A GENERALIZATION OF THE CONLEY-INDEX THEORY (*)

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SOMMARIO. - *Si presenta una generalizzazione della teoria dell'indice di Conley agli spazi di dimensione infinita. Si dà poi un'applicazione che generalizza un ben noto teorema di Krasnoselski.*

SUMMARY. - *We present a generalization of the Conley index theory to infinite dimensional spaces. We give an application which generalizes a well known theorem of Krasnoselski.*

Introduction

The Morse theory, in spite of the fact that has been the first to be generalized to a functional analytic contest, did not give as many results as the theory of Ljusternik and Schnirelman (at least in P.D.E.'s).

We think that there are two main reasons for this.

First the functional of variations occurring in many concrete problems are of class C^1 and not C^2 as the Morse theory requires.

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Second, the possible degeneracy of the critical points occurring in these problems causes troubles with Morse theory.

We feel that these difficulties can be avoided adapting the Conley ideas to a functional analytic contest.

For this reason we are developing a theory which, I hope, will be a new tool to treat variational problems.

In this paper we present part of this theory.

At the end of Section 3, we study a simple example obtaining a result which, as far as I know, cannot be obtained by other known theories.

1. The homotopic index

Let M be a metric space on which a semiflow η is defined i.e. a map

$$\eta : \mathbf{R}^+ \times M \to M$$

such that $\eta(0, x) = x$ and

$$\eta(t_1, \eta(t_2, x)) = \eta(t_1 + t_2, x) \quad (t_1, t_2 \in \mathbf{R}^+, x \in M).$$

When no ambiguity is possible we will write $x \cdot t$ instead of $\eta(t, x)$.

A semiflow which is defined for every $t \in \mathbf{R}$ is called a flow.

If X is any subset of M and T a positive constant we set

$$G^T(X) = G^T(X, \eta) = \{x \in M : x \cdot [0, T] \subset M\} \cap \{\bigcap_{t \geq 0} \eta(t, X)\}.$$

If η is a flow, clearly we have

$$G^T(X) = \{x \in M : x \cdot [-T, T] \subset X\} = \bigcap_{t \in [-T, T]} \eta(t, X).$$

Also we set

$$\Sigma = \Sigma(\eta) = \{X \subset M : X \text{ is closed and } \exists T > 0 \text{ s.t. } G^T(X, \eta) \subset \mathring{X}\},$$

where $\mathring{X}$ denotes the interior of X.

DEF. 1.1 - *A pair of closed subset of X, (N, N_0), with $N_0 \subset N$, is called* INDEX PAIR *if*

(i) $\overline{N - N_0} \in \Sigma$ *($\bar{A}$ denotes the closure of A)*;

(ii) N_0 *is positively invariant with respect to N (i.e. $x \in N_0$ and $x \cdot [0, t] \subset N \Rightarrow x \cdot [0, t] \subset N_0$)*;

(iii) *N_0 is an* EXIT *set for N (i. e. $x \in N$ and*

$x \cdot [0,t] \subset N \Rightarrow \exists\, t^* \in [0,t]$ *such that* $x \cdot t^* \in N_0$*).*

We say that (N, N_0) is an index pair for $X \in \Sigma$ if (iv) $\overline{N - N_0} \subset X$ *and there exists $T > 0$ such that $G^T(X) \subset \overline{N - N_0}$.*

Now it is necessary to recall some concepts from the homotopy theory.

If X is a topological space and A is a closed subset then X/A denotes the spaces obtained by X indentifying all the points of A.

Two spaces X/A and Y/B are called homotopic equivalent if there are maps $\varphi : X/A \to Y/B$ and $\psi : Y/B \to X/A$ such that $\varphi([A]) = [B]$; $\psi([B]) = [A]$ and such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic to the identity by homotopies which leaves the points $[A]$ and $[B]$ fixed respectively.

The class of all spaces homotopically equivalent to X/A is called homotopy type of X/A and denoted by $[X/A]$.

The homotopy type of X/X is denoted by $\underline{0}$; if X is a contractible space, the homotopy type of X/ϕ is denoted by $\underline{1}$.

Moreover, by convention, we set $\phi/\phi = \underline{0}$.

DEF. 1.2 - *For $X \in \Sigma$, the homotopy index of X is the homotopy type of an index pair (N, N_0) relative to X; in formula we write*

$$h(X) = h(X, \eta) = [N/N_0].$$

The Def. 1.2 makes sense if we prove that

(1.1)
(a) $\forall X \in \Sigma$ *there exists an index pair (N, N_0) for X;*
(b) *if (N, N_0) and $(\hat{N}, \hat{N}_0)$ are two index pairs relative to X, then* $[N/N_0] = [\hat{N}/\hat{N}_0]$.

In order to prove (1.1) some work is necessary. First, we need an other notation; for $T > 0$ we set

(1.2) $\Gamma^T(X) = \Gamma^T(X, \eta) = \{x \in G^T(X, \eta) : x \cdot [0,T] \cap \partial X \neq \phi\}.$

We need now a technical lemma:

LEMMA 1.3 - *Suppose that $X, Y \in \Sigma$; then*

(i) $X \subset Y \Rightarrow G^T(X) \subset G^T(Y)$ *for every* $T > 0$*;*

(ii) $T_1 > T > 0 \Rightarrow G^{T_1}(X) \subset G^T(X)$*;*

(iii) $G^{T_1 + T_2}(X) = G^{T_1}(G^{T_2}(X))$*;*

(iv) *if* $G^T(X) \subset \mathring{X}$ *then* $G^{2T}(X) \subset$ *int* $G^T(X)$*;*

(v) *if* $X \in \Sigma$ *then* $G^T(X)$ *and* $\eta(t, X)$ *belong to* Σ;

(vi) $G^T(X)$ *is closed*;

(vii) $\Gamma^T(X)$ *is closed*;

(viii) $\Gamma^T(X) \subset \partial G^T(X)$.

Proof. (i), (ii) and (iii) follow easily from the definition of $G^T(X)$.

(iv) First of all observe that

(1.3) if $x \in G^T(X)$, then $x \cdot t$ is defined for $t \in [-T, T]$ i.e. we can go back in time up to the point $-T$; and this by the definition of $G^T(X)$.

In order to prove (iv) we argue indirectly and we suppose that there exists $y \in G^{2T}(X) \cap \partial G^T(X)$. Then there exists a sequence $y_n \to y$ such that $y_n[-T, T] \not\subset X$. This implies that there exists times $t_n \in [-T, T]$ such that $y_n \cdot t_n \notin X$; we can extract a sequence t'_n such that $t'_n \to \bar{t}$, so we have that $y_n \cdot t'_n \to y \cdot \bar{t} \in \partial X$. Since $y \in G^{2T}(X)$, $y \cdot [-2T, 2T] \subset X$ and so $y \cdot \bar{t} \in G^T(X)$ (since $|\bar{t}| \leqslant T$).

