ON FUNCTIONAL CONVERGENCES (*)

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SOMMARIO. - *Partendo dalla ricerca di condizioni affinché una convergenza di successioni possa essere individuata da una funzione a valori reali, l'introduzione di due condizioni di tipo diagonale consente di caratterizzare le convergenze quasi-normabili, quindi quelle esistenti nelle topologie di spazio lineare. Dal risultato principale (Teorema 3) si traggono risultati riguardanti la metrizzabilità e diversi tipi di convergenze.*

SUMMARY. - *Starting with the research of conditions allowing a sequential convergence to be determined by a real-valued function, the introduction of two diagonal conditions leads to a characterization of quasi-normable spaces, like these of linear topologies. The main result (Theorem 3) is a source of new statements concerning metrizability and different kind of convergences.*

1. - The paper deals with sequential convergences on an arbitrary set X. The following question is taken as the starting point: under what conditions a given convergence G can be determined by a real-valued function? This leads to the question of a description of a convergence determined by a function satisfying the triangle inequality and, finally, by a metric.

Characterizations of those three cases of convergences G are given in terms of conditions for convergences G_0 such that $G_0^* = G$ (Theorems 1-3); the convergence G_0^* is the smallest convergence containing G_0 and satisfying the Urysohn condition. For the purpose of the characterizations, two simple diagonal conditions (D_1, D_2) are introduced. These conditions (assumed for G_0), which express

(*) Pervenuto in Redazione il 1° luglio 1983.
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a kind of uniformity, play a basic role in the theorems of the paper. For X being a linear space, a characterization of quasi-normable convergences is given (Theorem 4). This implies a characterization of convergences generated by a linear topology (Theorem 5).

The principal aim of the paper is to show that one can find relatively simple, sequential (countable) conditions that describe metric convergences, in particular. However, in some cases, the results of the paper may be applied. Especially, Theorem 4 is a useful criterion of metrizability. For example, from Theorem 4 it immediately follows that convergence in measure and Δ-convergence of Boehmians (see [3]) are metric convergences.

2. - Assume that X is an arbitrary set and G is a convergence on X, i.e. G is a subset of $X^N \times X$. Consequently, intersection or inclusion of convergences denote intersection or inclusion of the respective subsets of $X^N \times X$. If $\langle (x_n), x \rangle \in G$, then we say that the sequence (x_n) is convergent to x in G and write $x_n \to x(G)$ or, simply, $x_n \to x$. We shall also write $a_n \to a$ (or $f(x, x_n) \to 0$) to denote the ordinary convergence of real numbers. It will not lead to misunderstandings.

In further considerations, we shall use the following conditions:

F. If $x_n \to x$, then $y_n \to x$ for each subsequence (y_n) of (x_n);

U. If each subsequence (y_n) of (x_n) contains a subsequence (z_n) such that $z_n \to x$, then $x_n \to x$;

S. If $x_n = x$ for $n = 1, 2, \ldots$, then $x_n \to x$;

H. If $x_n \to x$ and $x_n \to y$, then $x = y$.

Given a convergence G, we define the convergence G^* in the following way:

$x_n \to x(G^*)$ iff each subsequence (y_n) of (x_n) contains a subsequence (z_n) such that $z_n \to x(G)$.

Evidently, if G fulfils condition F, then G^* is the smallest convergence containing G and fulfilling condition U. Moreover, G fulfils conditions S and H iff G^* does.

We shall say that a convergence G is a functional convergence if there exists a function $f: X \times X \to R$ such that

$$x_n \to x(G) \text{ iff } f(x, x_n) \to 0.$$

In this case, we shall also say that the function f generates the convergence G. Note that each functional convergence fulfils conditions F and U. Moreover, $f(x, x) = 0$ iff G fulfils condition S. If G fulfils conditions S and H, then $f(x, y) = 0$ iff $x = y$.

Without loss of generality, we shall assume that all functional convergences are generated by non-negative functions.

