A PRIORI BOUNDS AND EXISTENCE RESULTS FOR NONLINEAR EQUATIONS AT RESONANCE (*)

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SOMMARIO. - *Si presentano alcuni risultati riguardanti l'esistenza di limitazioni a priori per le soluzioni* (x, λ) *della equazione astratta* $Lx = (1-\lambda)\mathcal{A}x + \lambda\mathcal{N}x$, *dove* L *è lineare, Fredholm di indice zero, ed* $\mathcal{A}$ *e* $\mathcal{N}$ *sono operatori fra spazi normati. Usando un'opportuna teoria del grado, si ottengono teoremi di esistenza per equazioni astratte nonlineari in risonanza che consentono di provare l'esistenza di soluzioni periodiche per taluni sistemi differenziali del tipo di Liénard con argomenti deviati.*

SUMMARY. - *We present some results on the existence of a priori bounds for pairs* (x, λ) *satisfying the functional equation* $Lx = (1-\lambda)\mathcal{A}x + \lambda\mathcal{N}x$. L *is a linear Fredholm mapping of index zero, and* $\mathcal{A}$ *and* $\mathcal{N}$ *are (possibly) nonlinear maps between real normed spaces. The existence of a suitable coincidence degree theory is assumed, and some existence theorems for the equation* $Lx = \mathcal{N}x$ *are derived. As applications, we study the periodic problem for* N*-dimensional differential equation of Liénard type with deviating argument.*

1. Introduction

In this paper we deal with the problem of existence of solutions for functional equations of the form

$$Lx = \mathcal{N}x \tag{1.1}$$

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where, being X and Z real normed spaces, $L\colon \operatorname{dom} L \subseteq X \to Z$ is a linear Fredholm mapping of index zero and $\mathcal{N}\colon Y \subseteq X \to Z$ is a (possibly) nonlinear operator. It is well known that, in the present situation, the problem of existence of solutions of (1.1) may be reduced to the existence of fixed points of a convenient operator $\mathcal{N}'\colon X \to X$, which can be constructed using L and $\mathcal{N}$ (see [16]). Here, we shall study the equation

$$x = \mathcal{N}'x \tag{1.2}$$

in the framework of a suitable topological degree theory (see [2] and [12, Ch. V]). Therefore, we shall compare (1.2) with a uniquely solvable linear equation

$$x = \mathcal{A}'x \tag{1.3}$$

by embedding both (1.2) and (1.3) into a one-parameter family of equations of the form

$$x = (1-\lambda)\,\mathcal{A}'x + \lambda\mathcal{N}'x, \qquad \lambda \in [0,1]. \tag{1.4}$$

Now, the aim of the present paper is to prove some sufficient criteria which give a priori estimates for any possible solution of (1.4) and therefore make it possible to use topological degree methods for the solvability of (1.2).

The paper consists of five sections.

In Section 2 we give some sufficient conditions (Proposition 2.1 and Proposition 2.2) for the nonexistence of solutions $(x,\lambda) \in D \times [0,1]$ (or $(x,\lambda) \in D \times]0,1[$) to

$$Lx = (1-\lambda)\,\mathcal{A}x + \lambda\mathcal{N}x, \tag{1.5}$$

where $D \subseteq \operatorname{dom} L$ is fixed. The main assumptions required are of the following kind: (i) an hypothesis of definiteness of the quadratic form $u \to \langle Lu, u \rangle$ on $\operatorname{coim} L \doteq \operatorname{dom} L/\ker L$, where $\langle \cdot , \cdot \rangle$ is a suitable bilinear form defined on $\operatorname{Im} L \times \operatorname{coim} L$; (ii) a nondegeneracy assumption for the projection of (1.5) onto $Z/\operatorname{Im} L = \operatorname{coker} L$ (the bifurcation equation); (iii) a «one-sided growth restriction» for the projections of the operators $\mathcal{A}$ and $\mathcal{N}$ onto $\operatorname{Im} L$.

In Section 3 we assume the existence of a topological degree theory (in the sense of AMANN and WEISS [2]) for the study of (1.5) and we apply the preliminary results of Section 2 to get existence theorems (Theorems 3.1, 3.2, 3.3; see also Remark (3.2)) for (1.1). This procedure is made feasible trough an appropriate choice of the set D. More precisely, choosing $D = \operatorname{fr}\Omega \cap \operatorname{dom} L$, where $\Omega \subseteq X$ is open and bounded, we obtain existence of solutions to (1.1) in $\operatorname{cl}\Omega \cap \operatorname{dom} L$. Moreover, choosing $D \subseteq X$ in such a way that its complement $X \backslash D$ is bounded, we obtain a result ensuring that the solution set relative to (1.1) is nonempty and bounded.

In Section 4 we present some examples of abstract functional equations to which the previously developed results can be applied. For the sake of simplicity we restrict ourselves to the case of L-compact operators (for this definition, see MAWHIN [20]). The choice of this relevant class of operators is mainly due to the applications of the abstract theory to some differential equations of nonlinear mechanics, where this assumption is satisfied (see Section 5). However, L-k-set contractive [6] and L-condensing terms [25] can be considered as well.

More precisely, in Section 4 we examine at first the case in which $\ker L = \{0\}$ and $X \subseteq Z$ topologically and algebraically; in this situation, using a perturbation argument, we are able to find, as a consequence of our results in Section 3, an abstract theorem of resonance by WARD [26, Th. 1] which extended previous results by KANNAN and SCHUUR [10], [11]. Then we pass to the study of the situation in which L can be noninvertible, and present two theorems concerning the existence of solutions to the equations

$$Lx = (I - Q)\,\mathfrak{F}x + \mathfrak{A}x + e \tag{1.6}$$

and

$$Lx = (I - Q)\,\mathfrak{F}x + \mathfrak{G}x + e \tag{1.7}$$

where $I - Q$ is a continuous projection from Z onto $\operatorname{Im} L$, and where $\mathfrak{F}, \mathfrak{A}, \mathfrak{G}\colon X \to Z$ are L-completely continuous operators with $\mathfrak{A}$ linear, $\mathfrak{G}$ quasibounded (possibly nonlinear), and $\mathfrak{F}$ (possibly nonlinear) satisfying a suitable one-sided growth restriction; $e \in \operatorname{Im} L$ is also assumed.

The results which are achieved in relation to equations (1.6) and (1.7) permit us to obtain an extension of a theorem in MARTIN's book [14, Ch. IV, Prop. 6.3, p. 144] concerning the existence of solutions to Hammerstein equations in Hilbert spaces (see also AMANN [1]) as well as they can be applied to the problem of existence of periodic solutions to some forced nonlinear second order vector differential equations of Liénard type which appear, for instance, in nonlinear mechanics. Infact, it is well known that the problem of linear forced vibrations of a rigid body with constraints, springs, dashpots and actuators leads to a vector linear ordinary differential equation of the form

$$Mx'' + (d/dt)\,(Dx + Gx) + Bx = h(t) \tag{1.8}$$

where x is the vector of generalized coordinates, M is the nonsingular symmetric mass or inertia matrix, D is the symmetric damping matrix, G is the skewsymmetric matrix due to gyroscopic phenomena, B is the matrix of forces and h is the (time dependent) vector of external forcing. Consequently, in the last chapter (Section 5) we consider the following nonlinear equations with delayed argument generalizing (1.8)

$$x'' + (d/dt)\,(\nabla F(x) + V(t,x)) + \operatorname{diag} Bx(t-\sigma)^{\mathrm{T}} = h(t) \tag{1.9}$$

and

$$x'' + (d/dt)\,(\nabla F(x) + V(t,x)) + Bg(t, x(t-\sigma), x'(t-\sigma')) = h(t) \tag{1.10}$$

in which B is an $\mathrm{N} \times \mathrm{N}$ (possibly singular) matrix and h is a continuous periodic function with mean value zero.

In Theorem 5.1 we reduce equation (1.9) to an abstract equation of the type (1.6), with a suitable choice of function spaces. The existence theorem for (1.6) previously proved in Section 4 allows us to achieve an extension of some results of MAWHIN in [17] and CESARI and KANNAN [5, Th. 5.2 (c), (d)] concerning equation (1.9) without delay. In a similar way we deal with equation (1.10). In Theorem 5.2 we show that such equation can be written, in an equivalent form, like equation (1.7), with a pertinent choice of the abstract setting. In this case we also extend some recent results obtained in [9].

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2. A priori estimates for zeroes of linear homotopies in normed spaces

Let X, Z be real normed spaces, with corresponding norms $\|\cdot\|_X, \|\cdot\|_Z$, let $\operatorname{dom} L$ be a linear subspace of X and let

$$L\colon \operatorname{dom} L \subseteq X \to Z$$

be a linear map. Let $P\colon X \to X$ and $Q\colon Z \to Z$ be two projections (i.e. linear and idempotent operators) such that the sequence

$$X \xrightarrow{P} \operatorname{dom} L \xrightarrow{L} Z \xrightarrow{Q} Z$$

is exact (i.e. $\operatorname{Im} P = \ker L$ and $\operatorname{Im} L = \ker Q$). Whenever $x \in X$, we shall write $x = r + u$ $(r \in \ker L, u \in \ker P)$ according to the direct sum decomposition $X = \ker L \oplus \ker P$.

Let Y be a subset of X such that $\operatorname{dom} L \cap Y \neq \emptyset$ and let

$$\mathcal{A}, \mathcal{N}\colon Y \to Z$$

be two mappings. The linear homotopy joining $L - \mathcal{A}$ with $L - \mathcal{N}$ is the map

$$\mathcal{H}: [0,1] \times (\operatorname{dom} L \cap Y) \to Z$$

$$\mathcal{H}(\lambda, x) = Lx - (1-\lambda)\,\mathcal{A}x - \lambda \mathcal{N} x.$$

In the present section we derive two conditions sufficient to ensure that $\mathcal{H}$ does not have zeroes on suitable subsets of its domain. To achieve this goal we introduce the following assumptions (H0), (H0'), (H1), (H2).

(H0) *There exist a map* $\Pi: X \to \operatorname{Im} P$, *a bilinear form*

$$\langle \cdot\,,\cdot \rangle : \operatorname{Im} L \times (\operatorname{dom} L \cap \ker P) \to \mathbf{R},$$

and a number $\beta > 0$ *such that, for any* $x = r + u \in \operatorname{dom} L$,

$$\langle Lu, u \rangle \geqslant \beta \, \| x - \Pi x \|_X^2.$$

(H0') *There exist a map* $\Pi: X \to \operatorname{Im} P$ *and a bilinear form*

$$\langle \cdot\,,\cdot \rangle : \operatorname{Im} L \times (\operatorname{dom} L \cap \ker P) \to \mathbf{R}$$

such that, for any $x = r + u \in \operatorname{dom} L$,

$$\langle Lu, u \rangle \geqslant 0 \ \textit{and} \ \langle Lu, u \rangle = 0 \ \textit{if and only if} \ x = \Pi x.$$

Remark that, if $\Pi = P$, then (H0) (resp. (H0')) means that the quadratic form $u \to \langle Lu, u \rangle$ defined on $\operatorname{dom} L \cap \ker P$ is coercive (resp. positive definite). Obviously, if L is an invertible operator defined on X, we necessarily have $P = 0$ and $\Pi = 0$; therefore in this situation (H0) (resp. (H0')) means simply that L is coercive (resp. positive), a case which often occurs in applications. In section 5 we shall show also a case in which a useful choice for Π is a nonlinear map.

(H1) *There exist a set* $D_1 \subseteq \operatorname{dom} L \cap Y$ *and map*

$$\varphi : D_1 \to (\operatorname{Im} Q)^*$$

such that, for any $x \in D_1$,

$$(Q\mathcal{A}x, \varphi x)\,(Q\mathcal{N}x, \varphi x) > 0$$

(where (ξ, η) is the value of $\eta \in (\operatorname{Im} Q)^*$ (= the algebraic dual of $\operatorname{Im} Q$) at $\xi \in \operatorname{Im} Q$).

In view of applications, we can call (H1) an «abstract sign condition». The last assumption we introduce can be regarded as an «abstract one-sided growth restriction»:

(H2) *There exist a set* $D_2 \subseteq \operatorname{dom} L \cap Y$ *and numbers* $\varepsilon \in \,]0,1[$, $\alpha_i \in [0, \infty[$, $\nu_i \in [0, \infty[$ $(i = 1,2,3)$ *such that, for any* $x \in D_2$,

$$\langle (I - Q)\,\mathcal{A}x, u \rangle \leqslant \alpha_1 \langle Lu, u \rangle + \alpha_2 \langle Lu, u \rangle^{1-\varepsilon} + \alpha_3, \tag{2.2}$$

$$\langle (I - Q)\,\mathcal{N}x, u \rangle \leqslant \nu_1 \langle Lu, u \rangle + \nu_2 \langle Lu, u \rangle^{1-\varepsilon} + \nu_3 \tag{2.3}$$

(where $\langle \cdot , \cdot \rangle$ is the bilinear form introduced in (H0) or in (H0'); (H2) will be used only in connection either with (H0) or with (H0')).