And this contradicts our assumption that $G^T(X) \cap \partial X = \phi$.

(v) $G^T(X) \in \Sigma$ by (iv) and (vi), $\eta(t, X) \in \Sigma$ by the continuity of η.

(vi) We have $G^T(X) = \{\bigcap_{t \in [0,T]} \eta(t, X)\} \cap \{x \in X : x \cdot [0, T] \subset X\}$.

The first set in the above formula is closed since for every $t > 0$, $\eta(t, X)$ is closed. If we set $A_t = \eta(t, \cdot\,)$, then $\{x \in X : x \cdot [0, T] \subset X\} = \bigcap_{t \in [0,T]} A_t^{-1}(X)$.

So also this set is closed. Therefore $G^T(X)$ is closed.

(vii) Let $\{x_n\} \subset \Gamma^T(x)$ with $x_n \to \bar{x}$. Then there exists $t_n \in [0, T]$ such that $x_n \cdot t_n \in \partial X$. Let t'_n be a subsequence of t_n converging to some $\bar{t} \in [0, T]$ then $x_n \cdot t'_n \to \bar{x} \cdot \bar{t} \in \partial X$. Therefore $\bar{x} \in \Gamma^T(X)$.

(viii) Let $x \in \Gamma^T(X)$; then $\exists\, t \in [0, T]$ such that $x \cdot t \in \partial X$; thus there exists a sequence $y_n \in X^c$ (X^c denotes the complement of X in M) converging to $x \cdot t$. This implies that $y_n(-t) \to x$. But $y_n(-t) \notin G^T(x)$, therefore $x \in G^T(X)$.

Now we can prove (1.1)

THEOREM 1.4 - *(Existence of index pairs). Let* $X \in \Sigma$ *and let* T *be large enough that* $G^T(X) \subset \overset{\circ}{X}$. *Then*

$$(G^T(X), \Gamma^T(X))$$

is an index pair for X.

Proof. By Lemma 1.3 (vi), (vii), $G^T(X)$ and $\Gamma^T(X)$ are closed. We have to check points (i), (ii) and (iii) of Def. 1.2.

(i) By Lemma 1.3 (viii), $\overline{G^T(X) - \Gamma^T(X)} = G^T(X)$; so by Lemma (1.2), the conclusion follows.

(ii) Let $x \in \Gamma^T(X)$ and suppose that

(1.4) $\quad x \cdot [0, t] \subset G^T(X)$.

We want to prove that $X \cdot [0, t] \subset \Gamma^T(X)$. Suppose that this fact is not true; then there exists $\bar{t} \in [0, t]$ such that $x \cdot \bar{t} \notin \Gamma^T(X)$.

Now set

$$t^* = \inf\{\tau \in [0, t] : x \cdot \tau \notin \Gamma^T(X)\}.$$

Clearly $t^* \in [0, t)$ and

(1.5) (a) $x \cdot t^* \in \Gamma^T(X)$ since $\Gamma^T(X)$ is closed by Lemma 1.3 (vii);
(b) $x \cdot (t^* + \varepsilon_n) \notin \Gamma^T(X)$ (with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$).

If we set $y = x \cdot t^*$, by (1.5) and the definition of $\Gamma^T(X)$ we have

$$y \cdot [0, T] \cap \partial X \neq \phi,$$

$$y \cdot [\varepsilon_n, T] \cap \partial X = \phi.$$

From the above formulas we have that

(1.6) $\quad y \in \partial X$.

On the other hand, by (1.4), $y \in G^T(X)$ and by our assumptions $y \in \overset{\circ}{X}$; this fact contradicts (1.6).

(iii) It is trivial.

THEOREM 1.5 - *(Equivalence of index pairs) Let* (N, N_0) *and* $(\hat{N}, \hat{N}_0)$ *be two index pairs such that there exists* $T > 0$ *such that*

$$G^T(\overline{N - N_0}) \subset \overline{\hat{N} - \hat{N}_0}$$

and

$$G^T(\overline{\hat{N} - \hat{N}_0}) \subset \overline{N - N_0}.$$

Then $[N/N_0] = [\hat{N}/\hat{N}_0]$.

REMARK - The proof of Theorem 1.5 is essentially contained in Salamon [S]. He gave a short and elegant proof of Conley's theorem of equivalence of index pairs (in the compact contest). Salamon's proof can be adepted to our case.

Sketch of the proof of Theorem 1.5. We can suppose that $G^T(\overline{N - N_0}) \subset \text{int } \overline{\hat{N} - \hat{N}_0}$ and that $G^T(\overline{\hat{N} - \hat{N}_0}) \subset \text{int } (N - N_0)$ (if not it is enough to replace T by $2T$ and use Lemma 1.3 (iv)). Now let $f : N_1 / N_0 \to \hat{N} / \hat{N}_0$ be defined as follows

$$f([x]) = \begin{cases} [x \cdot 3T] \text{ if } x \cdot [0, 2T] \subset N_1 - N_0 \text{ or } x \cdot [T, 3T] \subset \hat{N}_1 - \hat{N}_0, \\ [\hat{N}_0] \text{ otherwise}. \end{cases}$$

The function f is continuous (for the details of the proof see [S] Lemma 4.7).

In an analogous way we can define a map

$$\hat{f} : \hat{N}/\hat{N}_0 \times [T, \infty] \to N/N_0 .$$

We have to prove that $\hat{f} \circ f$ and $f \circ \hat{f}$ are homotopic to the identity in N/N_0 and $\hat{N}/\hat{N}_0$ respectively.

For $t \in [0, T]$ define the map $h : [0, T] \times N/N_0 \to N/N_0$ as follows

$$h(t, [x]) = \begin{cases} [x \cdot 6t] \text{ if } x \cdot [0, 6t] \subset N_1 - N_0, \\ [N_0] \text{ otherwise}. \end{cases}$$

It is easy to show that h is continuous and that

$$h(T, [x]) = \hat{f} \circ f \text{ and } h(0, [x]) = \text{Id}_{N/N_0} .$$

In the same way it is possible to construct an homotopy $\hat{h} : [0, T] \times \hat{N}/\hat{N}_0 \to \hat{N}/\hat{N}_0$.

COROLLARY 1.6 - *If (N, N_0) and $(\hat{N}, \hat{N}_0)$ are two index pairs for X, then* $[N/N_0] = [\hat{N}, \hat{N}_0]$. *In particular* (1.1) (b) *holds.*

Proof. If (N, N_0) and $(\hat{N}, \hat{N}_0)$ are two index pairs for X we have

$$G^T(\overline{N - N_0}) \subset G^T(X) \subset \hat{N} - \hat{N}_0 \text{ by Def. 1.1}$$

and

$$G^T(\overline{\hat{N} - \hat{N}_0}) \subset G^T(X) \subset N - N_0 .$$

The conclusion follows from Theorem 1.5.

So at this point $h(x)$ is well defined. Another consequence of Theorem 1.5 is the following Corollary.