3. - Consider the following diagonal condition:

D_1. If $x_{mn} \to x$ as $n \to \infty$ for $m = 1, 2, \dots$, then $x_{nn} \to x$.

THEOREM 1. - *A convergence G is a functional convergence iff there exists a convergence G_0 fulfilling conditions* F *and* D_1 *such that $G_0^* = G$.*

Proof. Suppose that $x_n \to x(G)$ iff $f(x, x_n) \to 0$. We define the convergence G_0 as follows:

(1) $x_n \to x(G_0)$ iff $f(x, x_n) \leqslant 1/2^n$ for each $n = 1, 2, \dots$.

Evidently, the convergence G_0 fulfils conditions F and D_1 and $G_0 \subset G$, so $G_0^* \subset G^* = G$. On the other hand, each subsequence of a sequence (x_n), $x_n \to x(G)$ contains a subsequence (y_n) such that $f(x, y_n) \leqslant 1/2^n$ and thus $G \subset G_0^*$. Hence $G = G_0^*$.

Now, let G_0 fulfil conditions F and D_1 and let $G_0^* = G$. Moreover, let $\{(x_n^\alpha) : \alpha \in \mathfrak{A}\}$ be the family of all sequences tending to x in G_0. We define the function $f : X \times X \to R$ in the following way:

(2) $f(x, y) = \begin{cases} 1 & \text{if } y \neq x_n^\alpha \text{ for any natural } n \text{ and } \alpha \in \mathfrak{A} \\ \inf\{1/2^n : y = x_n^\alpha \text{ for some } \alpha \in \mathfrak{A}\} & \text{if the set is non-empty.} \end{cases}$

Let $x_n \to x(G_f)$ iff $f(x, x_n) \to 0$. We shall show that $G_f = G$. Since G_f fulfils condition U and $G_0 \subset G_f$, we have $G_0^* \subset G_f^* = G_f$. Let $x_n \to x(G_f)$, i.e. $f(x, x_n) \to 0$. Then each subsequence of (x_n) contains a subsequence (y_n) such that $f(x, y_n) \leqslant 1/2^n$. Since the convergence G_0 fulfils condition F we can find, by (2), a matrix (x_{mn}) $(m, n = 1, 2, \dots)$ such that $x_{mn} \to x(G_0)$ as $n \to \infty$ for $m = 1, 2, \dots$ and $x_{nn} = y_n$ for $n = 1, 2, \dots$ Therefore, by condition D_1, $y_n \to x(G_0)$ and, consequently, $x_n \to x(G_0^*)$. Hence $G_f \subset G_0^* = G$ and the proof is over.

It is easy to see that there are functional convergences which are not generated by any function satisfying the triangle inequality:

$$(\Delta) \qquad f(x, z) \leqslant f(x, y) + f(y, z).$$

Let us introduce the following diagonal condition:

D_2. If $x_{mn} \to x_m$ as $n \to \infty$ for $m = 1, 2, \dots$ and $x_n \to x$, then $x_{n+1, n+1} \to x$.

THEOREM 2. - *A convergence G fulfilling condition* S *is generated by a function satisfying the triangle inequality iff there exists a convergence G_0 fulfilling conditions* F *and* D_2 *such that $G_0^* = G$.*

Proof. Let G be a convergence generated by a function f satisfying inequality (Δ). Then the convergence G_0 defined by formula (1) fulfils conditions F and D_2, in view of inequality (Δ). As in Theorem 1, one can check that $G_0^* = G$.

Conversely, suppose that a convergence G_0 fulfils conditions F and D_2 and $G_0^* = G$. Let $\{(x_n^\alpha) : \alpha \in \mathfrak{A}\}$ be the family of all sequences converging to x in G_0.

We define the function $g : X \times X \to R$ in the following way:

$$(3)\quad g(x,y) = \begin{cases} 1 & \text{if } y \neq x_n^\alpha \text{ for any natural } n \text{ and } \alpha \in \mathfrak{A} \\ \inf\left\{\frac{1}{\sqrt{2^n}} : y = x_n^\alpha \text{ for some } \alpha \in \mathfrak{A}\right\} & \text{if the set is non-empty.} \end{cases}$$

Define the convergence $G_g : x_n \to x(G_g)$ iff $g(x, x_n) \to 0$. We shall show that $G_g = G_0^* = G$. Since $G_0 \subset G_g$ and the convergence G_g fulfils condition U, we have $G_0^* \subset G_g$.