Now we can state the following results.

Proposition 2.1. *Assume* (H0), (H1), (H2). *Suppose that*

(i) $\max\{\alpha_1, \nu_1\} < 1$;

(ii) there is a number $R > 0$ *such that*

$$D_2 \subseteq \{x \in X \mid \|x - \Pi x\|_X \geqslant R\},$$
$$\psi(\beta R^2) > 0,$$

where $\psi: [0, \infty[\to \mathbf{R}$ *is the function*

$$\psi(s) = (1 - \max\{\alpha_1, \nu_1\})\, s - \max\{\alpha_2, \nu_2\}\, s^{1-\varepsilon} - \max\{\alpha_3, \nu_3\}.$$

If $(\lambda, x) \in [0,1] \times (D_1 \cup D_2)$, *then* $Lx \neq (1-\lambda)\,\mathcal{A}x + \lambda \mathcal{N}x$.

Proposition 2.2. *Assume* (H0'), (H1), (H2). *Suppose that*

(i) $\max\{\alpha_1, \nu_1\} \leqslant 1$, $\min\{\alpha_1, \nu_1\} < 1, \alpha_2 = \nu_2 = 0$, $\alpha_3 = \nu_3 = 0$;

(ii) there is a number $R > 0$ *such that*

$$D_2 \subseteq \{x \in X \mid \|x - \Pi x\|_X \geqslant R\}.$$

If $(\lambda, x) \in]0,1[\times (D_1 \cup D_2)$, *then* $Lx \neq (1-\lambda)\,\mathcal{A}x + \lambda \mathcal{N}x$.

Proofs. We prove both Propositions 2.1 and 2.2 at the same time. Let $(\lambda, x) \in [0,1] \times (D_1 \cup D_2)$ verify

$$Lx = (1-\lambda)\,\mathcal{A}x + \lambda \mathcal{N}x, \tag{2.4}$$

or, equivalently,

$$0 = (1-\lambda)\, Q\mathcal{A}x + \lambda Q\mathcal{N}x, \tag{2.5}$$

$$Lu = (1-\lambda)\,(I-Q)\,\mathcal{A}x + \lambda(I-Q)\,\mathcal{N}x. \tag{2.6}$$

If $x \in D_1$, it follows from (2.5), using the map φ of (H1),

$$0 = (0, \varphi x) = (1-\lambda)\,(Q\mathcal{A}x, \varphi x) + \lambda\,(Q\mathcal{N}x, \varphi x)$$

which contradicts the fact that $(Q\mathcal{A}x, \varphi x)$ and $(Q\mathcal{N}x, \varphi x)$ are, in virtue of (2.1), non-zero real numbers with the same sign. Therefore $x \notin D_1$. If $x \in D_2$, using (H2) it follows from (2.6)

$$\begin{aligned}\langle Lu, u\rangle &= (1-\lambda)\,\langle (I-Q)\,\mathcal{A}x, u\rangle + \lambda\,\langle (I-Q)\,\mathcal{N}x, u\rangle \\ &\leqslant \big((1-\lambda)\alpha_1 + \lambda\nu_1\big)\,\langle Lu, u\rangle + \big((1-\lambda)\alpha_2 + \lambda\nu_2\big)\,\langle Lu, u\rangle^{1-\varepsilon} \\ &+ \big((1-\lambda)\alpha_3 + \lambda\nu_3\big).\end{aligned} \tag{2.7}$$

In the case of Proposition 2.1, we deduce from (2.7)

$$\langle Lu, u\rangle \leqslant \max\{\alpha_1, \nu_1\}\langle Lu, u\rangle + \max\{\alpha_2, \nu_2\}\langle Lu, u\rangle^{1-\varepsilon} + \max\{\alpha_3, \nu_3\},$$

and so, by the definition of the map ψ, we obtain

$$\psi(\langle Lu, u\rangle) \leqslant 0. \tag{2.8}$$

The assumptions on $\varepsilon, \alpha_i, \nu_i$ imply that $\psi(0) \leqslant 0$, that ψ is convex and that $\psi(s) \to \infty$ as $s \to \infty$. The assumption $\psi(\beta R^2) > 0$, jointly with (2.8), shows that

$$\langle Lu, u\rangle < \beta R^2.$$

Since $x \in D_2$, we obtain

$$\langle Lu, u\rangle < \beta R^2 \leqslant \beta \,\| x - \Pi x \|_X^2,$$

a contradiction with (H0). Therefore $x \notin D_2$, and Proposition 2.1 is proved.

In the case of Proposition 2.2, we deduce from (2.7)

$$\langle Lu, u\rangle \leqslant ((1-\lambda)\alpha_1 + \lambda\nu_1)\langle Lu, u\rangle,$$

and so, being $\lambda \in \,]0,1[$, we have

$$\langle Lu, u\rangle \leqslant 0.$$

The assumption (H0') implies that $x = \Pi x$, i.e. $\| x - \Pi x \|_X = 0$. Therefore $x \notin D_2$, and Proposition 2.2 is proved, too.

3. Some existence results for nonlinear equations in normed spaces

Throughout the present section *we assume that L is a linear Fredholm map of index zero* (i.e. $\operatorname{Im} L$ is closed and it has finite codimension, equal to the dimension of $\ker L$); obviously, the projections P and Q can be supposed to be continuous. Moreover *we assume that $\mathcal{A}$ and $\mathcal{N}$ are defined on $Y = X$, and that $\mathcal{A}$ is a linear map.*

Let $\Lambda: \operatorname{Im} Q \to \operatorname{Im} P$ be a linear isomorphism, and let

$$L^+: \ker Q \to \operatorname{dom} L \cap \ker P$$

be the right inverse of L relative to P and Q. It is well known (see Mawhin [16]) that a pair $(x, \lambda) \in [0,1] \times \operatorname{dom} L$ verifies

$$Lx = (1-\lambda)\,\mathcal{A}x + \lambda\mathcal{N}x \tag{3.1}$$

if and only if it verifies

$$x = (1-\lambda)\,\mathcal{A}'x + \lambda\mathcal{N}'x \tag{3.2}$$

where $\mathcal{A}' = P + (\Lambda Q + L^+(I-Q))\mathcal{A}$, $\mathcal{N}' = P + (\Lambda Q + L^+(I-Q))\mathcal{N}$.

Let us now introduce a key assumption.

Let ω be the set of all open bounded subsets of X. For any $\Omega \in \omega$, let $\mathrm{cl}\,\Omega$ and $\mathrm{fr}\,\Omega$ be, respectively, the closure and the boundary of Ω, and let $\mathcal{C}(\mathrm{cl}\,\Omega, X)$ be the linear space of all continuous mappings $\mathrm{cl}\,\Omega \to X$ with the topology of uniform convergence on $\mathrm{cl}\,\Omega$. *We assume that for any* $\Omega \in \omega$, *a subset* $\mathfrak{M}(\Omega)$ *of* $\mathcal{C}(\mathrm{cl}\,\Omega, X)$ *is selected in such a way that the following conditions hold:*

(D1) *for any* $\Omega \in \omega$, *the set* $\mathfrak{M}(\Omega)$ *is convex, and every map* $\mathfrak{f} \in \mathfrak{M}(\Omega)$ *maps bounded sets into bounded sets and it is proper* (i.e. the inverse image under $\mathfrak{f}$ of any compact set is compact);

(D2) *the class* $\mathfrak{M}(\omega) := \{\mathfrak{M}(\Omega) \mid \Omega \in \omega\}$ *is an admissible class of mappings in X, and there is a topological degree*

$$\mathfrak{d} := \{\mathfrak{d}(\cdot, \Omega) : \{\mathfrak{f} \in \mathfrak{M}(\Omega) \mid 0 \notin \mathfrak{f}(\mathrm{fr}\,\Omega)\} \to \mathbf{Z} \mid \Omega \in \omega\}$$

for $\mathfrak{M}(\Omega)$, *in the sense of* AMANN *and* WEISS [2] (see also [12, Ch. 5]);

(D3) *for any* $\Omega \in \omega$ *the maps* $I_{\mathrm{cl}\,\Omega} - \mathcal{A}'|_{\mathrm{cl}\,\Omega}$ *and* $I_{\mathrm{cl}\,\Omega} - \mathcal{N}'|_{\mathrm{cl}\,\Omega}$ *are in* $\mathfrak{M}(\Omega)$;

(D4) *if* $0 \in \Omega \in \omega$ *and if the linear map* $I - \mathcal{A}'$ *is* $1-1$, *then* $\mathfrak{d}(I - \mathcal{A}', \Omega) \neq 0$.

Therefore the usual technique of degree theory gives the following existence theorem of «continuation type»:

Theorem 3.0. *Suppose* $\ker(L - \mathcal{A}) = \{0\}$, *and assume that there is* $\Omega \in \omega$, *with* $0 \in \Omega$, *such that*

$$Lx \neq (1 - \lambda)\mathcal{A}x + \lambda\mathcal{N}x \tag{3.3}$$

for all $(\lambda, x) \in]0,1[\times (\mathrm{dom}\, L \cap \mathrm{fr}\,\Omega)$. *Then there exists at least one* $x \in \mathrm{dom}\, L \cap \mathrm{cl}\,\Omega$ *such that*

$$Lx = \mathcal{N}x. \tag{3.4}$$

The proof of the above result is the same as the proof of Théorème 1.2 in MAWHIN [20, p. 16] except for the fact that in [20] the set $\mathfrak{M}(\Omega)$ consists of the compact perturbations of the identity, that is $\mathfrak{d}$ is the Leray-Schauder degree. Obviously (see [12, Ch. VI], [6], [25], for instance) other possible choices for $\mathfrak{M}(\Omega)$, when X is a Banach space, are

$\mathfrak{M}(\Omega) = \{I_{\mathrm{cl}\,\Omega} - \psi \mid \psi \in \mathcal{C}(\mathrm{cl}\,\Omega, X)$ k-set-contractive, $k < 1\}$,
$\mathfrak{M}(\Omega) = \{I_{\mathrm{cl}\,\Omega} - \psi \mid \psi \in \mathcal{C}(\mathrm{cl}\,\Omega, X)$ condensing$\}$.

We remark that (D4) is automatically satisfied in these three cases. We remark also that it is possible to assume, instead of the existence of a degree theory in the sense of Amann and Weiss, the existence of some suitable theory of generalized degree: for instance the case in which $I - \mathcal{A}'$ and $I - \mathcal{N}'$ are A-proper maps can be studied in a similar manner (see [12, Ch. VII]).

Now we shall prove some existence theorems for a nonlinear equation of the form (3.4), using the results of the preceding section to ensure the validity of (3.3) in Theorem 3.0.

Theorem 3.1. *Suppose that* (H0) *holds for a continuous map* Π *such that* $\Pi(0) = 0$. *Assume that there exist positive numbers* a, b *such that* (H1) *holds with*

$$D_1 = \{x \in \operatorname{dom} L \mid \; \|\Pi x\|_X = a, \quad \|x - \Pi x\|_X \leqslant b\},$$

and that (H2) *holds with*

$$D_2 = \{x \in \operatorname{dom} L \mid \; \|\Pi x\|_X \leqslant a, \quad \|x - \Pi x\|_X = b\}.$$

Moreover, suppose

$$\max\{\alpha_1, \nu_1\} < 1 \text{ and } \psi(\beta b^2) > 0 \tag{3.5}$$

(where ψ is the map defined in Proposition 2.1).

Let

$$\Omega = \{x \in X \mid \; \|\Pi x\|_X < a, \quad \|x - \Pi x\|_X < b\}. \tag{3.6}$$

Then there exist at least one $x \in \operatorname{dom} L \cap \Omega$ *(so that* $\|x\|_X < a + b$*) such that*

$$Lx = \mathcal{N}x.$$

Proof. The continuity of Π, jointly with $\Pi(0) = 0$, implies that Ω is an open bounded subset of X containing 0. Moreover

$$\operatorname{dom} L \cap \operatorname{fr} \Omega \subseteq D_1 \cup D_2.$$

Using (3.5) and choosing $R = b$, the assumptions *(i)* and *(ii)* of Proposition 2.1 hold. So we can apply this result and obtain that

$$Lx \neq (1 - \lambda)\, \mathcal{A}x + \lambda \mathcal{N}x$$

whenever $(\lambda, x) \in [0, 1] \times (\operatorname{dom} L \cap \operatorname{fr} \Omega)$. In particular (setting $\lambda = 0$) the linear manifold $\ker(L - \mathcal{A})$ has empty intersection with $\operatorname{dom} L \cap \operatorname{fr} \Omega$ and so it must be equal to $\{0\}$. Applying Theorem 3.0 we get the result (remark that $Lx \neq \mathcal{N}x$ if $x \in \operatorname{dom} L \cap \operatorname{fr} \Omega$).