COROLLARY 1.7 - *Let $X, Y \in \Sigma$ and suppose that $\exists T > 0$ such that*
(1.7) $\quad G^T(X) \subset Y$ *and* $G^T(Y) \subset X$.

Then $h(X) = h(Y)$.

Proof. Let (N, N_0) and $(\hat{N}, \hat{N}_0)$ be two index pairs for X and Y respectively. Then

(1.9) $G^T(N - N_0) \subset G^T(X) \subset Y$ by Def. 1.1 and (1.7).

Since $(\hat{N}, \hat{N}_0)$ is an index pair for $Y, \exists T_1 > 0$ such that

$$G^{T_1}(Y) \subset \text{int}\,(\hat{N}_1 - \hat{N}_0)\,.$$

Therefore by the above formula, (1.9) and Lemma 1.2 (iii), we have that

$$G^{T+T_1}(\overline{N - N_0}) \subset \hat{N} - \hat{N}_0\,.$$

For the same reason there exists $T_2 > 0$ such that

$$G^{T+T_2}(\overline{\hat{N} - \hat{N}_0}) \subset N - N_0\,.$$

Thus by Theorem 1.5 (replacing T with $T + \max(T_1, T_2)$) the conclusion follows.

COROLLARY 1.8 - *For every $T > 0$, $h(G^T(X)) = h(X)$.*

Proof. Trivial.

COROLLARY 1.9 - *If there is $T > 0$ such that $G^T(X) = \phi$, then $h(X) = 0$.*

Notice that Corollary 1.9 cannot be inverted as the following example shows.

EXAMPLE 1.10 - Take

$$M = \mathbf{R}\,; \quad \eta(t, x) = x - t\,; \quad X = [0, +\infty)\,.$$

Then $h(x) = \underline{1}$ but $G^T(x) \neq \phi$ for every $T > 0$.

However there is a good test to see if the index of a set is $\underline{0}$.

THEOREM 1.11 - *Suppose that $X \in \Sigma$ and that*

(1.10) *for every $x \in X$, there is $t > 0$ such that $x \cdot t \notin X$.*

Then $h(x) = \underline{0}$.

We need some lemmas to prove Theorem 1.11.

LEMMA 1.12 - *Suppose that (N, N_0) is an index pair and that τ is a positive constant such that*

(1.11) $x \cdot [0, \tau] \subset N - N_0$.

Then there exists an open neighborhood V at x such that for every $y \in V \cap N$, $y \cdot [0, \tau] \subset N - N_0$.

Proof. We argue indirectly and suppose that the conclusion of the lemma is not true. Then there exists a sequence $x_n \to x$ $(x_n \in N - N_0)$ and a sequence $t_n \in [0, \tau]$ such that $x_n \cdot t_n \notin N - N_0$.

We set

$$\hat{t}_n = \sup\{t \in [0, t_n] \text{ such that } x \cdot [0, t] \subset N\}.$$

$\hat{t}_n$ is a bounded sequence; so we can suppose that it is convergent to some $\bar{t} \in [0, \tau]$.

By our construction, $x_n \cdot \hat{t}_n \in N_0$; so $x \cdot \bar{t} \in N_0$ since N_0 is closed. This last statement contradicts (i-ii); so the lemma is proved.

LEMMA 1.13 - *Let* $(N, N_0) = (G^T(x), \Gamma^T(x))$ *be an index pair for* X *(cf. Theorem* 1.4*) and suppose that* (1-10) *holds.*

We set

$$U = \{x \in N : \exists\, t \in [0, 2T] \text{ such that } x \cdot t \in N^c\},$$

where N^c *denotes* $M - N$.

Then U *satisfies the following properties:*

(i) U *is relatively open in* N;

(ii) *given two positive constants* $t_1 < t_2$ *such that* $x \cdot t_i \in U$ *and* $x \cdot [0, t_i] \subset N$ $(i = 1, 2)$ *then for every* $t \in [t_1, t_2]$, $x \cdot t \in U$;

(iii) $N_0 \subset U$;

(iv) $(\bar{U}, N_0)$ *is an index pair and* $[\bar{U}/N_0] = \underline{0}$.

Proof. (i) and (ii) are easy to check.

In order to prove (iii) we argue indirectly and suppose that there is $x \in N_0$ such that $x \cdot [0, 2T] \subset N$. Since N_0 is positively invariant with respect to N, $x \cdot [0, 2T] \subset N_0$. Then if we set $y = x \cdot T$, it follows that $y \in N_0$ and $y \in G^T(N)$. Since $G^T(N) \subset \overset{\circ}{N}$ by Lemma 1.3 (iii) and (iv) and $N_0 \subset \partial N$, by Lemma (1.3) (viii), we have obtained a contradiction.

Now let us prove (iv). First observe that $N_0 \subset \bar{U}$ by (iii). (i) of Def. 1.1 is satisfied since $\overline{\bar{U} - N_0} = \bar{U}$ and $G^{2T}(\bar{U}) = \phi \subset \text{int}(\bar{U})$. To check (ii), it is enough to observe that $\bar{U} \subset N$. (iii) follows directly by the definition of $\bar{U}$. So $(\bar{U}, N_0)$ is an index pair. $[\bar{U}/N_0] = h(U) = \underline{0}$ by Corollary (1.9).

Proof of Theorem 1.11 - Let N, N_0 and U as in Lemma 1.13. For every $x \in N$, we choose a $t(x) > 0$ such that $x \cdot [0, t(x)] \subset N$ and $x \cdot t(x) \in U$.

This is possible by (1.10) and Lemma 1.13 (iii). If $x \in U$ we choose $t(x) = 0$. Also if $x \notin U$, we can choose $t(x)$ such that $t(x) \notin N_0$.

Now for $x \in N - N_0$, let V_x be an open neighbourhood of N (open in the topology of N) such that

(1.12) for every $y \in V_x, y \cdot [0, t(x)] \subset N$ and $y \cdot t(x) \in U$.

This is possible by our choice of $t(x)$, Lemma 1.12, and Lemma 1.13 (i).

For $x \in N_0$, set $V_x = V$. Thus $\{V_x\}_{x \in N}$ is an open cover of N (open in the relative topology of N).

Let $\{V_i\}_{i \in I}$ be a locally finite refiniment of $\{V_x\}_{x \in N}$ which exists since N is a metric space.

Observe that, by our construction, for every $i \in I$, there exists $t_i \geqslant 0$ such that

(1.13) $\eta(t_i, v_i) \subset U$ and $\eta([0, t_i], V_i) \subset N$.

Now let $\{\beta_i(x)\}_{i \in I}$ be a partition of the unity relative to $\{V_i\}_{i \in I}$ i.e. a set of function $\beta_i : N \to \mathbf{R}$ whose support is V_i and $\sum_{i \in I} \beta_i(x) = 1$ for every $x \in N$. Such partition exists since N is a metric space.

