A REMARK ON ALEXANDER DUALITY AND THOM CLASSES (*)

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Sommario. - *Sia M una n-varietà differenziabile orientata e compatta, $A \subset M$ un chiuso, $U = M \backslash A$. Ad ogni $(n-k)$-varietà a bordo $(S, \partial S) \subset (M, U)$ si associa, per dualità di Alexander, una «k-forma» $\tau^{(S)} \in \bar{H}^k(A)$. Il teorema di isomorfismo di Thom permette poi di fornire una costruzione esplicita di $\tau^{(S)}$. Si discutono infine alcuni esempi concreti.*

Summary. - *Let M be an n-dimensional compact oriented differentiable manifold, $A \subset M$ a closed subset, $U = M \backslash A$. We associate to each $(n-k)$-submanifold with boundary $(S, \partial S) \subset (M, U)$ a «k-form» $\tau^{(S)} \in \bar{H}^k(A)$, via Alexander duality. Thom isomorphism theorem enables us to provide an explicit construction of $\tau^{(S)}$. Finally we discuss some concrete examples.*

1. Let M be an n-dimensional compact oriented differentiable manifold and $S \overset{i}{\hookrightarrow} M$ a closed $(n-k)$-dimensional submanifold of it. The well known Poincaré duality theorem associates to S a k-form η_S on M, its Poincaré dual, which satisfies

$$\int_S \omega = \int_M \omega \wedge \eta_S \text{ for any } \omega \in H^k_{DR}(M),$$

i.e. η_S represents the functional on $H^{n-k}_{DR}(M)$ defined by

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$$\omega \to \int_S \omega .$$

The explicit construction of η_S is usually achieved via Thom isomorphism theorem.

In this paper we extend this construction to the Alexander duality theorem which can be thought of as the analogue of Poincaré duality for relative (de Rham) cohomology groups $H^q_{DR}(X, Y)$.

More specifically, let A be a closed subset of M, $U = M \backslash A$, and $(S, \partial S) \subset (M, U)$ be an $(n-k)$-dimensional manifold with boundary. Under reasonable hypotheses on A we show that Alexander duality theorem associates to $(S, \partial S)$ a «k-form» $\tau^{(S)} \in \bar{H}^k(A)$ (see section 2 for notation) representing the functional F_S on $H^{n-k}_{DR}(M, U)$ defined by

$$F_S : (\omega, \theta) \to \int_S \omega - \int_{\partial S} \theta$$

(proposition 2.2); we will call $\tau^{(S)}$ the Alexander dual of S.

The use of Thom isomorphism theorem enables us to show that $\tau^{(S)}$, the dual Alexander form of S, can be identified with a limit of the Thom classes of the normal bundles of $(S \backslash \partial S) \cap V$, for V in an inductive family of open neighborhoods of A (theorem 3.1). Finally we apply theorem 3.1 to some simple examples ($M = S^2$ or $S^1 \times S^1$), for which we actually produce the form $\tau^{(S)}$.

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2. Let X be an n-dimensional oriented differentiable manifold and denote by $(\Omega^*(X), d)$ ($(\Omega^*_c(X), d)$ resp.) the graded complex of C^∞-differential forms with real coefficients (and compact supports resp.) and by $H^k_{DR}(X)$ ($H^k_{c,DR}(X)$ resp.) its de Rham cohomology vector spaces (with compact support resp.). Poincaré duality theorem states that

$$(H^k_{c,DR}(X))^* \cong H^{n-k}_{DR}(X) \quad k = 0, \ldots, n,$$

where $(-)^*$ denotes the dual vector space and where the pairing

$$H^k_{c,DR}(X) \times H^{n-k}_{DR}(X) \to \mathbf{R}$$

is induced by

$$(\omega, \theta) \to \int_X \omega \wedge \theta .$$

Provided X is of finite type, its cohomology vector spaces are finite dimensional, so one also has isomorphism

$$(2.1) \qquad H^k_{c,DR}(X) \cong (H^{n-k}_{DR}(X))^* \quad k = 0, \ldots, n.$$

REMARK 2.1 - For M an n-dimensional compact oriented differentiable manifold, isomorphism 2.1 can be used to associate to each $(n-k)$-dimensional closed oriented submanifold

$$i : S \hookrightarrow M \tag{2.2}$$

a cohomology class $\{\eta_S\} \in H^k_{DR}(M)$ which is called *Poincaré dual* of S, [1]: indeed integration over S defines a linear functional

$$\int_S - : H^{n-k}_{DR}(M) \to \mathbf{R}$$

which, in view of (2.1), is represented by a unique cohomology class $\{\eta_S\}$ in $H^{n-k}_{DR}(M)$ which satisfies

$$\int_S i^* \omega = \int_M \omega \wedge \eta_S \text{ for any } \omega \text{ in } H^k_{DR}(M),$$

where $i^* : H^k_{DR}(M) \to H^k_{DR}(S)$ is the restriction map induced by (2.2) (in the sequel we will often write ω for $i^*\omega$).