Theorem 3.2. *Suppose* (H0). *Assume that there exists a non-negative number* a *such that* (H1) *holds with*

$$D_1 = \{x \in \operatorname{dom} L \mid \; \|\Pi x\|_X > a\},$$

and that (H2) *holds with*

$$D_2 = \{x \in \operatorname{dom} L \mid \; \|\Pi x\|_X \leqslant a\}.$$

Moreover, suppose

$$\max\{\alpha_1, \nu_1\} < 1.$$

Then the set

$$\Sigma = \{x \in \operatorname{dom} L \mid Lx = \mathfrak{N}x\}$$

is non-empty and bounded, and $\|\Pi x\|_X \leq a$ *for any* $x \in \Sigma$.

Proof. We shall use Proposition 2.1 again. All the assumption in Proposition 2.1 are obviously satisfied, except for *(ii)*. But let R be any positive number such that $\psi(\beta R^2) > 0$ (ψ as in *(ii)*): if

$$D_2 \subseteq \{x \in X \mid \|x - \Pi x\|_X \geq R\}$$

we can directly apply Proposition 2.1. Otherwise, we can redefine

$$D_2 := \{x \in \operatorname{dom} L \mid \|\Pi x\|_X \leq a\} \cap \{x \in X \mid \|x - \Pi x\|_X \geq R\}$$

and apply Proposition 2.1 with this choice of D_2. In any case we have

$$Lx \neq (1-\lambda)\,\mathfrak{A}x + \lambda\mathfrak{N}x$$

for $(\lambda, x) \in [0,1] \times (D_1 \cup D_2) = [0,1] \times (\operatorname{dom} L \setminus \mathfrak{B})$, where $\mathfrak{B}$ is the bounded set

$$\mathfrak{B} = \{x \in \operatorname{dom} L \mid \|\Pi x\|_X \leq a, \quad \|x - \Pi x\|_X < R\}.$$

As particular cases, for $\lambda = 1$, we obtain that the solution set Σ is contained in $\mathfrak{B}$, and, for $\lambda = 0$, we obtain that $\ker(L - \mathfrak{A})$ is contained in $\mathfrak{B}$, so that it must be trivial. At last we fix Ω equal to any open ball containing $\mathfrak{B}$. Theorem 3.0 can be applied and we get that Σ is non-empty.

Theorem 3.3. *Suppose* (H0') *with* Π *continuous and* $\Pi(0) = 0$. *Assume that there exist positive numbers a, b such that* (H1) *and* (H2) *hold with*

$$D_1 = \{x \in \operatorname{dom} L \mid \|\Pi x\|_X = a, \quad \|x - \Pi x\|_X \leq b\},$$

$$D_2 = \{x \in \operatorname{dom} L \mid \|\Pi x\|_X \leq a, \quad \|x - \Pi x\|_X = b\},$$

and with

$$\alpha_1 < \nu_1 \leq 1,$$
$$\alpha_2 = \nu_2 = 0, \ \alpha_3 = \nu_3 = 0.$$

Let

$$\Omega = \{x \in X \mid \|\Pi x\|_X < a, \quad \|x - \Pi x\|_X < b\}.$$

Then there exist at least one $x \in \operatorname{dom} L \cap \operatorname{cl}\Omega$ *(so that* $\|x\|_X \leq a+b$*) such that*

$$Lx = \mathfrak{N}x.$$

Proof. First we use Proposition 2.2 with $R = b$, so that

$$Lx \neq (1-\lambda)\,\mathcal{A}x + \lambda \mathcal{N}x$$

whenever $(\lambda, x) \in]0,1[\times (\operatorname{dom} L \cap \operatorname{fr} \Omega)$. In order to apply Theorem 3.0 we have only to prove that the kernel of $L - \mathcal{A}$ is trivial. We claim that $\ker (L - \mathcal{A}) \cap \operatorname{fr} \Omega = \emptyset$, which implies the desired result. If we suppose that $x^* \in \operatorname{dom} L \cap \operatorname{fr} \Omega$ verifies $Lx^* = \mathcal{A}x^*$, we have that the pair

$$(\tfrac{1}{2}, x^*) \in]0,1[\times (\operatorname{dom} L \cap \operatorname{fr} \Omega)$$

verifies $Lx^* = (1 - \tfrac{1}{2})\,\mathcal{A}x^* + \tfrac{1}{2}\,\mathcal{A}x^*$, which is a contradiction with the conclusion of Proposition 2.2, used with $\mathcal{N} = \mathcal{A}$.

Remark 3.1. Proposition 2.2 can be applied to prove a version of Theorem 3.3 in which all solutions of $Lx = \mathcal{N}x$ are uniformly bounded (as Propositions 2.1 has been applied first to an existence result — Theorem 3.1 —, secondly to a version of this one — Theorem 3.2 — in which all possible solutions are uniformly bounded).

Remark 3.2. Further existence results for solutions to $Lx = \mathcal{N}x$, can be obtained by suitable definitions of D_1 and D_2 (according to more specific assumptions on L and $\mathcal{N}$). For instance, when $\mathcal{N}(X)$ is a bounded set, one can easily derive an existence result for abstract Landesman-Lazer type equations as in MAWHIN [19] (see also [3], [18],..., for different approaches to this problem).

4. Applications to abstract equations at resonance

We show in this section some applications of Theorem 3.2 and Theorem 3.3 to abstract equations at resonance. For the sake of simplicity we confine ourselves to the case in which the maps $\mathcal{A}'$ and $\mathcal{N}'$ are compact on bounded sets, that is, in the terminology of coincidence degree theory ([19], [20]), the case in which the maps $\mathcal{A}$ and $\mathcal{N}$ are L-compact on bounded sets. In this case the Leray-Schauder topological degree theory ensures the validity of (D1) - (D4).

We briefly discuss the case in which the kernel of L is trivial. In this case we necessarily have $P = 0$ and $Q = 0$. Therefore the map Π in (H0) or (H0') must be the constant 0. It follows that the set D_1, as defined in theorems 3.1, 3.2, 3.3 is empty, and that the set D_2 in theorems 3.1 and 3.3 consists of all points in $\operatorname{dom} L$ with a fixed positive norm b, whereas it coincides with the whole $\operatorname{dom} L$ in Theorem 3.2.

We can now summarize the results obtained in Section 3 in the particular case $\ker L = \{0\}$ as follows.

Let $L: \operatorname{dom} L \subseteq X \to Z$ be an invertible linear map, and let $\mathcal{N}: X \to Z$ be a map L-compact on bounded sets. We define $\mathcal{A}: X \to Z$ by setting $\mathcal{A} = 0$ (a natural choice due to the invertibility of L). Let $\langle \cdot , \cdot \rangle : Z \times \operatorname{dom} L \to \mathbf{R}$ be a bilinear form.

Corollary 4.1. *Assume that*

(i) there is a number $\beta > 0$ *such that* $\langle Lx, x \rangle \geqslant \beta \| x \|^2_X$ *for all* $x \in \operatorname{dom} L$,

(ii) there is a number $b > 0$ *such that*

$$\langle \mathcal{N}x, x \rangle \leqslant \nu_1 \langle Lx, x \rangle + \nu_2 \langle Lx, x \rangle^{1-\varepsilon} + \nu_3, \tag{4.1}$$

with $0 < \varepsilon < 1$, $0 \leqslant \nu_1 < 1$, $0 \leqslant \nu_2$, $0 \leqslant \nu_3$, *for all* $x \in \operatorname{dom} L$ *with* $\| x \|_X = b$,

(iii) $\psi(\beta b^2) > 0$.

Then there is at least one $x \in \operatorname{dom} L$ *with* $\| x \|_X < b$ *such that* $Lx = \mathcal{N}x$.

Moreover, if (4.1) *holds for all* $x \in \operatorname{dom} L$, *the assumption (iii) is not needed, being satisfied for all b sufficiently large.*

Corollary 4.2. *Assume that*

(i) $\langle Lx, x \rangle \geqslant 0$ *and* $\langle Lx, x \rangle = 0$ *implies* $x = 0$,

(ii) there is a number $b > 0$ *such that*

$$\langle \mathcal{N}x, x \rangle \leqslant \langle Lx, x \rangle,$$

for all $x \in \operatorname{dom} L$ *with* $\| x \|_X = b$.

Then there is a least one $x \in \operatorname{dom} L$ *with* $\| x \|_X \leqslant b$ *such that* $Lx = \mathcal{N}x$.

We remark that Corollary 4.1 can be used to prove the existence of a fixed point for a map $\mathcal{N} = ENE$, being X a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|_X$, $E: X \to X$ a linear, selfadjoint, completely continuous operator, and $N: X \to X$ a continuous map bounded on bounded sets.

In fact, using Corollary 4.1 with $L =$ *identity on* X, we obtain

Corollary 4.3. *Assume that there exist numbers* $\delta \in]0,2[$ *and* c_i $(i = 1,2,3)$ *such that, for every* $x \in X$,

$$\langle Nx, x \rangle \leqslant c_1 \| x \|^2_X + c_2 \| x \|^{2-\delta}_X + c_3 .$$

Suppose $c_1 \| E \|^2 < 1$. *Then the map ENE has a fixed point.*

For the proof, we have only to observe that

$$\langle ENEx, x\rangle = \langle NEx, Ex\rangle$$
$$\leqslant c_1 \| E \|^2 \langle x, x\rangle + c_2 \| E \|^{2-\delta} \langle x, x\rangle^{1-\delta/2} + c_3.$$

A direct proof of the existence of a fixed point of ENE can be found in MARTIN's book [14, IV.6] (see also AMANN [1]). This result has useful applications to the problem of the existence of solutions to integral Hammerstein equations with selfadjoint and positive kernels, since, under the same assumptions of Corollary 4.3, the map E^2N has a fixed point, namely Ex^*, where x^* is a fixed point of ENE.

Corollary 4.2 admits the following application.

Let X, Z be real Banach spaces with corresponding norms $\|\cdot\|_X, \|\cdot\|_Z$, and let H be a real Hilbert space with inner product $(\cdot\,,\cdot)_H$, such that the triple (Z, H, Z^*) is in normal position (see AMANN [1]) and $X \subseteq Z$ algebraically and topologically. Let $\langle z, z^*\rangle$ be the value of $z^* \in Z$ at $z \in Z$.

Corollary 4.4. *Let* $A\colon \operatorname{dom} A \subseteq X \to Z$ *be a linear operator such that*

(i) $\langle Ax, x\rangle \geqslant 0$ *for all* $x \in \operatorname{dom} A$,

(ii) there exists a number $c>0$ *such that the map*

$$L = A + cI : \operatorname{dom} A \subseteq X \to Z$$

is onto, has trivial kernel, and its inverse $L^{-1}: Z \to X$ *is completely continuous.*

Let $\mathfrak{M} : X \to Z$ *be a continuous map, bounded on bounded sets. Suppose that there exists* $b > 0$ *such that*

$$\langle Ax + \mathfrak{M}x, x\rangle \geqslant 0$$

whenever $x \in \operatorname{dom} A$ *and* $\| x \|_X = b$. *Then there is an* $x \in \operatorname{dom} A$ *with* $\| x \|_X \leqslant b$ *such that* $Ax + \mathfrak{M}x = 0$.

This result is equivalent to a result by WARD [26, Theorem 1], so that it contains the result of KANNAN and SCHUUR [10], [11]. To prove it, we set $L = A + cI$, $\mathfrak{N} = cI - \mathfrak{M}$, and we remark that $Ax + \mathfrak{M}x = 0$ and $Lx = \mathfrak{N}x$ have the same set of solutions. For any $x \in \operatorname{dom} L$ we have

$$\langle Lx, x\rangle = \langle Ax, x\rangle + c\langle x, x\rangle = \langle Ax, x\rangle + c(x, x)_H$$

so that we can apply directly Corollary 4.2.