Now set

$$\tau(x) = \sum_{i \in I} \beta_i(x)\, t_i .$$

Clearly $\tau(x)$ is a continuous function. We claim that

(1.14) $x \cdot \tau(x) \in U$.

In order to see this, fix $\bar{x} \in N$ and set

$$t_1(\bar{x}) = \min\{t_i : \bar{x} \in V_i\}; \quad t_2(\bar{x}) = \max\{t_i : \bar{x} \in V_i\}.$$

By (1.13), $\eta(t_i, \bar{x}) \in U$ $(i = 1, 2)$ and $\eta([0, t_1], \bar{x}) \subset N$.

Therefore (1.14) follows from Lemma 1.13 (ii).

Moreover observe that by our construction

(1.15) $\tau(x) = 0$ for every $x \in N$.

Now consider the map $h : [0, 1] \times N \to \bar{U}$ defined by

$$h(s, x) = \eta(s t(x), x).$$

h is an homotopy equivalence between N and $\bar{U}$, and by (1.15) it is also an homotopy equivalence between N/N_0 and $\bar{U}/N_0$.

Therefore, by Lemma 1.13 (iv)

$$h(x) = [N/N_0] = [\bar{U}/N_0] = \underline{0}.$$

REMARK 1.14 - Now, few words to compare the Conley index with our generalization.

A closed set X is called by Conley [C] an isolating neighbourhood if $I(X) \subset \overset{\circ}{X}$ where $I(X) = \{x \in X : x \cdot \mathbf{R} \subset X\}$ or, using our notation, $I(X) = \bigcap_{t \geqslant 0} G^T(X)$.

Let $\hat{\Sigma}$ be the family of isolating neighbourhoods in M; then if M is compact $\hat{\Sigma} = \Sigma$. If M is not compact, in generale, $\Sigma \subset \hat{\Sigma}$. So, in our approach, it was necessary to restrict the class of sets X for which to define index pairs (and introduce the operator $G^T(\cdot)$). Now, observe that the relationship (1.7) gives an equivalence relation on Σ (which we will denote by $\approx$).

Corollary 2.4 states that the index is constant on each equivalence class of $\approx$.

If M is compact, then $X \approx Y$ if and only if $I(x) = I(Y)$ (the easy proof of this is left to the reader).

So, when M is compact, h depends only on the maximal invariant set $I(X)$ contained in X; therefore it is an *index of isolated invariant sets*. Example 1.10 shows that this is not the case when the compactness is not assumed (in fact $h(x) = \underline{1}$ but $I(X) = \phi$).

Concluding, the Conley index is an index of isolated invariant sets; our generalization is an index of a class of closed set which has been chosen in order that the main properties of the Conley theory can be preserved.

2. The cohomologic index

Let $H^*(\cdot, \cdot)$ denote the Alexander Spanier cohomology with coefficients in some field F (cf. [Sp]).

We recall that the Alexander-Spanier cohomology satisfies the following property which is not shared by the singular cohomology theory.

THEOREM 2.1 - *Let (X, A) and (Y, B) two pairs of topological spaces. We suppose that X and Y are paracompact Hausdorff* *spaces and that A and B are closed in X and Y respectively. Moreover suppose that $X - A$ and $Y - B$ are homeomorphic. Then $H^*(X, A) \approx H^*(Y, B)$.*

Proof. See [Sp], Th. 5, pag. 318.

Now for every pairs of closed spaces (X, A) we set

$$p_t(X, A) = \sum_{q=0}^{\infty} [\dim H^q(X, A)]\, t^q .$$

$p(X, A)$ is a formal series whose coefficients are cardinal numbers, these numbers are known as Betti numbers.

If X is a compact manifold with boundary A, then $p(X, A)$ reduces to a polynomial called Poincaré or Betti polynomial. $p(X, A)$ is a topological invariant which carries part of the information contained in the cohomology algebras $H^*(X, A)$.

When $A = \phi$ we shall write $p(X)$ instead of $p(X, A)$. We shall denote by S the set of formal series with cardinal coefficients.

The following properties of $p(X, A)$ will be used to study the cohomological index.

LEMMA 2.2 - *Let (X, A) and (Y, B) be couples of closed subspaces of a metric space.*

Then

(i) $p(X, A) = p(X/A, [A])$;

(ii) *if* $X \cap Y = \phi$ *then*
$p(X \cup Y, A \cup B) = p(X, A) + p(Y, B)$;

(iii) $p((X, A) \times (Y, B)) = p(X, A) \cdot p(Y, B)$
where $(X, A) \times (Y, B) = (X \times Y, X \times B \cup Y \times A)$;

(iv) *if* $B \subset A$ *then there exists* $Q(t) \in S$ *s.t.*
$p_t(X, A) + p_t(A, B) = p_t(X, B) + (1 + t)\, Q(t)$.

Proof. (i) Let $\Pi : X \to X/[A]$ be the projection map. Then Π_{X-A} is a homeomorphism between $X - A$ and $X/[A] - [A]$. Thus the conclusion follows from Theorem 2.1.

(ii) trivial.

(iii) Since (X, A) and (Y, B) are closed pairs, there is an exact Mayer-Vietoris sequence for the $\bar{H}^*$ cohomology (cf. [Sp] pag. 291).

But every closed pair of Hausdorff-paracompact spaces is a tout pair for the Alexander-Spanier cohomology (cf. [Sp] pag. 315). Therefore $\bar{H}^* = H^*$ on such pairs. Therefore the Künneth formula can be applied to such pairs (cf. [Sp] pag. 249) and we get

$$H^*((X,A)\times(Y,B)) = H^*(X,A)\otimes H^*(Y,B).$$

From the above formula the conclusion follows.
(iv) Let us consider the exact sequence relative to the triple $B\subset A\subset X$:

$$(2.1)\qquad \cdots \xrightarrow{\delta_{q-1}} H^q(X,A) \xrightarrow{i_q} H^q(X,B) \xrightarrow{j_q} H^q(A,B) \xrightarrow{\delta_q} \cdots$$

and set $a_q = \dim(\ker\ i_q^*)$,

$b_q = \dim(\ker\ j_q^*)$,

$c_q = \dim(\ker\ \delta_q)$.

By the exactness of (2.1) we get

$\dim H^q(X,A) = c_{q-1} + a_q$ (with the convenction that $c_{-1}=0$),

$\dim H^q(X,B) = a_q + b_q$,

$\dim H^q(A,B) = b_q + c_q$.

Then we have

$$p(X,A) = \sum_{q=0}^{\infty} (c_{q-1}+a_q)\, t^q,$$

$$p(X,B) = \sum_{q=0}^{\infty} (a_q+b_q)\, t^q,$$

$$p(A,B) = \sum_{q=0}^{\infty} (b_q+c_q)\, t^q.$$

Then

$$p(X,A)+p(A,B) = p(X,B) + \sum_{q=0}^{\infty}(c_{q-1}+c_q)\, t^q =$$
$$= p(X,B) + (1+t)\sum_{q=0}^{\infty} c_q t^q.$$

The conclusion follows setting $Q(t) = \sum_{q=0} c_q t^q$.