It is our aim, in this section, to show how this construction extends to Alexander duality; to this purpose we need to interpretate it in terms of differential forms.

Let us state the usual Alexander duality theorem for singular cohomology with real coefficients, [2]:

PROPOSITION 2.1 - *Let M be a compact oriented n-manifold, A a closed subset of M, $U = M \backslash A$ its complement. Then the Čech-Alexander-Spanier cohomology group $\bar{H}^k(A)$ is dual to the relative cohomology group $H^{n-k}(M, U)$.*

In order to interpretate proposition 2.1 in terms of differential forms we will always assume A to admit an inductive family of open neighborhoods $\{V\}$, cofinal with the family of all open neighborhoods of A, which are manifolds with smooth boundaries ∂V. Under this hypothesis we can define

$$\bar{H}^k_{DR}(A) = \underset{V \supset A \text{ open}}{\text{ind lim}} \; H^k_{DR}(V).$$

On the other hand the relative cohomology too can be described in terms of differential forms as follows [1]: define

$$H^{n-k}_{DR}(M, U) = H^{n-k}(\Omega^*(M, U))$$

where the complex $\Omega^*(M, U)$ is defined by

$$\Omega^q(M, U) = \Omega^q(M) \oplus \Omega^{q-1}(U)$$

$$d(\omega, \theta) = (d\omega, j^*\omega - d\theta),$$

for $j : U \hookrightarrow M$.

Now proposition 2.1 can be restated for de Rham cohomology as follows:

PROPOSITION 2.2 - *Let M be a compact oriented differentiable n-manifold, A a closed subset of M such that $U = M \setminus A \overset{j}{\hookrightarrow} M$ is of finite type, $\mathcal{V} = \{V\}$ an inductive family of open neighborhoods of A as above. Then there is an isomorphism*

$$P_{M,U} : \bar{H}^k_{DR}(A) \to (H^{n-k}_{DR}(M,U))^* \tag{2.3}$$

which can be described as follows: let

$$\tau = \operatorname{ind}\lim_{\mathcal{V}} \{\tau_V\} \in \bar{H}^k_{DR}(A), \{\tau_V\} \in H^k_{DR}(V)$$

and $\eta = \{(\omega, \theta)\} \in H^{n-k}_{DR}(M,U)$, i.e. $\{\omega\} \in H^{n-k}_{DR}(M)$ and

$$j^* \omega = d\theta, \tag{2.4}$$

then

$$P_{M,U}(\tau)(\eta) = \lim_{\mathcal{V}} \{\textstyle\int_V \omega \wedge \tau_V - \int_{\partial V} \theta \wedge \tau_V\} \tag{2.5}$$

(notice that, since τ is defined by an inductive limit, τ_V can be thought of as defined in an open set properly containing V itself).

Proof. The existence of isomorphism (2.3) follows as in the standard proof of proposition 2.1. We only need to show (2.5). As usual consider the following diagram (in which we suppress the subscripts DR)

$$\begin{array}{ccccccccccc}
\to & H^k_c(U) & \to & H^k(M) & \to & \bar{H}^k(A) & \to & H^{k+1}_c(U) & \to & H^{k+1}(M) & \to \\
 & \downarrow P_U & & \downarrow P_M & \boxed{1} & \downarrow P_{M,U} & \boxed{2} & \downarrow P_U & & \downarrow P_M & \\
\to & (H^{n-k}(U))^* & \to & (H^{n-k}(M))^* & \to & (H^{n-k}(M,U))^* & \to & (H^{n-k-1}(U))^* & \to & (H^{n-k-1}(M))^* & \to
\end{array}$$

where all P_U 's and P_M 's are Poincaré isomorphisms as in (2.1). By (2.4) and Stokes' theorem it easily follows that $P_{M,U}$ in (2.5) is well defined. Moreover both squares $\boxed{1}$ and $\boxed{2}$ sign-commute, so that the thesis is achieved by a standard application of Five Lemma. □

From now on, since we only deal with de Rham cohomology, we will write $H^k(-)$ instead of $H^k_{DR}(-)$.