We shall apply now the results of Section 3 to some cases in which the kernel of L is not necessarely trivial. More precisely we shall discuss two cases in which the map $\mathfrak{N}$ can be decomposed into a sum $\mathfrak{N}_1 + \mathfrak{N}_2$, where $\mathfrak{N}_1$ has its range contained into the image of L whereas $\mathfrak{N}_2$ is a suitable map, linear in the first case, nonlinear but quasibounded in the second one.

Let the spaces X, Z, the Fredholm mapping L, the projections P, Q, be as in Section 3. Let $\mathfrak{F}, \mathfrak{A}, \mathfrak{G} : X \to Z$ be mappings L-compact on bounded sets. Suppose that $\mathfrak{A}$ is linear, and that $\mathfrak{G}$ is quasibounded, i.e. the quasinorm of $\mathfrak{G}$

$$\| \mathfrak{G} \| = \inf\{c \geq 0 \mid (\exists\, b \geq 0)\ (\forall x \in X)\ \| \mathfrak{G}x \|_Z \leq c \| x \|_X + b\}$$

is finite. Let e be a fixed element in $\operatorname{Im} L$. We look for solutions $x \in \operatorname{dom} L$ to the abstract equations at resonance

$$Lx = (I - Q)\, \mathfrak{F}x + \mathfrak{A}x + e \qquad (4.2)_L$$

and

$$Lx = (I - Q)\, \mathfrak{F}x + \mathfrak{G}x + e. \qquad (4.2)_N$$

Theorem 4.1. *Suppose that*

(i) $\ker Q\mathfrak{A} = \ker P$,

(ii) there is a continuous bilinear form

$$\langle \cdot , \cdot \rangle : \operatorname{Im} L \times (\operatorname{dom} L \cap \ker P) \to \mathbf{R}$$

such that $u \to \langle Lu, u \rangle$ *is coercive on* $\operatorname{dom} L \cap \ker P$ *and the inequality* $\langle (I - Q)\, \mathfrak{F}u, u \rangle \leq \mu_1 \langle Lu, u \rangle + \mu_2 \langle Lu, u \rangle^{1-\varepsilon} + \mu_3$ *holds with* $0 < \varepsilon < 1$ *and* $\mu_i \geq 0$ $(i = 1,2,3)$ *for all* $u \in \operatorname{dom} L \cap \ker P$,

(iii) $\alpha + \mu_1 < 1$, *where*

$$\alpha = \sup\{\langle Lu, u \rangle^{-1} \langle \mathfrak{A}u, u \rangle \mid u \in \operatorname{dom} L \cap \ker P,\ \| u \|_X = 1\}.$$

Then the set of all solutions $x \in \operatorname{dom} L$ *to* $(4.2)_L$ *is non-empty, bounded, and contained in* $\ker P$.

Proof. The map $\mathfrak{N} : X \to Z$ defined by $\mathfrak{N}x = (I - Q)\, \mathfrak{F}x + \mathfrak{A}x + e$ is L-compact on bounded sets. Let us apply Theorem 3.2. To verify (H0) we choose $\Pi = P$; the existence of $\beta > 0$ such that, for all $u \in \operatorname{dom} L \cap \ker P$,

$$\langle Lu, u \rangle \geq \beta \| u \|^2_X \qquad (4.3)$$

follows from the coercivity assumption. Let us choose $a = 0$. To verify (H1) we observe that $D_1 = \operatorname{dom} L \setminus \ker P$, and we define $\varphi : D_1 \to (\operatorname{Im} Q)^*$ by setting, for all $\varepsilon \in \operatorname{Im} Q$ and $x \in D_1$,

$$(\varepsilon, \varphi x) := \Sigma_j\, \varepsilon_j\, (Q\mathfrak{A}x)_j,$$

where η_j is the j-th coordinate of $\eta \in \operatorname{Im} Q$ with respect to a fixed finite-dimensional orthonormal basis in $\operatorname{Im} Q$. If $x \in D_1$ we have

$$(Q\mathfrak{A}x, \varphi x)\ (Q\mathfrak{N}x, \varphi x)$$

$$= (Q\mathfrak{A}x, \varphi x)\ (Q(I - Q)\, \mathfrak{F}x + Q\mathfrak{A}x + Qe, \varphi x)$$

$$= (Q\mathcal{A}x, \varphi x)^2$$
$$= (\Sigma_j (Q\mathcal{A}x)_j^2)^2 \geqslant 0.$$

The assumption *(i)* implies $Q\mathcal{A}x \neq 0$, being $Px \neq 0$. Therefore (H1) holds.

To verify (H2) we observe first that $D_2 = \operatorname{dom} L \cap \ker P$. For any $u \in D_2$, using *(i)*, we have

$$\begin{aligned} \langle (I-Q)\,\mathcal{A}u, u\rangle &= \langle \mathcal{A}u, u\rangle - \langle Q\mathcal{A}u, u\rangle \\ &= \langle \mathcal{A}u, u\rangle \leqslant \alpha \langle Lu, u\rangle \end{aligned} \tag{4.4}$$

and so, using *(ii)* and (4.4),

$$\begin{aligned} &\langle (I-Q)\,\mathcal{N}u, u\rangle \\ &\quad = \langle (I-Q)^2 \mathcal{F}u, u\rangle + \langle (I-Q)\,\mathcal{A}u, u\rangle + \langle (I-Q)\,e, u\rangle \\ &\quad = \langle (I-Q)\,\mathcal{F}u, u\rangle + \langle \mathcal{A}u, u\rangle + \langle e, u\rangle \\ &\quad \leqslant (\mu_1 + \alpha)\,\langle Lu, u\rangle + \mu_2 \langle Lu, u\rangle^{1-\varepsilon} + \langle e, u\rangle. \end{aligned} \tag{4.5}$$

Inequality *(iii)* allows us to fix a number $\zeta > 0$ such that

$$(\mu_1 + \alpha) + \zeta\beta^{-1/2} s^* \| e \|_Z < 1 \tag{4.6}$$

where $s^* = \sup\{ |\langle v, w\rangle| \mid \| w \|_X = \| v \|_Z = 1\}$. Let $K(\zeta)$ be a number such that $t^{1/2} \leqslant K(\zeta) + \zeta t$ for any real $t \geqslant 0$. From (4.5) we obtain, using (4.3),

$$\begin{aligned} \langle (I-Q)\,\mathcal{N}u, u\rangle &\leqslant (\mu_1 + \alpha)\,\langle Lu, u\rangle + \mu_2 \langle Lu, u\rangle^{1-\varepsilon} + s^* \| e \|_Z \| u \|_X \\ &\leqslant \big((\mu_1 + \alpha) + \zeta\beta^{-1/2} s^* \| e \|_Z \big)\,\langle Lu, u\rangle + \mu_2 \langle Lu, u\rangle^{1-\varepsilon} \\ &\quad + \big(\beta^{-1/2} s^* \| e \|_Z K(\zeta) + \mu_3\big). \end{aligned}$$

Therefore (H2) holds. Using (4.6) we get the result.

Theorem 4.2. *Suppose that*

(i) the assumption (H0) *holds,*

(ii) there is a number $\omega > 0$ *such that, for any* $u \in \operatorname{dom} L \cap \ker P$ *and* $z \in \operatorname{Im} L$ *with* $\| z \|_Z = 1$,

$$\langle Lu, u\rangle^{1/2} \geqslant \omega \langle z, u\rangle,$$

(iii) there is a linear map $\mathcal{A}^* : X \to Z$, *L-compact on bounded sets, a number* $a > 0$ *and a map*

$$\varphi : D_1 = \{x \in \operatorname{dom} L \mid \| \Pi x \|_X > a\} \to (\operatorname{Im} Q)^*$$

such that, for every $x \in D_1$,

$$(Q\mathcal{A}^* x, \varphi x)\ (Q\mathcal{G}x, \varphi x) > 0,$$

(iv) there are numbers $\mu_i \geqslant 0$ $(i=1,2,3)$ *such that, for any* x *in the set*

$$D_2 = \{x \in \operatorname{dom} L \mid \|\Pi x\|_X \leqslant a\}$$

one has

$$\langle (I-Q)\,\mathcal{F}x, u\rangle \leqslant \mu_1 \langle Lu, u\rangle + \mu_2 \langle Lu, u\rangle^{1/2} + \mu_3,$$

(v) $\omega^{-1}\beta^{-1/2}\|\mathcal{G}\| + \mu_1 < 1.$

Then the set of all solutions $x \in \operatorname{dom} L$ *to* $(4.2)_N$ *is non-empty and bounded.*

Proof. We shall apply Theorem 3.2, choosing

$$\mathcal{N} : X \to Z,\ \mathcal{N}x = (I-Q)\,\mathcal{F}x + \mathcal{G}x + e,$$
$$\mathcal{A} : X \to Z,\ \mathcal{A}x = \delta(1 + \|\mathcal{A}^*\|)^{-1}\mathcal{A}^*x,$$

where δ is a positive number which will be fixed later. Now (H0) is an assumption. Consider *(iii)*: if $x \in D_1$ we have

$$(Q\mathcal{A}x, \varphi x)\ (Q\mathcal{N}x, \varphi x)$$
$$= \delta(1 + \|\mathcal{A}^*\|)^{-1}\,(Q\mathcal{A}^*x, \varphi x)\ (Q\mathcal{G}x, \varphi x) > 0.$$

Therefore (H1) holds for each $x \in D_1$. Consider *(iv)*: for any $x \in D_2$, using *(ii)*, we have

$$\begin{aligned}
&\langle (I-Q)\,\mathcal{A}x, u\rangle \\
&\quad\leqslant \omega^{-1}\|(I-Q)\,\mathcal{A}x\|_Z \langle Lu, u\rangle^{1/2} \\
&\quad\leqslant \omega^{-1}\|\mathcal{A}\|\,(\|x - \Pi x\|_X + \|\Pi x\|_X)\,\langle Lu, u\rangle^{1/2} \\
&\quad\leqslant \omega^{-1}\delta(1 + \|\mathcal{A}^*\|)^{-1}\|\mathcal{A}^*\|\,\beta^{-1/2}\langle Lu, u\rangle \\
&\qquad + \omega^{-1}\delta(1 + \|\mathcal{A}^*\|)^{-1}\|\mathcal{A}^*\|\,a\,\langle Lu, u\rangle^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
&\langle (I-Q)\,\mathcal{N}x, u\rangle \\
&\quad= \langle (I-Q)^2\mathcal{F}x, u\rangle + \langle (I-Q)\,\mathcal{G}x, u\rangle + \langle e, u\rangle \\
&\quad\leqslant \mu_1\langle Lu, u\rangle + \mu_2\langle Lu, u\rangle^{1/2} + \mu_3 \\
&\qquad + \omega^{-1}\|(I-Q)\,\mathcal{G}x\|_Z\langle Lu, u\rangle^{1/2} + \omega^{-1}\|e\|_Z\langle Lu, u\rangle^{1/2}.
\end{aligned}$$

But now there is a number $K(\delta)$ such that

$$\begin{aligned}
&\|(I-Q)\,\mathcal{G}x\|_Z \leqslant \|\mathcal{G}x\|_Z \\
&\quad\leqslant (\|\mathcal{G}\| + \delta)\,\|x\|_X + K(\delta) \\
&\quad\leqslant (\|\mathcal{G}\| + \delta)\,(\|x - \Pi x\|_X + K(\delta) \\
&\quad\leqslant (\|\mathcal{G}\| + \delta)\,\beta^{-1/2}\langle Lu, u\rangle^{1/2} + (\|\mathcal{G}\| + \delta)\ a + K(\delta).
\end{aligned}$$

Therefore (H2) holds for any $x \in D_2$ with

$$\begin{aligned}
\varepsilon &= 1/2 \\
\alpha_1 &= \delta\omega^{-1}\,(1 + \|\mathcal{A}^*\|)^{-1}\|\mathcal{A}^*\|\,\beta^{-1/2}, \\
\nu_1 &= \mu_1 + \omega^{-1}\,(\|\mathcal{G}\| + \delta)\,\beta^{-1/2} = \omega^{-1}\beta^{-1/2}\|\mathcal{G}\| + \mu_1 + \delta\omega^{-1}\beta^{-1/2}.
\end{aligned}$$

Obviously, we can choose a priori the number δ so small that $\max\{\alpha_1, \nu_1\} = \nu_1$ (in virtue of $(1+\|\mathcal{A}^*\|)^{-1}\|\mathcal{A}^*\| < 1$), and, simultaneously, $\nu_1 < 1$ (in virtue of the inequality (ν)). The theorem is proved.

In the following section we shall show some applications of the preceding existence theorems for abstract equations to the problem of existence of periodic solutions to vector differential equations with deviating arguments.