Notice that the formula (iv) holds even if some of the coefficients are infinite cardinal numbers.

We can now define the cohomologic index:

DEF. 2.3 - *The cohomologic index is a map*

$$i:\Sigma(\eta)\to S$$

defined by

$$i_t(X,\eta) = p_t(N,N_0)$$

where (N,N_0) *is an index pair for* X.

When no ambiguity is possible we shall write $i(X)$ instead of $i_t(X, \eta)$.

REMARK 2.4 - By Lemma 2.2 (i), $p(N, N_0) = p(N/N_0, [N_0])$; so the cohomologic index depends only on $h(x)$; thus it is well defined by (1.1) (a) and (b).

The above remark implies that the cohomologic index carries less information than the homotopic index. Neverthless is more usefull since it is much easier to deal with. The following theorem illustrates the first properties of the cohomologic index.

THEOREM 2.5 - *The cohomologic index satisfies the following properties:*

(i) *if $X \in \Sigma$ and for every $x \in X$, there is $t > 0$ such that $x \cdot t \notin X$, then $i(X) = 0$;*

(ii) *if $X \in \Sigma$ is contractible and positively invariant, then $i(X) = 1$;*

(iii) *if $X, Y \in \Sigma$ and $X \cap Y = \phi$ then $i(X \cup Y) = i(X) + i(Y)$;*

(iv) *if η_i is a semiflow on M_i $(i = 1, 2)$, then a semiflow $\eta_1 \times \eta_2$ is defined on $M_1 \times M_2$ as follows*

$$(\eta_1 \times \eta_2)\,(t, (x_1, x_2)) = (\eta_1(t, x_2), \eta_2(t, x_2));$$

then if $X_i \in \Sigma(\eta_i)$ $(i = 1, 2)$, we have that

$$X_1 \times X_2 \in \Sigma(M_1 \times M_2, \eta_1 \times \eta_2)$$

and

$$i(X_1 \times X_2, \eta_1 \times \eta_2) = i(X_1, \eta_1) \cdot i(X_2, \eta_2).$$

Proof. (i) follows from Theorem 1.11; (ii) follows by the fact that $H^q(X) = 1$ if and only if $q = 0$. (iii) and (iv) follow by Lemma 2.3 (ii) and (iii) respectively.

Next we are going to prove a property of the index wich is a generalization of the classical Morse inequalities.

DEF. 2.6 - *Take $X_1, X_2 \in \Sigma$ with $\overset{\circ}{X}_1 \cap \overset{\circ}{X}_2 = \phi$. We say that X_2 is over X_1 if there exists $T > 0$ such that $X_1 \cap G^T(X_1 \cup X_2)$ is positively invariant with respect to $G^T(X_1 \cup X_2)$.*

If X_2 is over X_1 or X_1 is over X_2, then we say that X_1 and X_2 are η-connected. Otherwise we say that they are η-disconnected.

EXAMPLES 2.7 - I: If $X_1 \cap X_2 = \phi$, then X_1 and X_2 are η-disconnected.

II: Let f be a Liapunov function for (M, η) and let c be a constant which is a regular value for f (i.e. $f(x) = c \Rightarrow f'(x) \neq 0$). We set $X_1 = \{x \in M : f(x) \leqslant c\}$; $X_2 = \{x \in M : f(x) \geqslant c\}$. Then $X_1, X_2 \in \Sigma$ and X_2 is over X_1.

DEF. 2.8 - Let $X \in \Sigma$. *A family of sets* $\{X_k\}_{k \in N}$ *is called a* MORSE DECOMPOSITION *of* X, *if*

(i) $X = \bigcup_{k=1}^{N} X_k$;

(ii) $X_k \in \Sigma$ *for* $k = 1, \dots, N$;

(iii) $\overset{\circ}{X}_k \cap \overset{\circ}{X}_h = \phi$ *for* $k \neq h$;

(iv) X_{h+1} *is over* $\bigcup_{k=1}^{h} X_k$ *for* $h = 1, \dots, N-1$.

EXAMPLE 2.9 - Let f be a Liapunov function for (M, η) and let $c_1 < c_2 < \dots < c_{N-1}$ be a sequence of regular values for f.

Let $c_0 = -\infty$ and $c_N = +\infty$ and

$$x_k = \{x \in X : c_{k-1} \leqslant f(x) \leqslant c_k\}.$$

Then X_k is a Morse decomposition of X.

The next theorem states one of the most important property of the cohomological index (as far as the applications are concerned).

THEOREM 2.10 - *If* X_k *id a Morse decomposition of* X, *then there exists* $Q \in S$ *such that*

$$\sum_{k=1}^{N} i(X_k) = i(X) + (1+t)\, Q(t)\,.$$

In order to prove Theorem 2.10 some lemmas are necessary.

LEMMA 2.11 - *Let* $X = X_1 \cup X_2$ *and suppose that* X_2 *is over* X_1. *Then there exist closed spaces* $N_0 \subset N_1 \subset N_2$ *such that* (N_2, N_0), (N_2, N_1), (N_1, N_0) *are index pairs for* X, X_2 *and* X_1 *respectively.*

Proof. Take T big enough in order that

$$(2.4) \quad \begin{cases} \text{(a) } X_1 \cap G^T(X) \text{ is positively invariant with respect to } G^T(X); \\ \text{(b) } (G^T(X);\ \Gamma^T(X)) \text{ is an index pair for } X; \\ \text{(c) } G^T(X_1) \subset \overset{\circ}{X}_1. \end{cases}$$

We set

$$N_0 = \Gamma^T(X),$$
$$N_1 = (X_1 \cap G^T(X)) \cup \Gamma^T(X),$$
$$N_2 = G^T(X).$$

We want to prove that N_0, N_1, N_2 satisfy the required properties. We now prove that (N_1, N_0) is an index pair for X.

Let is check (i) of Def. 1.1. Since $\overline{N - N_0} = X_1 \cap G^T(X)$

(2.5) $\quad G^T(\overline{N - N_0}) = G^T(X_1 \cap G^T(X)) \subset G^T(X_1) \subset \mathring{X}_1$ by (2.4) (c).

Also by Lemma 1.3 (i), (iii) and (v)

(2.6) $\quad G^T(\overline{N - N_0}) \subset G^T(G^T(X)) = G^{2T}(X) \subset \text{int } G^T(X).$

Then by (2.5) and (2.6).

$$G^T(\overline{N - N_0}) \subset \text{int } (N - N_0).$$

(ii) of Def. 1.1 holds since $(X_1 \cap G^T(X))$ is positively invariant in $G^T(X)$ by definition and $\Gamma^T(X)$ is positively invariant in $G^T(X)$ by Theorem 1.4.