We now apply proposition 2.2 to extend remark 2.1 to Alexander duality: a more precise version of this will be dealt with in section 3 with the use of Thom isomorphism.

For (M, U) as above, let $S \subset M$ be a closed $(n-k)$-manifold with boundary $\partial S \subset U$. The pair $(S, \partial S)$ defines a linear functional F_S on $H^{n-k}(M,U)$ by

$$F_S : (\omega, \theta) \to \textstyle\int_S \omega - \int_{\partial S} \theta .$$

Indeed if (ω, θ) is a coboundary, i.e.

$$(\omega, \theta) = (d\varphi, j^* \varphi - d\psi), \text{ for } \{(\varphi, \psi)\} \text{ in } H^{n-k-1}(M,U),$$

it is, since $\varphi = j^*\varphi$ in $\partial S \subset U$,

$$\int_S d\varphi - \int_{\partial S} j^*\varphi + \int_{\partial S} d\psi = \int_{\partial S}\varphi - \int_{\partial S}\varphi + \int_{\partial\partial S}\psi = 0 .$$

Hence, by proposition 2.2, there exists a unique «form»

$$\tau^{(S)} \in \check{H}^k(A),$$

defined by an inductive family $\{\tau_V^{(S)}\} \in H^k(V)$ such that

$$F_S\{(\omega,\theta)\} = \int_S \omega - \int_{\partial S}\theta = \lim_{\mathcal{V}}\{\int_V \omega\wedge\tau_V^{(S)} - \int_{\partial V}\theta\wedge\tau_V^{(S)}\}. \tag{2.6}$$

In analogy with remark 2.1 we will call $\tau^{(S)}$ the *Alexander dual* of the pair $(S, \partial S)$; notice that, in case $U = \emptyset$, this construction reduces to the one in remark 2.1.

3. In this section we use Thom isomorphism to explicitely describe the Alexander dual $\tau^{(S)}$ introduced in section 2.

Let X be an n-dimensional oriented differentiable manifold and

$$\pi : E \to X \tag{3.1}$$

a rank q oriented vector bundle on X. Denote by $\Omega^*_{cv}(E)$ the complex of forms on E with compact support in the vertical direction, and with $H^*_{cv}(E)$ its cohomology which is usually called *compact vertical cohomology*. Thom isomorphism theorem shows that there exists an isomorphism

$$Th : H^*(X) \to H^{*+q}_{cv}(E);$$

under this isomorphism the image of $1 \in H^0(X) = \mathbf{R}$ determines a cohomology class Φ in $H^q_{cv}(E)$ called the Thom class of the oriented vector bundle E, [1]. Notice that Th is the inverse of isomorphism

$$\pi_* : H^{*+q}_{cv}(E) \to H^*(X)$$

induced by integration of forms along fibers. Finally, let us recall the well known *projection formula: for ω in $\Omega^p_{cv}(E)$* and σ in $\Omega^{m+q-p}_c(X)$, it is $\int_E (\pi^*\sigma)\wedge\omega = \int_X \sigma\wedge\pi_*\omega$, where π^* is the map induced in cohomology by (3.1).

Let (M, U) be a pair as in proposition 2.2 and $(S, \partial S)$ as in section 2; since both S and A are compact, $S \cap A$ only has finitely many connected components: for the sake of simplicity we will assume, throughout the sequel, $S \cap A$ to be connected (it is easy to modify our reasonings for the general case). Denote by $S^\circ = S \backslash \partial S$ and by $N = N_{S^\circ, M}$ the normal bundle of S° in M. Consider an inductive family $\mathcal{V} = \{V\}$ of open neighborhoods of A such that ∂V is smooth, $V \cap \partial S = \emptyset$, and $V \cap S$ is connected. The restriction $N|_{S^\circ \cap V}$ is still a bundle of rank $k = \operatorname{codim} S$

(3.2) $$\pi_V : N|_{S^\circ \cap V} \to S^\circ \cap V$$

over $S^\circ \cap V$; hence we can apply Thom isomorphism theorem to (3.2) to get a k – form

$$\Phi_V \in H^k_{cv}(N|_{S^\circ \cap V}),$$

the Thom class of (3.2).