5. Applications to the existence of periodic solutions for differential delay equations

Let $\mathbf{R}^N$ $(N \geqslant 1)$ be the N-dimensional real euclidean space, with euclidean inner product $(\cdot|\cdot)$ and corresponding norm $|\cdot|$. Let $T > 0$ be fixed and let $\omega := 2\pi/T$. Let us define the spaces H^k. We first define, for $k \geqslant 0$, the real vector spaces

$$\mathcal{C}^k_T := \{x : \mathbf{R} \to \mathbf{R}^N \mid x \text{ is } T\text{-periodic and of class } \mathcal{C}^k\}.$$

Next we define on $\mathcal{C}^0_T$ the inner product

$$(x|y)_2 := T^{-1}\int_0^T (x(t)\,|\,y(t))\,dt$$

and the corresponding norm $|x|_2 := (x|x)_2^{1/2}$. We then define

$$H^0_T = \text{the completion of } \mathcal{C}^0_T \text{ with respect to } |\cdot|_2;$$

(H^0_T is canonically isomorphic to the Hilbert space $L^2([0,T],\mathbf{R}^N)$).

For any $k \geqslant 1$ we denote by H^k_T the vector subspace of H^0_T consisting of all mappings $x \in \mathcal{C}^{k-1}_T$ with $x^{(k-1)}$ absolutely continuous and $x^{(k)} \in H^0_T$. If $x \in H^0_T$ we shall write simply $\int x$ instead of $\int_0^T x(t)\,dt$ and $\bar{x}$ will denote the mean value $T^{-1}\int x$ of x.

We recall that, for any $x \in H^1_T$,

$$|x|_2^2 \leqslant \omega^{-2}|x'|_2^2 + |\bar{x}| \tag{5.1}$$

(the ALMANSI - TONELLI - WIRTINGER inequality: see [24]),

$$\sup\{|x(t)| \mid t \in \mathbf{R}\} \leqslant 3^{-1/2}\pi\omega^{-1}|x'|_2 + |\bar{x}| \tag{5.2}$$

(a particular case of an inequality due to CESARI: see [15, I.1.4]).

Let $\mathcal{M}_N$ be the real algebra of all $N \times N$ real constant matrices. If $A = [a_{ij}] \in \mathcal{M}_N$ then $\operatorname{diag}(A)$ is the vector $(a_{11}, a_{22}, \ldots, a_{NN})$, i.e. the principal diagonal of A, A^T is the transpose of A and $\operatorname{spec}(A)$ is the spectrum of A. Moreover, we introduce the norms

$\|A\| := (\max \operatorname{spec}(A^T A))^{1/2}$ (the spectral norm of A),

$\| A \|_1 := \max_j \Sigma_i | a_{ij} |$ (the maximum absolute column sum norm of A),
$\| A \|_e := (\mathrm{N}^{-1} \Sigma_{ij} | a_{ij} |^2)^{1/2}$ (the normalized euclidean norm of A).

It can be easily seen that, for all $A, B \in \mathfrak{M}_\mathrm{N}$,

$$| \mathrm{diag}(AB) | \leqslant \mathrm{N}^{1/2} \| A \| \; \| B \|_e . \tag{5.3}$$

If $x \in \mathcal{C}^0_T$, $t \in \mathbf{R}$ and $\tau = [\tau_{ij}] \in \mathfrak{M}_\mathrm{N}$, then we define

$$x(t - \tau) = [x_j(t - \tau_{ij})],$$

i.e. $x(t - \tau)$ is the matrix whose (i, j)-entry is $x_j(t - \tau_{ij})$. We remark that, if $A = [a_{ij}] \in \mathfrak{M}_\mathrm{N}$, the i-th component of the vector

$$\mathrm{diag}\, Ax(t - \sigma)^\mathrm{T}$$

is

$$a_{i1} x_1(t - \sigma_{i1}) + a_{i2} x_2(t - \sigma_{i2}) + \ldots + a_{iN} x_N(t - \sigma_{iN}) \; (i = 1, \ldots, \mathrm{N}).$$

Therefore, if $\sigma = 0$, we have

$$\mathrm{diag}\, Ax(t - 0)^\mathrm{T} = Ax(t).$$

Let us introduce now a map $F : \mathbf{R}^\mathrm{N} \to \mathbf{R}$ of class $\mathcal{C}^2$, a map $V : \mathbf{R} \times \mathbf{R}^\mathrm{N} \to \mathbf{R}^\mathrm{N}$ of class $\mathcal{C}^1$ and T-periodic in the first variable, a (possibly singular) real $\mathrm{N} \times \mathrm{N}$ constant matrix $B = [b_{ij}]$ and a map $h \in H^0_T$.

In both the following Theorems 5.1 and 5.2 we will assume:

(j) $\bar{h} = 0$;

(jj) *there exists a continuous T-periodic map* $\ell : \mathbf{R} \to \mathbf{R}$ *and a number* $k \geqslant 0$ *such that, for all* $(t, x) \in \mathbf{R} \times \mathbf{R}^\mathrm{N}$,
$$| V(t, x) | \leqslant \ell(t) | x | + k .$$

The first result in this section is the following

Theorem 5.1. *Assume (j), (jj) and uppose that*

$$b(V)\, \omega + a(B) < \omega^2,$$

where

$$b(V) := \min \{ 3^{-1/2} \pi \, | \ell |_2 , \sup | \ell(\cdot) | \},$$

$$a(B) := \begin{cases} \max \left(\mathrm{spec}\,(1/2\; (B + B^\mathrm{T})) \cup \{0\} \right), & \text{when } \sigma = 0, \\ (\Sigma_{ij} | b_{ij} |^2)^{1/2}, & \text{otherwise.} \end{cases}$$

Then there exists at least one $x \in H^2_T$ *such that* $\bar{x} = 0$ *and*

$$x'' + (d/dt)\, (\nabla F(x) + V(t, x)) + \mathrm{diag}\, Bx(t - \sigma)^T = h(t). \tag{5.4}$$

Proof. We shall apply Theorem 4.1. Let us fix a number

$$\theta \in [0, (\omega^2 - (b(V)\,\omega + a(B)))/2\, [\setminus \operatorname{spec}(B).$$

Obviously, a map $x \in H^2_T$ is a solution of (5.4) with mean value $\bar{x}=0$ if and only if

$$x'' + (d/dt)\,(\nabla F(x) + V(t,x)) + \theta \operatorname{diag}(x - \bar{x})\,(t - \sigma)$$
$$+ \operatorname{diag}(B - \theta I)\,x(t - \sigma)^{\mathrm{T}} = h(t) \tag{5.5}$$

and $\bar{x} = 0$. Therefore, if we define

$X := H^1_T$, with norm $\|x\|_X := \max\{|x|_2, |x'|_2\}$,

$Z := H^0_T$, with norm $\|z\|_Z := |z|_2$,

$\operatorname{dom} L := H^2_T$, $Lx := -x''$,

$P : X \to X$, $Q : Z \to Z$, $Px := \bar{x}$, $Qz := \bar{z}$,

$\mathcal{A} : X \to Z$, $\mathcal{A}x := \operatorname{diag}(B - \theta I)\,x(t - \sigma)^{\mathrm{T}}$,

$\mathcal{F} : X \to Z$, $\mathcal{F}x := (d/dt)\,(\nabla F(x) + V(t,x)) + \theta \operatorname{diag} x(t - \sigma)$,

$e := -h(\cdot)$,

we obtain that $x \in H^2_T$ is a solution of (5.4) with $\bar{x} = 0$ if and only if $x \in \operatorname{dom} L \cap \ker P$ and

$$Lx = (I - Q)\,\mathcal{F}x + \mathcal{A}x + e. \tag{5.6}$$

Now it is a classical fact that L is a linear Fredholm map of index zero with $\ker L = \operatorname{Im} P$, $\operatorname{Im} L = \ker Q$.

The right inverse of L

$$L^+ : \operatorname{Im} L \to X$$

is completely continuous. Hence in order to prove the L-compactness on bounded sets of $\mathcal{A}, \mathcal{F}$, we have only to check that $\mathcal{A}, \mathcal{F}$ are continuous and map bounded sets into bounded sets. This easily follows from the chain rule, the continuty of $(\partial/\partial x)\,(\nabla F(x))$, $(\partial/\partial t)\,V(t,x)$, $(\partial/\partial x)\,V(t,x)$, and the fact that the topology induced on H^1_T by its own norm is finer than the topology induced by the uniform convergence.

Clearly $e \in \operatorname{Im} L = \ker Q$.

Let us verify assumptions *(i) – (iii)* of Theorem 4.1.

(i) - Let $x \in H^1_T$. We have the *i*-th component of $Q\mathcal{A}x$ *is*

$$(Q\mathcal{A}x)_i = T \textstyle\int (\mathcal{A}x)_i = T^{-1} \int \Sigma_k\,(b_{ik} - \theta)\,x_k(\cdot - \sigma_{ik})$$
$$= \Sigma_k (b_{ik} - \theta)\,T^{-1} \textstyle\int x_k(\cdot) = ((B - \theta I)\,\bar{x})_i,$$

that is

$$Q\mathcal{A}x = B\bar{x} - \theta\bar{x}.$$

The choice $\theta \notin \operatorname{spec}(B)$ implies that, for any $x \in H^1_T$, $B\bar{x} - \theta\bar{x} = 0$

if an only if $\bar{x} = 0$, that is $Q\mathcal{A}x = 0$ if and only if $Px = 0$.

In order to check *(ii)* and *(iii)*, we shall use the following two estimates.

First we evaluate, for any $u \in H^1_T$ with $\bar{u} = 0$.

$$(\operatorname{diag} u(\cdot - \sigma) \mid u)_2 = T^{-1} \int \Sigma_i u_i(\cdot - \sigma_{ii}) u_i(\cdot)$$
$$\leqslant (T^{-1} \int \Sigma_i \mid u_i(\cdot - \sigma_{ii}) \mid^2)^{1/2} \mid u \mid_2 = \mid u \mid_2^2.$$

Moreover, let

$$v = (v_1, v_2, \ldots, v_N) \in H^1_T,$$
$$v_i(\cdot) := \Sigma_j b_{ij} u_j(\cdot - \sigma_{ij}) \qquad (i = 1, \ldots, N).$$

Then we have

$$(\operatorname{diag} B\, u(\cdot - \sigma)^T \mid u)_2 = (v \mid u)_2$$
$$\leqslant \mid v \mid_2 \mid u \mid_2$$
$$= (T^{-1} \int \Sigma_i \mid \Sigma_j b_{ij} u_j(\cdot - \sigma_{ij}) \mid^2)^{1/2} \mid u \mid_2$$
$$\leqslant (T^{-1} \int \Sigma_i (\Sigma_j \mid b_{ij} \mid^2)(\Sigma_j \mid u_j(\cdot - \sigma_{ij}) \mid^2))^{1/2} \mid u \mid_2$$
$$= (T^{-1} \Sigma_i (\Sigma_j \mid b_{ij} \mid^2)(\Sigma_j \int \mid u_j(\cdot) \mid^2))^{1/2} \mid u \mid_2$$
$$= (\Sigma_{ij} \mid b_{ij} \mid^2)^{1/2} \mid u \mid_2^2.$$

(ii). - We define $\langle \cdot, \cdot \rangle := (\cdot|\cdot)_2$. The inequality (5.1) implies

$$\langle Lu, u \rangle = \mid u' \mid_2^2 \geqslant (\min\{1, \omega\})^2 \cdot \| u \|_X^2$$

for any $u \in \operatorname{dom} L \cap \ker P$, i.e. $u \to \langle Lu, u \rangle$ is coercive on $\operatorname{dom} L \cap \ker P$.