Now, let $x_n \to x(G_g)$. Then each subsequence of (x_n) contains a subsequence (y_n) such that $g(x, y_n) \leqslant \frac{1}{\sqrt{2^{n+1}}}$. Since the convergence G_0 fulfils condition F, we can find, by (3), a matrix (x_{mn}) $(m,n = 1,2,\ldots)$ such that $x_{mn} \to x(G_0)$ as $n \to \infty$ for $m = 1,2,\ldots$ and $x_{n+1,n+1} = y_n$ for $n = 1,2,\ldots$. Therefore, by conditions S and D_2, $y_n \to x(G_0)$ and, consequently, $x_n \to x(G_0^*)$. Hence $G_g \subset G_0^*$ and the identity $G_g = G_0^* = G$ is proved.

We shall show that for each $x, y, z \in X$

$$(4)\quad g(x,z) \leqslant \sqrt{2}\max[g(x,y), g(y,z)].$$

In fact. We shall consider three cases.

The first case: $\max[g(x,y), g(y,z)] = 1$.
Then, evidently, (4) holds.

The second case: $\max[g(x,y), g(y,z)] = 0$.
Then, by (3), the sequences $y, y, \ldots$ and $z, z, \ldots$ are convergent in G_0 to x and y, respectively. Hence, by condition D_2, the sequence $z, z, \ldots$ is convergent to x in G_0 and, consequently, $g(x,z) = 0$.

The third case: $\max[g(x,y), g(y,z)] = \frac{1}{\sqrt{2^k}}$ for some natural k.
Then, by (3) and condition F for G_0, there is a matrix (x_{mn}) $(m,n = 1,2,\ldots)$ such that $x_{mn} \to x_m(G_0)$ as $n \to \infty$ for $m = 1,2,\ldots$, $x_m \to x(G_0)$ and $x_{kk} = z$. Therefore, by condition D_2, we have

$g(x,z) \leqslant \frac{1}{\sqrt{2^{k-1}}}$, which implies (4), as desired.

In view of (4), we obtain

(5) $g(x,z) \leqslant 2\max[g(x,t), g(t,y), g(y,z)]$ for each $x,t,y,z \in X$.

Inequality (5) implies, by induction (see similar proofs in [1] p. 300 and [2]) that for each natural k and $t_i \in X$ $(i=1,2,\dots,k)$

(6) $g(x,y) \leqslant 2\,(g(x,t_1) + g(t_1,t_2) + \dots + g(t_k,y))$.

Let

$$f(x,y) = \inf\{g(x,t_1) + g(t_1,t_2) + \dots + g(t_k,y) : t_1,\dots,t_k \in X, k \in N\}.$$

Of course, the function f fulfils inequality (Δ).

In view of (6), we have

$$\tfrac{1}{2}\, g(x,y) \leqslant f(x,y) \leqslant g(x,y).$$

Thus, defining $x_n \to x(G_f)$ iff $f(x,x_n) \to 0$, we have $G_g = G_f$ and, consequently, $G_f = G$, which completes the proof.

The following simple example proves that there are functional convergences generated by functions satisfying inequality (Δ) that are not generated by any symmetric function.

EXAMPLE 1. - Let $X = R$ and let the convergence G be defined as follows:

(7) $x_n \to x(G)$ iff $x_n \to x$ and $x_n \geqslant x$ for $n > N_0$

where $x_n \to x$ denotes the usual convergence in R.

Then the function

$$f(x,y) = \begin{cases} y - x & \text{if } y \geqslant x \\ 1 & \text{if } y < x \end{cases}$$

generates G and satisfies inequality (Δ).

On the other hand, no symmetric function generates the convergence G.