Now consider a suitably small tubular neighborhood T of S° in M, and suppose A to satisfy the following condition:

the family $\mathfrak{V}$ can be chosen such that the intersection $T_V = T \cap V$ is still a tubular neighborhood of $S^\circ \cap V$ in V, with the same fibers as T.

The examples which we will provide in the sequel show that this hypothesis is easily verified; actually, at least when A has dimension n, it reduces to regularity conditions on $\partial A \cap S$.

Now identify $H^k_{cv}(N|_{S^\circ \cap V})$ with $H^k_{cv}(T_V)$ and consider the map

$$e : H^k_{cv}(T_V) \to H^k(V)$$

defined as extension by zero. Under this identification, Φ_V defines an element

$$e(\Phi_V) = \psi_V \in H^k(V);$$

notice that, because of the assumptions on $\mathfrak{V}$, each ψ_V is the Thom class of the bundle $N|_{S^\circ \cap V}$ and the family $\{\psi_V\}$ defines an element $\psi \in \bar{H}^k(A)$.

THEOREM 3.1 - *ψ is the Alexander dual $\tau^{(S)}$ of the pair $(S, \partial S)$, with respect to (M, U).*

Proof. In view of (2.6) it is enough to show that, for any $\{(\omega, \theta)\}$ in $H^{n-k}(M, U)$, it is

$$\int_S \omega - \int_{\partial S} \theta = \lim_{\mathfrak{V}} \{\int_V \omega \wedge \psi_V - \int_{\partial V} \theta \wedge \psi_V\}.$$

Indeed one has, for $V \in \mathfrak{V}$

$$\int_S \omega - \int_{\partial S} \theta = \int_{S \cap V} \omega + \int_{S \cap (M \setminus V)} \omega - \int_{\partial S \cap V} \theta - \int_{\partial S \cap (M \setminus V)} \theta =$$

$$= \int_{S \cap V} \omega + \int_{S \cap (M \setminus V)} d\theta - \int_{\partial S} \theta = \int_{S \cap V} \omega + \int_{\partial (S \cap (M \setminus V))} \theta - \int_{\partial S} \theta =$$

$$= \int_{S \cap V} \omega + \int_{\partial S} \theta + \int_{S \cap \partial (M \setminus V)} \theta - \int_{\partial S} \theta = \int_{S \cap V} \omega - \int_{S \cap \partial V} \theta .$$

On the other hand, if one denotes by $i : S^\circ \hookrightarrow T$, the inclusion regarded as the zero section of the bundle $\pi : T \to S^\circ$, the induced maps π^* and i^* are inverse isomorphisms in cohomology, so that

$$\omega = \pi^* i^* \omega + d\rho, \text{ for some } \rho .$$

Hence

$\int_V \omega\wedge\psi_V - \int_{\partial V} \theta\wedge\psi_V = \int_V \pi^* i^* \omega\wedge\psi_V + \int_V d\rho\wedge\psi_V - \int_{\partial V} \theta\wedge\psi_V =$

$= \int_{T_V} \pi^* i^* \omega\wedge\psi_V + \int_{T_V} d\rho\wedge\psi_V - \int_{T_V \cap \partial V} \theta\wedge\psi_V .$

By projection formula, the last expression becomes:

$\int_{S\cap V} \omega + \int_{\partial T_V} \rho\wedge\psi_V - \int_{T_V \cap \partial V} \theta\wedge\psi_V =$

$= \int_{S\cap V} \omega + \int_{T_V \cap \partial V} \rho\wedge\psi_V - \int_{T_V \cap \partial V} \theta\wedge\psi_V =$

$= \int_{S\cap V} \omega + \int_{\partial V\cap S} \rho - \int_{\partial V\cap S} \theta = \int_{S\cap V} \omega + \int_{\partial(V\cap S)} \rho - \int_{\partial V\cap S} \theta =$

$= \int_{S\cap V} \omega + \int_{V\cap S} d\rho - \int_{\partial V\cap S} \theta = \int_{S\cap V} \omega - \int_{\partial V\cap S} \theta ,$

since $\omega = \pi^* i^* \omega$ on S, so that, on S, $d\rho = 0$. □

We conclude the paper by supplying some examples in which the Alexander dual τ is explicitely constructed.