Moreover, using both (5.1) and (5.2) we have, for each $u \in \operatorname{dom} L \cap \ker P$,

$$\langle (I - Q)\mathcal{F}u, u \rangle$$
$$= T^{-1} \int ((d/dt) \nabla F(u) \mid u) + T^{-1} \int ((d/dt) V(t,u) \mid u)$$
$$+ T^{-1} \int (\theta \operatorname{diag} u(\cdot - \sigma) \mid u)$$
$$= 0 - T^{-1} \int (V(t,u) \mid u') + \theta T^{-1} \int (\operatorname{diag} u(\cdot - \sigma) \mid u)$$
$$\leqslant T^{-1} \int \ell(\cdot) \mid u \mid \mid u' \mid + T^{-1} \int k \mid u' \mid + \theta \mid u \mid_2^2$$
$$\leqslant (\min\{3^{-1/2} \pi \omega^{-1} \mid \ell \mid_2, \omega^{-1} \sup \mid \ell(\cdot) \mid\} + \theta \omega^{-2}) \mid u' \mid_2^2 + k \mid u' \mid_2$$
$$= (b(V)\omega^{-1} + \theta \omega^{-2}) \langle Lu, u \rangle + k \langle Lu, u \rangle^{1/2}.$$

Therefore we can choose $\varepsilon = 1/2$ and

$$\mu_1 = b(V)\omega^{-1} + \theta\omega^{-2}, \ \mu_2 = k, \mu_3 = 0.$$

(iii). - Let us estimate α. For every $u \in \operatorname{dom} L \cap \ker P$ we have, for $\sigma \neq 0$,

$$\langle \mathcal{A}u, u \rangle$$
$$= (\operatorname{diag} Bu(\cdot - \sigma)^T \mid u)_2 - \theta (\operatorname{diag} u(\cdot - \sigma) \mid u)_2$$

$$\leqslant (\Sigma_{ij} | b_{ij} |^2)^{1/2} | u |_2^2 + \theta | u |_2^2$$
$$\leqslant \big(a(B)\omega^{-2} + \theta\big) | u' |_2^2 = \big(a(B)\omega^{-2} + \theta\big) \langle Lu, u \rangle.$$

If $\sigma = 0$ we obtain

$$\langle \mathcal{A}u, u \rangle$$
$$= (Bu|u)_2 - \theta(u | u)_2 = (\tfrac{1}{2}(B + B^T) u | u)_2 - \theta | u |_2^2$$
$$\leqslant a(B) | u |_2^2 + 0 \leqslant a(B)\omega^{-2} | u' |_2^2$$
$$\leqslant \big(a(B)\omega^{-2} + \theta\big) \langle Lu, u \rangle,$$

so that $\alpha \leqslant a(B)\omega^{-2} + \theta$. By virtue of the choice of θ we obtain

$$\alpha + \mu_1 \leqslant \big(b(V)\omega + a(B) + 2\theta\big)\omega^{-2} < 1.$$

We conclude that (5.6) has at least one solution $x \in \operatorname{dom} L \cap \ker P$.

Remark 5.1. Theorem 4.1 shows also that (under the assumptions of Theorem 5.1) the set of all possible solutions $x \in H_T^2$ of (5.4) with mean value zero is bounded in H_T^1. More precisely, if $x \in H_T^2$ is a solution of (5.4) with $\bar{x} = 0$, one has

$$| x' |_2 \leqslant \rho^* := \big(\omega^{-2} - (b(V)\omega + a(B))\big)^{-1} \cdot \big(k\omega^2 + | h |_2 \omega\big),$$

from which one can easily obtain bounds for $| x |_2$ and $\sup | x(\cdot) |$, namely $| x |_2 \leqslant \rho^* \omega^{-1}$ and $\sup | x(\cdot) | \leqslant 3^{-1/2} \pi \omega^{-1} \rho^*$.

Of course, (5.4) may have an unbounded set of solutions, actually $\ker B$, if $\sigma = 0$, $F = 0$, $V = 0$, $h = 0$, and B is singular.

Remark 5.2. Our Theorem 5.1 extends a result of MAWHIN [17, Theorem 2], where V is identically zero, $B = [b_{ij}]$ is nonsingular and either it is negative definite or $(\Sigma_{ij} | b_{ij} |^2)^{1/2} < \omega^2$.

Remark 5.3. The inequality $b(V)\omega + a(B) < \omega^2$ is «sharp». Indeed, if $N = 2$, let $T := 2\pi$, and

$$\sigma := \begin{bmatrix} \pi & \pi/2 \\ 3\pi/2 & \pi \end{bmatrix}, \quad B := \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}, \quad F := 0, \ V := 0.$$

The system $x'' + \operatorname{diag} Bx(t - \sigma)^T = 0$, i.e.,

$$\begin{cases} x''_1 - \tfrac{1}{2} x_1 (t - \pi) + \tfrac{1}{2} x_2 (t - \pi/2) = 0 \\ x''_2 + \tfrac{1}{2} x_1 (t - 3\pi/2) - \tfrac{1}{2} x_2 (t - \pi) = 0 \end{cases}$$

has the nontrivial solution $x = (\cos(t), -\sin(t))$. Therefore (Fredholm Alternative) there exists $h \in H_{2\pi}^0$ with $\bar{h} = 0$ such that

$$x'' + \operatorname{diag} Bx(t - \sigma)^T = h$$

does not have a solution in $H_{2\pi}^2$. Being $b(V) = 0$, one has

$$a(B) = \omega^2 \qquad (\text{i.e. } | -1/2 |^2 + | 1/2 |^2 + | 1/2 |^2 + | -1/2 |^2 = 1).$$

We remark also that spec $(B) = \{-1, 0\}$ and $B = B^T$. This shows that, in the case $\sigma \neq 0$, the condition on $(\Sigma |b_{ij}|^2)^{1/2}$ cannot be substituted (as in the case $\sigma = 0$) with the negative (semi-)definiteness of B. Actually, in the present case, the problem

$$x'' + Bx = 0, \quad x \in H^2_{2\pi},$$

is uniquely solvable.

A second example concerns the case $B = 0$. Let $T = 2\pi$, *and*

$$\sigma := 0,\ B := 0,\ F := 0,\ V(\cdot, x) := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x.$$

One can immediately check that the system

$$x'' + (d/dt)\, V(t,x) = (\sin(t), \cos(t))$$

does not have 2π-periodic solutions. However, being $a(B) = 0$, one has

$$b(V)\omega = \omega^2.$$

A third example, with $B \neq 0$, $V \neq 0$, and $\sigma = 0$, can be found in [8, Remark 3.4].

Remark 5.4. In order to illustrate the definition of $b(V)$, let us look for 2-periodic solutions $x \in H^2_2$ with mean value zero to the scalar equation

$$x'' + (d/dt)\,(v_n(t)x) + ax = h(t),$$

where $h \in H^0_2$, $a \in \mathbf{R}$, and, for each $n = 1, 2, \ldots$, v_n is 2-periodic of class $\mathcal{C}^1$, and satisfies

$v_n(t) = 0$ for $t \in [0, 1 - (1/n)] \cup [1 + (1/n), 2]$,
$v_n(1) = n^{1/2}$
$0 \leqslant v_n(t) \leqslant n^{1/2}$ for $t \in [0, 2]$.

For each n we have $\sup |v_n(\cdot)| = n^{1/2}$, $|v_n|_2 \leqslant 1$, so that

$$b(V) = \min\{3^{-1/2}\pi\, |v_n|_2, \sup |v_n(\cdot)|\} \leqslant \min\{3^{-1/2}\pi, n^{1/2}\}.$$

More precisely,

$b(V) = n^{1/2}$, when $n \leqslant 3$,
$b(V) \leqslant 3^{-1/2}\pi$, when $n \geqslant 4$.

Hence the possible use of just one norm in the definition ob $b(V)$ would have restricted the range of application of Theorem 5.1.

Remark 5.5. If h is continuous, then Theorem 5.1 proves the existence of a classical (i.e. of class $\mathcal{C}^2$) T-periodic solution of (5.4).

We shall consider now the case in which the linear «restoring field» in equation (5.4) is substituted by a nonlinear term with delayed arguments.

Let $g : \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$ be a continuous map, T-periodic in the first variable. If $A = [a_{ij}]$, $A' = [a'_{ij}]$, and $t \in \mathbf{R}$, we define

$$\begin{aligned} g(t, A, A') := \big(& g_1(t; a_{11}, a_{12}, \dots, a_{1N}; a'_{11}, a'_{12}, \dots, a'_{1N}), \\ & g_2(t; a_{21}, a_{22}, \dots, a_{2N}; a'_{21}, a'_{22}, \dots, a'_{2N}), \\ & \dots\dots\dots\dots, \\ & g_N(t; a_{N1}, a_{N2}, \dots, a_{NN}; a'_{N1}, a'_{N2}, \dots, a'_{NN}) \big). \end{aligned}$$

We shall look for solutions $x \in H^2_T$ to the following delay differential equation

$$x'' + (d/dt)\,(\nabla F(x) + V(t,x)) + Bg(t, x(t-\sigma), x'(t-\sigma')) = h(t), \tag{5.7}$$

where $\sigma = [\sigma_{ij}]$ and $\sigma' = [\sigma'_{ij}]$ are fixed matrices of $\mathfrak{M}_N$.

In the sequel we shall relate the matrix σ with a fixed matrix $\sigma^* \in \mathfrak{M}_N$ choosen in such a way that

$$\sigma^* \in \mathfrak{M}^*_N, \quad \|\sigma - \sigma^*\|_1 = \min\{\|\sigma - \tau\|_1 \mid \tau \in \mathfrak{M}^*_N\},$$

where $\mathfrak{M}^*_N$ is the linear manifold of $\mathfrak{M}_N$ consisting of all matrices $\tau = [\tau_{ij}]$ such that $\tau_{ik} = \tau_{ih}$ for any i, h, k. Namely, $\tau \in \mathfrak{M}^*_N$ if and only if the columns of τ are equal one to each other. (For some properties of σ^*, see Remark 5.12 after Theorem 5.2).

At last, we introduce for any matrix $\tau = [\tau_{ij}] \in \mathfrak{M}_N$ (actually $\tau = \sigma$ or $\tau = \sigma'$) the sets

$$\mathfrak{J}(\tau) := \left\{ (h,k,j) \in \{1, 2, \dots, N\}^3 \mid \tau_{hj} = \tau_{kj} \right\},$$

$$\mathfrak{W}(\tau) := \left\{ A = [a_{ij}] \in \mathfrak{M}_N \mid a_{hj} = a_{kj} \text{ whenever } (h,k,j) \in \mathfrak{J}(\tau) \right\}.$$

We will prove

Theorem 5.2. *Suppose that (j) and (jj) of Theorem 5.1 hold. Moreover, assume that:*

(jjj) there exist an orthogonal matrix $U \in \mathfrak{M}_N$ and number $M > 0$ such that, for each index i,

$$g_i(t,x,y)\,(Ux)_i > 0$$

holds for all $(t,x,y) \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N$ with $|(Ux)_i| > M$;

(jv) there are numbers $b > 0$, $c > 0$, $d \geqslant 0$, $p \in [1,2]$ such that

$$|g(t,A,A')| \leqslant (b^p \|A\|^p_e + c^p \|A'\|^p_e)^{1/p} + d$$

holds for all $t \in \mathbf{R}$ and all $(A, A') \in \mathfrak{W}(\sigma) \times \mathfrak{W}(\sigma')$;

(v) the inequality

$$\left[\left(\frac{b}{\omega/\gamma}\right)^p + \left(\frac{c}{\omega}\right)^p\right]^{1/p} \| B \| < 1 - \gamma \sup | \ell(\cdot)|,$$

holds, with

$$\gamma = \gamma(\sigma) := 2\omega^{-1} + T^{1/2} \| \sigma - \sigma^* \|_1^{1/2}$$

Then there is at least one $x \in H_T^2$ *such that*

$$x'' + (d/dt)(\nabla F(x) + V(t,x)) + Bg(t, x(t-\sigma), x'(t-\sigma')) = h(t).$$

Proof. We use Theorem 4.2. We choose first a number

$$\theta \in [0,1[\setminus \operatorname{spec}(B)$$

to be specified later, and we define

$X := H_T^1$, with norm $\| x \|_X := (b^p | x |_2^p + c^p | x' |_2^p)^{1/p}$,

$Z := H_T^0$, with norm $\| z \|_Z := | z |_2$,

$\operatorname{dom} L := H_T^2$, $Lx := -x''$,

$P : X \to X$, $Q : Z \to Z$, $Px := \bar{x}$, $Qz := \bar{z}$,

$\Gamma : X \to Z$, $\Gamma x := g(\cdot, x(\cdot - \sigma), x'(\cdot - \sigma'))$,

$\mathcal{G} : X \to Z$, $\mathcal{G}x := B\Gamma x - \theta\Gamma x$,

$\mathcal{F} : X \to Z$, $\mathcal{F}x := (d/dt)(\nabla F(x) + V(t,x)) + \theta\Gamma x$,

$e := -h(\cdot)$.

We need only to show that the abstract equation

$$Lx = (I - Q)\mathcal{F}x + \mathcal{G}x + e \tag{5.8}$$

has a solution $x \in \operatorname{dom} L$. In fact, if $x \in \operatorname{dom} L$ verifies (5.8), then $x \in H_T^2$ and the equality

$$x'' + (d/dt)(\nabla F(x) + V(t,x)) + \theta\Gamma x - \theta\overline{\Gamma x} + B\Gamma x - \theta\Gamma x = h(t) \tag{5.9}$$

holds. Integrating both members of (5.9) on $[0, T]$ we obtain

$$B \int \Gamma x = \theta \int \Gamma x.$$

Since $\theta \notin \operatorname{spec}(B)$, we necessarily have $\int \Gamma x = 0$, so that (5.9) becomes exactly (5.7). In order to prove that (5.8) has a solution $x \in \operatorname{dom} L$, we shall apply Theorem 4.2.