Now let us check (iii) of Def. 1.1. If $x \in N_1$ and it leaves N_1 at some times, it has to leave $G^T(X)$ also, since N_1 is positively invariant in $G^T(X)$. Thus there exists t^* such that $x \cdot t^* \in \Gamma^T(X)$ since $\Gamma^T(X)$ is an exit set for $G^T(X)$.

Finally, since $G^T(X_1) \subset N_1 - N_0$, (iv) of Def. 1.1 holds. Let us check that (N_2, N_1) is an index pair for X_2.

$$\overline{N_2 - N_1} = \overline{G^T(X) - X_1} = G^T(X) \cap X_2,$$

then arguing as we have done for $G^T(X) \cap X_1$, it follows that $\overline{N_2 - N_1} \in \Sigma$.

(ii) of Def. 1.1 holds since N_1 is positively invariant in N_2,

(iii) holds since $N_1 \supset \Gamma^T(X)$ and $\Gamma^T(X)$ is an exit set for N_2,

(iv) follows by the fact that

$$G^T(X_2) \subset \overline{N_2 - N_1}.$$

COROLLARY 2.12 - *If $X = X_2 \cup X_1$ and X_2 is over X_1, then there exists $Q \in S$ such that*

$$i(X_1) + i(X_2) = i(X) + (1 + t)\, Q(t)\,.$$

Proof. By Lemma 2.2 (iv) applied to the triple N_0, N_1, N_2 defined in Lemma 2.10 we have

$$p(N_2, N_1) + p(N_1, N_0) = p(N_2, N_0) + (1 + t)\, Q(t).$$

The conclusion follows by Lemma 2.10 and the definition of the cohomological index.

REMARK 2.13 - It is easy to check that if X_1 and X_2 are η-disconnected, then, for T large enough

$$G^T(X_1 \cup X_2) = G^T(X_1) \cup G^T(X_2) \text{ and } G^T(X_1) \cap G^T(X_2) = \phi.$$

Then

$$\begin{aligned} i(X) &= i(G^T(X_1 \cup X_2)) \text{ by Corollary 1.8} \\ &= i(G^T(X_1)) + i(G^T(X_2) \text{ by Theorem 2.5 (iii)} \\ &= i(X_1) + i(X_2). \end{aligned}$$

Comparing this result with Corollary 2.12 we deduce that $Q(t) \neq 0$ implies that X_1 and X_2 are η-connected.

Proof of Th. 2.10. We argue by induction. For $N = 2$ it is true since it is nothing else but Corollary 2.12.

We can suppose that it is true for $N - 1$; so there exists $Q_1 \in S$ such that

$$\sum_{k=1}^{N-1} i(X_k) = i(\bigcup_{k=1}^{N-1} X_k) + (1 + t)\, Q_1(t).$$

Now, since X_N is over $\bigcup_{k=1}^{N-1} X_k$, applying Corollary 2.12 an other time, we get

$$i(X_N) + i(\bigcup_{k=1}^{N-1} X_k) = i(X) + (1 + t)\, Q_2(t) \text{ with } Q_2(t) \in S.$$

Then the conclusion follows with $Q(t) = Q_1(t) + Q_2(t)$.

3. The generalized index and compactness

In this section we analyse some situation usefull in the applications:

DEF. 3.1 - *An index theory is a triple* $\{M, \eta, \Gamma\}$ *where* (M, η) *is a semiflow and* Γ *is a subset of* Σ *such that* $Y \subset X \in \Gamma$ *and* $Y \in \Sigma$ *implies* $Y \in \Gamma$.

Axiom A. We say that $\{M, \eta, \Gamma\}$ satisfies Axiom A if for every $X, Y \in \Gamma$ we have

$$I(X) = I(Y) \Rightarrow h(X) = h(Y)$$

where $I(Z) = \bigcap_{t \geq 0} G^T(Z) = \{x \in Z : x \cdot \mathbf{R} \subset Z\}$.

Axiom B. We say that $\{M, \eta, \Gamma\}$ satisfies Axion B if

$$X \in \Gamma \text{ and } I(X) = \phi \Rightarrow h(X) = \underline{0}.$$

Observe that Axiom A implies Axiom B. If Axiom A holds then we have an *index theory for isolated invariant sets* as the original Conley theory (cf. Remark 1.14).

However Axiom A is too strong to be usefull in certain class of variational problems. Axiom B instead has the right «amount of compactness» to be applied to the class of problems we have in mind.

These consideration will be discussed and made precise in this section.

DEF. 3.2 - *If S is an isolated invariant set we define*

$$h(S) = \lim_{U \in \hat{\Gamma}} i(U) \quad \textit{where } \hat{\Gamma} = \{X \in \Gamma : S \subset X\}$$

if the limit exists (i.e. if $h(U)$ is constant for U «small»).

It is not difficult to check that this is the case if Axiom A is satisfied. In the same way we can define $i(S)$.

REMARK 3.3 - Let $\{X_1, X_2\}$ be a Morse decomposition of X, and suppose that

$S_i = I(X_i)$; $i(S_i)$ is defined and that $i(S_i) = i(X_i)$ $(i = 1, 2)$. Thus by Corollary 2.11 we have

$$i(S_1) + i(S_2) = i(X) + (1 + t)\, Q(t).$$

Suppose now that $Q(t) \neq 0$.

Then, if Axiom A is satisfied, we have that

$$S = I(X) \neq S_1 \cup S_2$$

and more precisely $S_1 \cup S_2 \subset S$. If we set $C = S - (S_1 \cup S_2)$ we can state that if X_1 and X_2 are η-connected, then S_1 and S_2 are «connected» by an invariant set C.

This is not the case if only Axiom B is satisfied. In fact in this case it might happen that X_1 and X_2 are η-connected but $C = \phi$. Intuitively, we can say that the connection C «lies at infinity».

REMARK 3.3' - By Def. 3.2 and Lemma 2.2 (ii) if $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \phi$ we have that $i(S_1 \cup S_2) = i(S_1) + i(S_2)$. Now let us investigate some criteria to know if Axiom A or Axiom B are satisfied.

LEMMA 3.4 - *Let $\{M, \eta, \Gamma\}$ be an index theory and suppose that for every $X \in \Gamma$ and any neighbourhood U of $I(X)$ there exists $T > 0$ such that*

$$G^T(X) \in U.$$

Then $\{M, \eta, \Gamma\}$ satisfies Axiom A.

Proof. Let $X, Y \in \Gamma$ and suppose that $S = I(X) = I(Y)$. Then, X and Y are neighbourhood of S, and so, also $X \cap Y$ is a neighbourhood of S.

Then by our assumption there is a $T > 0$ such that

$$G^T(X) \subset X \cap Y \subset Y \text{ and } G^T(Y) \subset X \cap Y \subset X .$$

Therefore by Corollary 1.7, $h(X) = h(Y)$.

COROLLARY 3.5 - *Suppose that M is locally compact and that Γ is the family of compact subsets of M. Then $\{M, \eta, \Gamma\}$ satisfies Axiom A.*

Proof. Take $X \in \Gamma$ and let $S = I(X) = \bigcap_{T \geqslant 0} G^T(X)$.