In fact. Suppose, on the contrary, that there is a function $g : X \times X \to R$ such that $x_n \to x(G)$ iff $g(x,x_n) \to 0$ and $g(y,x) = g(x,y)$.

In view of (7), for each x we have $\inf\limits_{y<x} g(x,y) > 0$. Denoting $X_k = \{x : \inf\limits_{y<x} g(x,y) > 1/k\}$ we have $\bigcup\limits_{k=1}^{\infty} X_k = X$, so there is K such that X_K is uncountable. Therefore there is a sequence (x_n) contained in X_K such that $X_K \cap \langle x_0, x_n \rangle$ is uncountable and $x_0 < x_{n+1} < x_n$ for $n = 1,2,\dots$. Hence there is x such that $x_n \to x(G)$, i.e. $g(x,x_n) \to 0$. But $g(x,x_n) = g(x_n,x) > 1/K$. We get a contradiction.

However, a characterization of convergences generated by symmetric functions is possible. Suppose that a symmetric function f generates a convergence G such that for each x there is a sequence (x_n) which is convergent to x (in particular, G satisfies condition S).

Then the convergence G_0 defined by formula (1) satisfies the following condition:

(*) $x_n \to x(G_0)$ iff there is a matrix (x_{mn}) $(m, n = 1, 2, \ldots)$ such that for each n $x_{mn} \to x_n$ as $m \to \infty$ and $x_{nn} = x$.

On the other hand, if a convergence G_0 fulfils condition (*) and function f is given by formula (2) (or (3)), then

$$f(x, x_n) \to 0 \text{ iff } f(x_n, x) \to 0$$

and, consequently, f can be replaced by a symmetric function generating the same convergence.

In particular, we obtain

THEOREM 3. - *A convergence G is metrizable (i.e. generated by a metric) iff there is a convergence G_0 fulfilling conditions* F, S, H, D_2, (*) *such that $G_0^* = G$.*

4. - Now, let X be a linear space. We shall additionally consider the following condition of linearity:

L. If $x_n \to x$, $y_n \to y$, $a_n \to a$, $b_n \to b$, then $a_n x_n + b_n y_n \to ax + by$.

If a convergence G fulfils conditions S and L and a function f generates G, then defining $q(x) = f(0, x)$ we have $q(x_n - x) \to 0$ iff $x_n - x \to 0(G)$, i.e.

$$q(x_n - x) \to 0 \text{ iff } x_n \to x(G).$$

In this case, we shall also say that the function q generates the convergence G.

In [2], it is shown that if a convergence G fulfils conditions S and L and G is generated by a function $q: X \to R$, then there is an equivalent quasi-norm on X, i.e. a non-negative function $p: X \to R$ with properties:

$$\begin{aligned} x_n \to x(G) &\text{ iff } p(x_n - x) \to 0, \\ p(0) &= 0, \\ p(x) &= p(-x), \\ p(x+y) &\leqslant p(x) + p(y), \\ a_n \to a, p(x_n - x) \to 0 &\text{ implies } p(a_n x_n - ax) \to 0. \end{aligned} \tag{8}$$

Hence, by virtue of Theorem 1, we obtain

THEOREM 4. - *A convergence G on a linear space is generated by a quasi-norm iff G fulfils conditions* S *and* L *and there is a convergence G_0 satisfying conditions* F *and* D_1 *such that $G_0^* = G$.*

REMARK - The above considerations remain true in an adequately reduced form in commutative group. In particular, condition L is reduced to the following condition:

If $x_n \to x$ and $y_n \to y$, then $x_n - y_n \to x - y$ and the quasi-norm p does not possess the last of properties (8).