We prove first that the inequalities

$$\| \Gamma x \|_Z \leqslant \| x \|_X + d \tag{5.10}$$

$$\| \mathcal{G}x \|_Z \leqslant (\| B \| + \theta) \| x \|_X + (\| B \| + \theta) d \tag{5.11}$$

hold for all $x \in X$. In fact, if $x \in X$ we easily see that

$$| \| x(\cdot - \sigma) \|_e |_2^2 = T^{-1} \int \| x(\cdot - \sigma) \|_e^2$$

$$= T^{-1}\int N^{-1}\Sigma_{ij}|x_j(\cdot-\sigma_{ij})|^2 = T^{-1}N^{-1}\Sigma_{ij}\int|x_j(\cdot-\sigma_{ij})|^2$$
$$= T^{-1}N^{-1}\Sigma_{ij}\int|x_j(\cdot)|^2 = T^{-1}N^{-1}N\int\Sigma_j|x_j(\cdot)|^2$$
$$= T^{-1}\int|x(\cdot)|^2 = |x|_2^2,$$

and, similarly, that

$$|\,\|x'(\cdot-\sigma')\|_e\,|_2^2 = |x'|_2^2.$$

We remark that, if $\sigma_{hj} = \sigma_{kj}$ for some triple (h, k, j) of indexes, then $x_j(\cdot - \sigma_{hj}) = x_j(\cdot - \sigma_{kj})$; equivalently, we have, for all t, $x(t-\sigma) \in \mathfrak{W}(\sigma)$, and, similarly, $x'(t-\sigma') \in \mathfrak{W}(\sigma')$.

Therefore we can use assumption (jv) to evaluate $\|\Gamma x\|_Z$. Actually, we have

$$\|\Gamma x\|_Z = \left(T^{-1}\int|g(\cdot, x(\cdot-\sigma), x'(\cdot-\sigma'))|^2\right)$$
$$\leqslant \left(T^{-1}\int(b^p\|x(\cdot-\sigma)\|_e^p + c^p\|x'(\cdot-\sigma')\|_e^p)^{2/p}\right) + d$$
$$\leqslant\left[\left(T^{-1}\int|b^p\|x(\cdot-\sigma)\|_e^p|^{2/p}\right)^{p/2} + \left(T^{-1}\int|c^p\|x'(\cdot-\sigma')\|_e^p|^{2/p}\right)^{p/2}\right]^{1/p} + d$$
$$= \left[b^p\left(T^{-1}\int\|x(\cdot-\sigma)\|_e^2\right)^{p/2} + c^p\left(T^{-1}\int\|x'(\cdot-\sigma')\|_e^2\right)^{p/2}\right]^{1/p} + d$$
$$= (b^p|x|_2^p + c^p|x'|_2^p)^{1/p} + d$$
$$= \|x\|_X + d,$$

from which we easily obtain

$$\|\mathcal{G}x\|_Z = \|(B-I)\,\Gamma x\|_Z \leqslant (\|B\|+\theta)\|x\|_X + (\|B\|+\theta)\,d.$$

Hence the inequalities (5.10) and (5.11) are proved.

The above computations show, in particular, that Γ (as substitution operator) is continuous, and bounded on bounded sets. As in the proof of Theorem 5.1 we can now see that $\mathcal{F}, \mathcal{G}$ are both L-compact on bounded sets; moreover, (5.11) shows that $\mathcal{G}$ is quasibounded with quasinorm

$$\|\mathcal{G}\| \leqslant \|B\| + \theta. \tag{5.12}$$

Obviously, $e \in \operatorname{Im} L = \ker Q$.

Let us verify assumptions $(i) - (v)$ of Theorem 4.2.

(i). - We have to verify (H0). For every $x \in X$, let $\Phi x: \mathbf{R} \to \mathbf{R}^N$ be the continuous T-periodic map defined by

$$(\Phi x)\,(\cdot) := \operatorname{diag}(Ux(\cdot-\sigma)^{\mathrm{T}})$$

and, for any $i = 1, \ldots, N$, let $t_i = t_i(x)$ be the minimum of all $t \in [0, T]$ at which the map $|(\Phi x)_i(\cdot)|$ reaches its minimum value, that is

$$t_i = \min\{t \in [0,T] \mid |(\Phi x)_i(t)| = \min|(\Phi x)_i(\cdot)|\}. \tag{5.13}$$

We remark that, being $U = [u_{ij}]$,

$$(\Phi x)_i(\cdot) = \Sigma_j u_{ij} x_j(\cdot - \sigma_{ij}).$$

We define

$$\Pi : X \to \operatorname{Im} P, \ \Pi x := U^{-1}\big((\Phi x)_1(t_1), (\Phi x)_2(t_2), \dots, (\Phi x)_N(t_N)\big),$$
$$\langle\cdot,\cdot\rangle := (\cdot|\cdot)_2,$$

and we claim that

$$\|x - \Pi x\|_X^2 \leqslant [b^p(2\omega^{-1} + T^{1/2}\|\sigma - \sigma^*\|_1^{1/2})^p + c^p]^{2/p}\langle Lu, u\rangle. \quad (5.14)$$

In fact, we recall first that

$$\|x - \Pi x\|_X^p = b^p|x - \Pi x|_2^p + c^p|x'|_2^p. \quad (5.15)$$

Next we evaluate $|x - \Pi x|_2$. To do this we introduce the vector

$$w = (w_1, w_2, \dots, w_N),$$

whose i-th component is

$$w_i := \Sigma_j u_{ij} x_j(t_i - \sigma^*_i).$$

We have

$$\begin{aligned} |x - \Pi x|_2 &= |Ux - U\Pi x|_2 \\ &\leqslant |Ux - w|_2 + |w - U\Pi x|_2 \\ &= \big(T^{-1}\textstyle\int\Sigma_i|\Sigma_j u_{ij}(x_j(\cdot) - x_j(t_i - \sigma^*_i))|^2\big)^{1/2} \\ &\quad + \big(\Sigma_i|\Sigma_j u_{ij}(x_j(t_i - \sigma^*_i) - x_j(t_i - \sigma_{ij}))|^2\big)^{1/2} \\ &= \big(T^{-1}\Sigma_i\textstyle\int|y_i(\cdot)|^2\big)^{1/2} + |\operatorname{diag}(U(S^* - S))|, \end{aligned}$$

where

$$\begin{aligned} &y_i(\cdot) := \Sigma_j u_{ij}(x_j(\cdot) - x_j(t_i - \sigma^*_i)), \ i = 1, \dots, N, \\ &S^* := [s^*_{ij}], \ s^*_{ij} := x_i(t_j - \sigma^*_j), \\ &S \ := [s_{ij}], \ s_{ij} := x_i(t_j - \sigma_{ij}). \end{aligned}$$

Each map $y_i(\cdot)$ vanishes at $t_i - \sigma^*_i$. Therefore, using Poincaré-Picard inequality, we obtain

$$\begin{aligned} (T^{-1}\Sigma_i\textstyle\int|y_i(\cdot)|^2)^{1/2} &\leqslant 2\omega^{-1}(T^{-1}\Sigma_i\textstyle\int|y'_i(\cdot)|^2)^{1/2} \\ &= 2\omega^{-1}(T^{-1}\Sigma_i\textstyle\int|\Sigma_j u_{ij} x'_j(\cdot)|^2)^{1/2} \\ &= 2\omega^{-1}|Ux'|_2 \\ &= 2\omega^{-1}|x'|_2 \end{aligned} \quad (5.17)$$

Moreover, using (5.3), we have

$$|\operatorname{diag}(U(S^* - S))| \leqslant N^{1/2}\|U\|\cdot\|S^* - S\|_e = N^{1/2}\|S^* - S\|_e. \quad (5.18)$$

(Recall that $\|U\| = 1$ because U is orthogonal). Therefore, from (5.16) and using (5.17) and (5.18), we have

$$|x - \Pi x|_2 \leqslant 2\omega^{-1}|x'|_2 + N^{1/2}\| S^* - S\|_e . \qquad (5.19)$$

At last let us evaluate $\| S^* - S\|_e$. We have

$$\begin{aligned} N^{1/2}\| S^* - S\|_e &= (\Sigma_{ij}| x_i(t_j - \sigma^*_j) - x_i(t_j - \sigma_{ji})|^2)^{1/2} \\ &\leqslant (\Sigma_{ij}|\sigma_{ji} - \sigma^*_j| \int | x'_i(\cdot)|^2)^{1/2} \\ &= (\Sigma_i(\Sigma_j|\sigma_{ji} - \sigma^*_j|) \int | x'_i(\cdot)|^2)^{1/2} \\ &\leqslant (\max_i \Sigma_j |\sigma_{ji} - \sigma^*_j|)^{1/2} (\int\Sigma_i | x'_i(\cdot)|^2)^{1/2} \\ &= \|\sigma - \sigma^*\|_1^{1/2} \cdot T^{1/2}|x'|_2 . \end{aligned} \qquad (5.20)$$

We derive, from (5.15), (5.19), (5.20), that

$$\| x - \Pi x\|_X^2 \leqslant [b^p(2\omega^{-1} + T^{1/2}\|\sigma - \sigma^*\|_1^{1/2})^p + c^p]^{2/p} \cdot |u'|_2^2 ,$$

i.e. (5.14) holds.

(ii). - This assumption is nothing but inequality (5.1) with weight. Namely, let $\omega = 2\pi/T$, $u \in H^2_T$, $z \in H^0_T$ with $\bar{u} = 0$, $|z|_2 = 1$. Then (5.1) implies $\langle Lu, u\rangle^{1/2} = |u'|_2 \geqslant \omega |u|_2 \geqslant \omega(z|u)_2 = \omega\langle z, u\rangle$.

(iii). - Let us define the linear L-completely continuous map

$$\mathcal{A}^* : X \to Z,\ \mathcal{A}^* x := (B - \theta I)\,\mathrm{diag}(Ux(\cdot - \sigma)^{\mathrm{T}}),$$

and let

$$a := b M N^{1/2} .$$

Since

$$\| x\|_X = (b^p |\Pi x|_2^p)^{1/p} = b|\Pi x|_2 = b|\Pi x| ,$$

we have

$$\begin{aligned} D_1 &= \{x \in \mathrm{dom}\, L \mid \|\Pi x\|_X > a\} \\ &= \{x \in H^2_T \mid |\Pi x| > ab^{-1}\} = \{x \in H^2_T \mid |U\Pi x| > MN^{1/2}\}. \end{aligned}$$

Since $\mathrm{Im}\, Q$ consists of all constant mappings $\mathbf{R} \to \mathbf{R}^N$, we can identify both $\mathrm{Im}\, Q$ and $(\mathrm{Im}\, Q^*)$ with $\mathbf{R}^N$, and choose the euclidean inner product as bilinear pairing. Let e_j, $j = 1, 2, \dots, N$, be the j-th vector of the canonical orthonormal basis in $\mathbf{R}^N$. We shall define the map

$$\varphi : D_1 = \{\in H^2_T \mid |U\Pi x| > MN^{1/2}\} \to (\mathrm{Im}\, Q)^* = \mathbf{R}^N$$

setting

$$\varphi x := (B^{\mathrm{T}} - \theta I)^{-1} e_{j(x)} ,$$

where the index $j(x)$ is defined in the following way:

$$j(x) := \min\{j \in \{1, \dots, N\} \mid \text{for every } t \in \mathbf{R},\ |(\Phi x)_j(t)| > M\}.$$

Of course we have to verify that $j(x)$ is well-defined. Namely, if

$$\{j \in \{1, \dots, N\} \mid \text{for every } t \in \mathbf{R},\ |(\Phi x)_j(t)| > M\} = \varnothing ,$$

and recalling the definition (5.13) of $t_j = t_j(x)$, we have that for every index j

$$|(\Phi x)_j(t_j)| \leqslant M,$$

and so, by definition of Πx,

$$\begin{aligned} |U\Pi x| &= |((\Phi x)_1(t_1), \ldots, (\Phi x)_N(t_N))| \\ &= (\Sigma_j |(\Phi x)_j(t_j)|^2)^{1/2} \leqslant M N^{1/2}, \end{aligned}$$

which is a contradiction with $x \in D_1 = \{x \in H^2_T \mid |U\Pi x| > MN^{1/2}\}$.