Since S is compact then $\{G^T(X)\}_{T \geqslant 0}$ is a fondamental family of neighbourhoods of S. Therefore the assumptions of Lemma 3.4 is satisfied. The conclusion follows from Lemma 3.4.

COROLLARY 3.5' - *Suppose that $\{M, \eta, \Gamma\}$ satisfies the following assumption:*

given a sequence $x_n \in M$ and a sequence $t_n \to +\infty$ such that

(3.1) *$x_n \cdot [0, t_n] \subset X$ for some $X \in \Gamma$, then the sequence $x_n \cdot t_n$ is precompact.*

Then $\{M, \eta, \Gamma\}$ satisfies Axiom A.

Proof. Take $x \in \Gamma$ and $S = I(X)$. We want to verify the assumption of Lemma 3.3. We argue indirectly and suppose that there exists a neighbourhood U of S such that for every $T > 0$

$$G^T(X) \not\subset U .$$

Then there exists a sequence $t_n \to +\infty$ and $x_n \in X$ such that

$$x_n \in G^{t_n}(X) - U .$$

If we set $y_n = x_n(-t_n)$, then $y_n \cdot [0, t_n] \subset X$; thus $\{y_n, t_n\}$ satisfies the assumption (3.1). Therefore $y_n \cdot t_n$ is convergent (may be considering a subsequence). But $y_n \cdot t_n = x_n$. So $x_n \to \bar{x}$. By its construction, $\bar{x} \cdot \mathbf{R} \subset X$, therefore $\bar{x} \in S$. This is a contradiction since we have assumed that $x_n \not\subset U$.

REMARK 3.6 - The assumption (3.1) has been introduced by Rybakowsky in [R], where he constructed a Conley-index theory for flows satisfying (3.1). We refer the reader to the paper of Rybakowsky to see the application of this theory.

Now we will tourn our attention to Axiom B.

LEMMA 3.7 - *Let $\{M, \eta, \Gamma\}$ be an index theory and suppose that*

(3.2) *$x \cdot [0, \infty] \subset X$ for some $X \in \Sigma$, then there exists $t_n \to +\infty$ such that $x \cdot t_n$ is convergent.*

Then $\{M, \eta, \Gamma\}$ satisfies Axiom B (i).

Proof. We have to prove that $h(X) \neq 0 \Rightarrow I(X) \neq \phi$.

If $h(X) \neq 0$, by Theorem 1.11, there exists $x \in X$ such that $x \cdot [0, +\infty] \subset X$. Then, by (3.2) $y_n = x_n \cdot t_n$ is a convergent sequence to some $\bar{y}$.

Since $y_n \in G^{t_n}(X)$, then, as it can be checked easily, $\bar{y} \cdot \mathbf{R} \subset X$. Then $I(X) \neq \phi$.

We now discuss a tipical situation in which Axiom B is satisfied.

Let M a Hilbert (or Banach) manifold and let $f \in C^1(M, \mathbf{R})$ which satisfies the compactness assumption of Palais and Smale i.e.

(3.3) is x_n is a sequence such that $f(x_n)$ is bounded and $f'(x_n) \to 0$, then x_n is precompact.

If (3.3) is satisfied it is possible to construct a vector field F on M such that

(i) the equation $\begin{cases} \dot{x} = -F(x) & \text{has a unique solution for every } t \in \mathbf{R}, \\ x(0) = x_0; \end{cases}$

(3.4) (ii) $\langle F(x), f'(x) \rangle \geqslant \alpha(\|f'\|)$ where α is a strictly monotone function with $\alpha(0) = 0$;

(iii) $\|F\|$ is bounded.

Now we consider the index theory $\{M, \eta, \Gamma\}$ where is the flow relative to the vector field $F(x)$ (i.e. $\eta(t, x_0)$ is the solution of equation 1) and

$$\Gamma = \{X \in \Sigma(\eta) : f|_X \text{ is bounded below}\}.$$

THEOREM 3.8 - *$\{M, \eta, \Gamma\}$ as defined above satisfies Axiom B.*

Proof. We have to check (3.2). Suppose that $x \cdot [0, \infty) \subset X$. Then since $f|X$ is bounded below, and f is decreasing along trajectories:

(3.5) $f(x \cdot t)$ is bounded for $t \in [0, \infty)$.

$$\frac{d}{dt} f(x \cdot t) = -\langle F(x \cdot t), f'(x \cdot t) \rangle \leqslant -\alpha(\|f'(x \cdot t)\|).$$

By the above formula and (3.4), it follows that there is a sequence $t^n \to +\infty$ such that

$$\alpha(\|f'(x \cdot t_n)\|) \to 0$$

and by our assumption on α we have that

$$\|f'(x \cdot t_n)\| \to 0 .$$

Moreover, by (3.5), $f(x \cdot t_n)$ is bounded. So by the above formula and (3.3) $x \cdot t_n$ has a converging subsequence and (3.2) is satisfied.

Another interesting property of flows of this type is the following one

LEMMA 3.9 - *Let K_c be a critical set at level c, i.e.*

(i) $\forall x \in K_c, f(x) = c, f'(x) = 0$,
and let U be a closed neighbourhood of K_c such that

(ii) $\forall x \in U - K_c, f'(x) \neq 0$,

(iii) $f|_U$ *is bounded.*
Then if U is an other neighbourhood of K_c which satisfies (ii) *and* (iii)
$h(U) = h(V)$.

Proof. By (3.3) K_c is compact. Thus there exists $\varepsilon > 0$ such that

(3.6) (a) $N_\varepsilon(K_c) \subset U \cap V$

where $N_\varepsilon(K_c) = \{x \in M : \text{dist}\,(x, K_c) \leqslant \varepsilon\}$.

Now we want to show that there is $T > 0$ such that

(3.6) (b) $G^T(U) \subset N_\varepsilon(K_c)$.

If Z is any neighbourhood of K_c, then there is $T > 0$, depending on Z, such that

$$(3.7) \quad \begin{cases} \text{if } x \in G^T(U) - Z, \text{ there exists } t_1 < 0 < t_2 \text{ such that} \\ x \cdot t_i \in Z \quad i = 1, 2 . \end{cases}$$

(3.7) is a consequence of (3.4) (ii) not too difficult to verify.

Now take $Z = N_{\varepsilon/2}(K_c)$; then by (3.7), for every $x \in G^T(U) - N_\varepsilon(K_c)$ there are times $t_1(x) < 0 < t_2(x)$ such that $x \cdot t_i(x) \in N_{\varepsilon/2}(K_c)$ $(i = 1, 2)$. Also we can suppose that

$$x \cdot [t_1(x), t_2(x)] \subset \overline{U - N_{\varepsilon/2}(K_c)}.$$

Now set $M = \max\{\|F(x)\| : x \in U\}$.