EXAMPLE 1 - Let M be the two-dimensional torus, represented on the (x, y) plane by the square with identified edges:

$$0 \leqslant x \leqslant 2\pi \ , \ 0 \leqslant y \leqslant 2\pi .$$

Define $A = \{ (x,y) \in M : x = \pi \}$, $U = M \backslash A$, and

$$S = \{ (x,y) \in M : y = \pi \text{ and } \pi/2 \leqslant x \leqslant 3\pi/2 \},$$

so that $\partial S = \{ (\pi/2, \pi) , (3\pi/2, \pi) \}$.

The family $\mathfrak{V}$ can be chosen, accordingly with the previous assumptions, to be $\mathfrak{V} = \{V_n\}_{n=1,2,\dots}$ where

$$V_n = \{ (x,y) \in M : \pi - \pi/4n < x < \pi + \pi/4n \}.$$

Following the procedure employed in the proof of theorem 3.1, one sees that the Alexander dual $\tau^{(S)} \in \bar{H}^k(A)$ of the pair $(S, \partial S)$ is

$$\tau^{(S)} = \lim \psi_n ,$$

where

$$\psi_n = \lambda_n f_n (x,y)\, dy$$

for

$$f_n(x,y) = \begin{cases} \exp(1/(y-3\pi/2) - 1/(y-\pi/2)) & y \in (\pi/2, 3\pi/2) \\ 0 & y \in [0,\pi/2] \cup [3\pi/2, 2\pi] \end{cases}$$

and

$$\lambda_n^{-1} = \int_0^{2\pi} f_n(x,y)\, dy .$$

It is now easy to verify that such a $\tau^{(S)}$ satisfies (2.6), for any $\{ (\omega, \theta) \}$ in $H^1(M, U) = \mathbf{R}$.

EXAMPLE 1' - A quite similar reasoning applies if in example 1 we substitute A with the annulus

$$A = \{ (x, y) \in M : 4\pi/5 \leqslant x \leqslant 6\pi/5 \}.$$

EXAMPLE 2 - Let M be S^2 and suppose $A \subset M$ to be a closed 2-disk. In such a case, since $H^q(S^2, D^2)$ is non trivial but for $q = 2$, we will take S to be two-dimensional, in which case, due to the assumptions, either $S \cap A = \emptyset$, or $S \supset A$. If $S \cap A = \emptyset$, one can suppose T_V to be empty, $\psi_V = 0$, and therefore $\tau^{(S)} = 0$, which agrees with

$$F_S\{(\omega, \theta)\} = \int_S \omega - \int_{\partial S} \theta = \int_S d\theta - \int_{\partial S} \theta = 0.$$

If $S \supset A$, each V is eventually contained in S; moreover, the bundle $N|_{S^\circ \cap V} = N|_V$ has rank zero, so that ψ_V is the zero form constantly equal to one, and its limit is $\tau^{(S)} = 1$ on A.

EXAMPLE 3 - Let $M = S^2 \subset \mathbf{R}^3$ be the unit sphere in polar coordinates

$$\rho = 1, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad -\pi/2 \leqslant \theta \leqslant \pi/2,$$

A be the annulus $(-\pi/4 \leqslant \theta \leqslant \pi/4)$ and S be the arc

$$(\varphi = \pi, -\pi/3 \leqslant \theta \leqslant \pi/3).$$

Reasoning as above, one gets $\tau^{(S)} = \lim \psi_\varepsilon$, where ψ_ε is defined on

$$V_\varepsilon = \{ (\rho, \varphi, \theta) : -\pi/4 - \varepsilon < \theta < \pi/4 + \varepsilon, \rho = 1 \}$$

by

$$\varphi_\varepsilon(\rho, \varphi, \theta) = \lambda_\varepsilon f_\varepsilon(\rho, \varphi, \theta)\, d\varphi$$

for

$$f_\varepsilon(\rho, \varphi, \theta) = \begin{cases} \exp(1/(\varphi - 3\pi/2) - 1/(\varphi - \pi/2)) & \varphi \in (\pi/2, 3\pi/2) \\ 0 & \varphi \in [0, \pi/2] \cup [3\pi/2, 2\pi] \end{cases}$$

and

$$\lambda_\varepsilon^{-1} = \int_0^{2\pi} f_\varepsilon(\rho, \varphi, \theta)\, d\varphi.$$

It is not difficult to generalize both examples 2 and 3 to higher dimensional cases.

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[2] MASSEY W. S., *Singular Homology Theory*, Springer Verlag, New York 1980.