At last, let $x \in D_1$. We have

$$\begin{aligned} &(Q\mathcal{A}^*x, \varphi x) = (Q\mathcal{A}^*x \mid \varphi x) \\ &= (T^{-1} \textstyle\int (B - \theta I) \operatorname{diag}(Ux(\cdot - \sigma)^T) \mid (B^T - \theta I)^{-1} e_{j(x)}) \\ &= T^{-1} \textstyle\int (\Phi x)_{j(x)}, \end{aligned}$$

$$\begin{aligned} &(Q\mathcal{G}x, \varphi x) \\ &= (T^{-1} \textstyle\int (B - \theta I) \Gamma x \mid (B^T - \theta I)^{-1} e_{j(x)}) \\ &= T^{-1} \textstyle\int (g(\cdot, x(\cdot - \sigma), x'(\cdot - \sigma')))_{j(x)} \\ &= T^{-1} \textstyle\int g_{j(x)} (\cdot; x_1(\cdot - \sigma_{j(x)1}), \ldots, x_N(\cdot - \sigma_{j(x)N}); x'_1(\cdot - \sigma'_{j(x)1}), \ldots) \end{aligned}$$

The choice of $j(x)$ implies that

$$|\Sigma_k u_{j(x)k} x_k(t - \sigma_{j(x)k})| = |(\Phi x)_{j(x)}(t)| > M$$

for each $t \in \mathbf{R}$. Using the assumption *(jjj)* we obtain that

$$g_{j(x)}(t; x_1(t - \sigma_{j(x)1}), \ldots, x_N(t - \sigma_{j(x)N}); y) \cdot (\Phi x)_{j(x)}(t) > 0$$

for all $t \in \mathbf{R}$ and all $y \in \mathbf{R}^N$. In particular this implies that

$$(Q\mathcal{A}^*x, \varphi x)\,(Q\mathcal{G}x, \varphi x) > 0.$$

(iv). - Let $x \in H^2_T$ with $\|\Pi x\|_X = b\,|\Pi x|_2 \leqslant a$ (i.e. $x \in D_2$), and let $u = (I - P)x$. We have, using (5.10),

$$\begin{aligned} &\langle (I - Q)\mathcal{F}x, u\rangle \\ &\quad = ((d/dt)(\nabla F(x) + V(\cdot, x)) + \theta\Gamma x - \theta\overline{\Gamma x} \mid u)_2 \\ &\quad = T^{-1} \textstyle\int ((d/dt)\nabla F(x) \mid u) + T^{-1} \textstyle\int ((d/dt) V(\cdot, x) \mid u) \\ &\qquad + T^{-1} \textstyle\int \theta (\Gamma x - \overline{\Gamma x} \mid u) \\ &\quad = -T^{-1} \textstyle\int (V(\cdot, x) \mid u') + \theta T^{-1} \textstyle\int (\Gamma x \mid u) \\ &\quad \leqslant T^{-1} \textstyle\int \ell(\cdot) |x| \cdot |u'| + T^{-1} \textstyle\int k |u'| \\ &\qquad + \theta (b^p |x|_2^p + c^p |x'|_2^p)^{1/p} |u|_2 + \theta d |u|_2 \\ &\quad \leqslant \sup |\ell(\cdot)| \cdot |x|_2 |u'|_2 + \theta c_1 |x|_2 |u|_2 + \theta c_2 |x'|_2 |u|_2 + c_3 |u'|_2 \end{aligned}$$

where c_1, c_2, c_3 are constants depending on b, c, p, N, k, d, ω. Moreover, using (5.19), (5.20), we obtain

$$
\begin{aligned}
&\langle (I-Q)\mathfrak{F}x, u\rangle \\
&\quad \leqslant (\sup|\ell(\cdot)| + \theta c_1 \omega^{-1})(|x-\Pi x|_2 + |\Pi x|_2)|u'|_2 \\
&\quad + \theta c_2 \omega^{-1}|u'|_2^2 + c_3|u'|_2 \\
&\quad \leqslant (\sup|\ell(\cdot)| + \theta c_1 \omega^{-1})(2\omega^{-1}|x'|_2 \\
&\qquad + T^{1/2}\|\sigma-\sigma^*\|_1^{1/2}|x'|_2 + ab^{-1})|u'|_2 \\
&\quad + \theta c_2 \omega^{-1}|u'|_2^2 + c_3|u'|_2 \\
&\quad \leqslant (\gamma \sup|l(\cdot)| + \theta c_4)|u'|_2^2 + c_3|u'|_2,
\end{aligned}
$$

where c_4 is a constant explicitly computable. Therefore *(iv)* holds with

$$
\begin{aligned}
&\mu_1 := \gamma \sup|\ell(\cdot)| + \theta c_4 \qquad (5.21)\\
&\mu_2 := c_3,\ \varepsilon := 1/2,\ \mu_3 := 0.
\end{aligned}
$$

(v). - The part of the proof concerning *(i)* shows that the constant β of inequality *(v)* of Theorem 4.2 can be defined as follows:

$$
\begin{aligned}
\beta^{-1/2} &:= [b^p(2\omega^{-1} + T^{1/2}\|\sigma-\sigma^*\|_1^{1/2})^p + c^p]^{1/p} \\
&= (b^p\gamma^p + c^p)^{1/p} \qquad (5.22)
\end{aligned}
$$

Using (5.12), (5.21), (5.22), we have

$$
\begin{aligned}
&\omega^{-1}\beta^{-1/2}\|\mathcal{G}\| + \mu_1 \\
&\quad = \omega^{-1}(b^p\gamma^p + c^p)^{1/p}(\|B\| + \theta) + \gamma\sup|\ell(\cdot)| + \theta c_4 \\
&\quad = \left[\left(\frac{b}{\omega/\gamma}\right)^p + \left(\frac{c}{\omega}\right)^p\right]^{1/p}\|B\| + \gamma\sup|\ell(\cdot)| + \theta c_5,
\end{aligned}
$$

where c_5 is a computable real constant. In virtue of the inequality *(v)*, we can choose θ, at the beginning of the proof, in such a way that

$$\omega^{-1}\beta^{-1/2}\|\mathcal{G}\| + \mu_1 < 1.$$

The proof is complete.

Remark 5.6. The case of a nonconstant delay matrix $\sigma(t) = [\sigma_{ij}(t)]$, where each entry $\sigma_{ij}(\cdot): \mathbf{R} \to \mathbf{R}$ is a continuous and T-periodic function, can be treated through the same perturbation argument we have used in [7] and following the same lines of the proof or Theorem 5.2.

Remark 5.7. Theorem 5.2 (even if it permits us to deal with equation (5.7) which is more general than (5.4) does not contain Theorem 5.1 as a corollary. Indeed, if we want to apply Theorem 5.2 to equation (5.4) and we assume the same hypotheses as in Theorem 5.2, we have to choose

$$\sigma = \sigma' = 0,\ g(t,x,y) = x,\ U = I,\ b = 1,\ c = d = 0,$$

and p any number in the interval $[1,2]$.

Then, according to assumption *(v)* in Theorem 5.2, we would have existence of a solution $x \in H^2_T$ for the equation (5.4) provided that

$$\| B \| + 2 (\sup | \ell(\cdot)|) \omega < \omega^2/2 ,$$

while, according to Theorem 5.1, it would be sufficient to require

$$\| B \| + (\sup | \ell(\cdot)|) \omega < \omega^2$$

which is sharper.

Remark 5.8. It is easy to see that the usual quasiboundedness condition on the map g

(jv') $|g(t,x,y)| \leqslant (b^p |x|^p + c^p |y|^p)^{1/p} + d ,$

for all $x, y \in \mathbf{R}^N$ and $t \in \mathbf{R}$,

is equivalent to the assumption *(jv)* in the particular case in which

$$\mathfrak{J}(\sigma) = \mathfrak{J}(\sigma') = \{1, \ldots, N\}^3 ,$$

i.e. if in the matrix σ, as well as in σ', all the rows are equal one to each other.

Remark 5.9. Following the proof of Theorem 5.2, it is not difficult to see that if we assume $V(t,x) = V(x)$ satisfying

(jj') there are two constants $\ell \geqslant 0$ and $k \geqslant 0$ usch that

$$|V(x) - V(y)| \leqslant \ell |x - y| + k$$

holds, for any $x, y \in \mathbf{R}^N$,

then the growth restiction *(v)* on the coefficients may be improved into

$$(v') \qquad \left[\left(\frac{b}{\omega/\gamma} \right)^p + \left(\frac{c}{\omega} \right)^p \right]^{1/p} \| B \| < 1 - \ell\omega^{-1}$$

and the same existence result holds.

Remark 5.10. We note that a classical solution is obtained in the case of a continuous forcing term h.

Remark 5.11. If $N = 1$, Theorem 5.2 gives the following result

Corollary 5.1. *Let $f_1 : \mathbf{R} \to \mathbf{R}$ and $g_1 : \mathbf{R}^3 \to \mathbf{R}$ be continuous functions, with g_1 T-periodic in the first variable and let $h_1 : \mathbf{R} \to \mathbf{R}$ be continuous and T-periodic.*

Assume that

(k) there exists a number $M_1 > 0$ such that

$$(g_1(t,x,y) - \bar{h}_1)\,\mathrm{sign}\, x > 0 \quad (< 0)$$

for every $t \in \mathbf{R}, y \in \mathbf{R}$ *and* $x \in \mathbf{R}$ *such that* $|x| \geqslant M_1$;

(kk) there are numbers $b_1 > 0, c_1 > 0, d_1 \geqslant 0, p_1 \in [1,2]$ *such that for every* $t, x, y \in \mathbf{R}$,

$$|g_1(t,x,y)| \leqslant \left({b_1}^{p_1} |x|^{p_1} + {c_1}^{p_1} |y|^{p_1} \right)^{1/p_1} + d_1;$$

(kkk) $(2b_1/\omega)^{p_1} + {c_1}^{p_1} < \omega^{p_1}$.

Then, for any $\sigma, \sigma' \in \mathbf{R}$, *the generalized Liénard delay-differential equation*

$$x'' + f_1(x)\,x' + g_1(t, x(t-\sigma), x'(t-\sigma')) = h_1(t)$$

has at least one T-periodic solution (of class $\mathcal{C}^2$*).*

In fact, Theorem 5.2 applies with the positions

$$F(x) := \textstyle\int_0^x (\int_0^t f_1(s)\,ds)\,dt,$$

$$V(t,x) := 0, \; B = 1,$$

$$g(t,x,y) := g_1(t,x,y) - \bar{h}_1$$

$$h(t) := h_1(t) - \bar{h}_1.$$

All the assumptions of Theorem 5.2 can be easily checked.

Moreover it can be easily seen that the assumption *(k)* may be replaced with the more general one

(k') there exists a number $M_1 > 0$ *such that*

$$\left(\textstyle\int (g_1(\cdot, x(\cdot - \sigma), x'(\cdot - \sigma')) - h_1(\cdot))\right) \mathrm{sign}(x(t)) \geqslant 0 \quad (\leqslant 0)$$

for any $x \in \mathcal{C}^1_T$, *such that* $|x(t)| \geqslant M_1$, *for all t.*

Corollary 5.1 extends a result of MARTELLI and SCHUUR [13] (for related results, see [9], [21], [22], [23]. In [21] and [23] sharper estimates than *(kkk)* have been obtained in the case $\sigma = \sigma' = 0$ and $g = g(t,x)$ not dependent on x').

Remark 5.12. Obviously, there is no loss of generality if we suppose, for each i, j,

$$|\sigma_{ij}| \leqslant T/2.$$

This implies that the constant $\|\sigma - \sigma^*\|_1$ which appears in the definition of $\gamma = \gamma(\sigma)$ satisfies the inequality

$$\|\sigma - \sigma^*\|_1 \leqslant NT/2.$$

From this one can easily obtain sufficient conditions for the existence of periodic solutions $x \in H^2_T$ to the equation (5.7), whatever σ

may be.

The practical computation of the matrix σ^* which minimizes the convex (nondifferentiable) map

$$\tau \to \| \sigma - \tau \|_1$$

on the manifold $\mathfrak{M}_N^*$ seems not to be trivial.

There are examples, for any $N \geqslant 2$ in which the minimizing matrix σ^* is not unique. Moreover it can be shown that, if $N = 2$, one can choose $\sigma^* = [\sigma^*_{ij}]$, with $\sigma^*_{ij} = N^{-1} \Sigma_j \sigma_{ij}$, but this is not true for $N \geqslant 3$.

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