We have that

(3.8) $t_1(x) \leqslant \dfrac{-\varepsilon}{2M}$ and $t_2(x) \geqslant \dfrac{\varepsilon}{2M}$ independently of x (this is the time necessary to go from $N_{\varepsilon/2}(K_c)$ to $U - N_\varepsilon(K_c))$.

Now let

$$b = \min\{-Df(x)\} : x \in \overline{U - N_{\varepsilon/2}(K_c)}$$

where $Df(x) = \frac{d}{dt} f(x \cdot t)|_{t=0}$.

Thus by (3.8) it follows that

$$(3.9) \qquad f(x \cdot t_1(x)) - f(x \cdot t_2(x)) = \int_{t_2(x)}^{t_1(x)} \frac{d}{dt} f(tx)\, dt \geqslant \frac{\varepsilon b}{M}.$$

Now set

$$W = \{x \in B_{\varepsilon/2} : c - \frac{b\varepsilon}{3M} \leqslant f(x) \leqslant c + \frac{b\varepsilon}{3M}\}.$$

Then, according to (3.7), if $x \in G^T(U) - N_\varepsilon(K_c)$, it needs to visit W two times $t_1 \leqslant \bar{t}_1(x) < 0 < \bar{t}_1(x) \leqslant t_2$.

But this is not possible by (3.9). Thus

$$G^T(U) - N_\varepsilon(K_c) = \phi$$

for T big enough. This proves (3.6).

By Corollary 1.7, (3.6) (a) and (3.6) (b) it follows that

$$h(U) = h(N_\varepsilon(K_c)).$$

In the same way we can prove that

$$h(V) = h(N_\varepsilon(K_c))$$

from which the conclusion follows.

COROLLARY 3.10 - *If K_c is an isolated critical set of f then $h(K_c)$ and $i(K_c)$ are well defined.*

Proof. Compare Lemma 3.9 and Def. 3.2.

COROLLARY 3.11 - *Let $\{M, \eta, \Gamma\}$ be an index theory as in Theorem 3.8. Let $X \in \Gamma$ and suppose that X contains a finite number of critical sets $K_{c_1}, \dots, K_{c_N}$ as in Lemma 3.9. Then*

$$\sum_{k=1}^{N} i(K_{c_k}) = i(X) + (i + t)\, Q(t)\, , Q(t) \in S\, .$$

Proof. Suppose that $c_1 < c_2 < \dots < c_N$ and let

$$b_k \in (c_k, c_{k+1}) \text{ for } k = 1, \dots, N-1\, ;$$

$$b_N = c_N + 1 \, ; \; b_0 = -\infty \, ; \; b_{N+1} = +\infty \, ;$$

$$X_k = \{x \in X : b_{k-1} \leqslant f(x) \leqslant b_k\} \;\; k = 1, \ldots, N+1 \, .$$

Then by Theorem (2.9) and Example (2.9)

$$\sum_{k=1}^{N+1} i(X_k) = i(X) + (1+t)\, Q(t) \, , Q(t) \in S \, .$$

By Theorem 2.5 (i), $i(X_{N+1}) = \underline{0}$. By Lemma 3.9 and Corollary 3.10 $i(X_k) = i(K_{c_k})$.

Thus the conclusion holds in the case that the c_i's are different from each other. If this is not the case, just use Remark 3.3′.

Now, in order to show how to use the generalised Conley index we consider an example which seems that cannot be treated with other known theories.

Let S be the unit sphere in an Hilbert space H and let $f \in C^1(H, \mathbf{R})$. We suppose that

$$(3.10) \quad \begin{cases} \text{(a)} \; f' \text{ is a compact operator,} \\ \text{(b)} \; \langle f'(x), x \rangle > 0 \text{ for all } x \in S \, . \end{cases}$$

We want to estimate the critical points of f on S, i.e. the solutions of the nonlinear eigenvalue problem

$$(3.11) \quad \begin{cases} x \in S \, , \\ f'(x) = \lambda x \, . \end{cases}$$

By a well known theorem of Krasnoselsky $f|_S$ satisfies (3.3) (the Palais-Smale assumption).

When f is even, Krasnoselsky proves that (3.11) has infinitely many solution.

The proof is based on the fact that $S/\sim$ ($\sim$ denotes the equivalence relation induced by the antipodal map) is the real projective space and its Liyustenik-Schnirelmann category is known; in fact it is $+\infty$.

We suppose that f is invariant with respect to the action of a finite group G of unitary representation (i.e. $f(gx) = f(x) \; \forall g \in G$).

For example if f is even, it is invariant for the actiom of the group $\mathbf{Z}_2 = \{Id, -Id\}$.

If $G \neq \mathbf{Z}_2$ the L-S category of S/G is not known (except than in some particular case when the group acts freely).

However, using the generalized Conley index we can get the following result:

THEOREM 3.12 - *Let O_x be the orbit of the group relative to x, i.e.*

$$O_x = \{gx : g \in G\}.$$

We suppose that O_x is multiple of some number p (e.g. if the action of the group is free, p is the order of the group). Then the problem (3.10) *has infinitely many solutions.*

The proof of Theorem 3.12 is based on a lemma whose complete prove is not given here.

LEMMA 3.13 - *If x is an isolated critical point of f, then $i(x)$ is a polynomial (i.e. a finite series).*

Idea of the proof. Since f' is compact it can be «well» approximated by finite dimensional operators. Since it is possible to prove that the index is stable under small perturbations, it turns out that $i(x)$ is the index of a finite dimensional problem, say of dimension n. Therefore $i(x)$ is a polynomial of degree at most n.

Proof of Theorem 3.12. First of all remember that S is contractible. Then by Theorem 2.5 (ii), $i(S) = 1$. Now we argue indirectly and suppose that f has only a finite number of critical points $x_1, \ldots, x_N$.

Then by Corollary 3.11 we have

$$\sum_{k=1}^{N} i(x_k) = 1 + (1 + t)\, Q(t)\,, Q(t) \in S\,. \tag{3.12}$$

If x_h and x_k lie on the same orbit of G (i.e. $x_h = gx_k$), then

$$i(x_k) = i(x_h)\ ;$$

in fact if $U \in \Gamma$ is a neighbourhood of x_k such that $i(U) = i(x_k)$ then gU (for some $g \in G$) is a neighbourhood of x_h and $i(U) = i(gU)$ since g is a homeomorphism.

Thus the critical points of f can be arranged in N/p sets of p elements each such that two critical points belonging to the same set have the same index.

So we have that

$$\sum_{k=1}^{N} i(x_k) = p \sum_{l=1}^{N/p} i(x_l)$$

where the x_l's are chosen each for every set.

Then by (3.12) and (3.13) we get

$$p \sum_{l=1}^{N/p} i(x_l) = 1 + (1 + t)\,, Q(t) \in S\,.$$

Now choosing $t = p - 1$, by the above formula we get

$$p \cdot (\text{integer number}) = 1 + p \cdot (\text{integer number}).$$

This is a contradiction. So the theorem is proved.

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