Evidently, convergences in topological vector, Hausdorff spaces fulfil all the conditions F, U, S, H, L. On the other hand, there are convergences fulfilling conditions F, U, S, H, L which are not generated by any linear topology (see example in [4]). A natural question arises: under what conditions a directly given convergence fulfilling all the conditions F, U, S, H, L is generated by a linear topology? Applying Theorem 4, we shall obtain the following result:

THEOREM 5. - *A convergence G on a linear space is generated by a linear topology iff there exists a family of convergences $\{G_\beta\}$ $(\beta \in \mathfrak{B})$ such that the convergences G_β fulfil conditions* F, S, D_1, *the convergences G^*_β fulfil condition* L *and* $G = \bigcap_{\beta \in \mathfrak{B}} G^*_\beta$.

Proof. It is well known (see [1] p. 302 or [2]) that the convergence G in a topological vector space can be generated by a family P of quasi-norms, i.e. $x_n - x(G)$ iff $p(x_n - x) \to 0$ for each $p \in P$. Then for every convergence G_p such that $x_n \to x(G_p)$ iff $p(x_n - x) \to 0$ there is, by Theorem 4, a convergence G_{p_0} fulfilling conditions F, S, D_1 such that $G^*_{p_0} = G_p$. Since G_p fulfils condition L, we obtain the required form of G.

To prove the converse implication, note that for each $\beta \in \mathfrak{B}$ the convergence G^*_β fulfils condition S. Thus, by Theorem 4, for each β there is a quasi-norm p_β which generates the convergence G_β. Then the topology generated by the family of quasi-norms $\{p_\beta\}$, $(\beta \in \mathfrak{B})$ possesses the properties we need.

Now, we shall produce an example which gives the negative answer to the following problem posed by J. Burzyk:

Suppose that a convergence G fulfils conditions F, U, S, H, L and, additionally, the following one:

B. For each sequence (x_n) there are real numbers $a_n \neq 0$ $(n = 1, 2, \dots)$, such that $a_n x_n \to 0$.

Must the convergence G be generated by a linear topology?

EXAMPLE 2.(1) - Let a set $\{e_i, e_{ij} : i, j = 1, 2, \dots\}$ be the base of a linear space X, i.e.

$$x \in X \text{ iff } x = \sum_i a_i e_i + \sum_{i,j} a_{ij} e_{ij},$$

(1) Another example (not published) was independently given by J. Burzyk.

where only a finite number of real a_i and a_{ij} $(i, j = 1, 2, \ldots)$ is different from zero.

We introduce a convergence G on X.

Let $x_n = \sum_i a_{i,n} e_i + \sum_{i,j} a_{ij,n} e_{ij}$ for $n = 1, 2, \ldots$. We adopt $x_n \to 0(G)$ iff the following three conditions are satisfied:

1^0 $a_{ij,n} \to 0$ for each i, j,

2^0 $\sum_j a_{ij,n} - a_{i,n} \to 0$ for each i,

3^0 there is a positive number K such that $|a_{i,n}| < K$ for each i, n.

Moreover, we define

$$x_n \to x(G) \text{ iff } x_n - x \to 0(G).$$

One can easily verify that the convergence G fulfils all the conditions F, U, S, H, L, B.

We shall show that the convergence G is not generated by any linear topology.

Consider the following matrix:

$$\begin{array}{c} e_{11}, \quad e_{12}, \quad e_{13}, \ldots \\ 2e_{21}, \quad 2e_{22}, \quad 2e_{23}, \ldots \\ \vdots \\ ke_{k1}, \quad ke_{k2}, \quad ke_{k3}, \ldots \\ \vdots \end{array} \tag{9}$$

By the definition of G, for each $k = 1, 2, \ldots$ the sequence (x^k_n) where $x^k_n = ke_{kn} - ke_k$, tends to zero. In other words, k–th row of the matrix (9) is convergent to ke_k in G. Moreover, each diagonal of the matrix (9) (i.e. a sequence (ne_{np_n}), where (p_n) is an increasing sequence of natural numbers) tends to zero, whereas the sequence of limits (ne_n) does not tend to zero in G, because it does not fulfil condition 3^0 of the definition.

On the other hand, it is easy to prove that such a situation is impossible in any topology satisfying condition T_3.

Therefore no linear topology generates the convergence G.

I wish to thank Dr. A. Kamiński for his valuable remarks.

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