,* Confer. Sem. Mat. Univ. Bari n. *132* (1973).

[16] A. MARINO, *Evolution equations and multiplicity of critical points with*

respect to an obstacle, (Bologna, 1985), Res. Notes in Math., Pitman, in press.

[17] A. MARINO, D. SCOLOZZI, *Geodetiche con ostacolo,* Boll. Un. Mat. Ital. B (6) *2* (1983), 1-31.

[18] J. B. MC LEOD, R. E. L. TURNER, *Bifurcation for nondifferentiable operators with an application to elasticity,* Arch. Rational Mech. Anal. *63* (1976), 1-45.

[19] P. H. RABINOWITZ, *A bifurcation theorem for potential operators,* J. Funct. Anal. *25* (1977), 412-424.

[20] R. C. RIDDELL, *Eigenvalue problems for nonlinear elliptic variational inequalities,* Nonlinear Anal. *3* (1979), 1-33.

[21] J. T. SCHWARTZ, *Nonlinear functional analysis,* Gordon & Breach, New York, 1969.