

THE PSEUDOSTRICT TOPOLOGY ON FUNCTION SPACES (*)

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SOMMARIO. - Si introduce la topologia «pseudostretta» β' sullo spazio $C_b(X)$ di tutte le funzioni continue e limitate definite su uno spazio X completamente regolare, e si discutono le sue proprietà e relazioni con le note topologie «strette» su $C_b(X)$. Una di tali proprietà è che le misure, elementi dello spazio duale di $(C_b(X), \beta')$, sono disintegrabili.

SUMMARY. - The pseudostrict topology β' on $C_b(X)$ the space of all bounded continuous functions on a completely regular space X is introduced and its properties and relations to known strict topologies on $C_b(X)$ are discussed. One such property is that measures in the dual of $(C_b(X), \beta')$ admit disintegration.

1. Introduction. The material in this note was motivated by an attempt to extend a disintegration theorem of Edgar [3] which holds for tight Baire measures on a completely regular space X to a wider class of measures by appealing directly to the Dunford Pettis theorem, which seems to provide an immediate access to vector valued density functions. This requires the representation of such measures as members of the positive cone of the dual of $C_b(X)$, the space of bounded continuous functions on X , under a suitable locally convex topology. In § 3 we introduce the pseudo-

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strict topology β' on $C_b(X)$, relate it to known strict topologies on $C_b(X)$ and show that the dual of $C_b(X)$ under β' can be identified with the space of all pseudotight signed Baire measures on X , i.e. signed Baire measures admitting thick σ -pseudocompact subsets of X . A disintegration theorem for pseudotight measures is given in §4 and, questions of completeness and Stone-Weierstrass property of $(C_b(X), \beta')$ are discussed in §5.

2. Preliminary material. For a completely regular Hausdorff space X we denote by $C_b(X)$ the space of all real-valued bounded continuous functions on X . A Baire measure on X is a finite, non-negative σ -additive set function defined on all Baire subsets of X . By a signed Baire measure we mean the difference of two Baire measures. We denote by $M(X)$ the vector space of all signed Baire measures on X . We write $M_\tau(X)$ and $M_t(X)$ for the spaces of all τ -additive and all tight signed Baire measures respectively. The process of integration enables us to identify $M(X)$, $M_\tau(X)$ and $M_t(X)$ with certain subspaces of the dual of $C_b(X)$ under the uniform norm topology. Locally convex topologies on $C_b(X)$ giving $M(X)$, $M_\tau(X)$ and $M_t(X)$ as dual spaces were given in [1], [2], [4] and [7]. We adopt the terminology of [7] where these topologies are denoted by β_0 , β and β_1 , and are called the substrict, the strict and the superstrict topologies respectively. β_0 , β and β_1 are the topologies on $C_b(X)$ of uniform convergence on uniformly tight subsets of M_t , uniformly τ -additive subsets of M , and uniformly σ -additive subsets of M , respectively.

We say that a signed Baire measure μ is pseudotight if for any $\varepsilon > 0$ there exists a pseudocompact set $K \subset X$ such that

$$|\mu|_*(X \setminus K) < \varepsilon.$$

We denote by $M_p(X)$ the space of all pseudotight signed Baire measures. And a subset H of $M(X)$ is said to be uniformly pseudotight if it is uniformly bounded and for given $\varepsilon > 0$ exists a pseudocompact set K such that $|\mu|(E) < \varepsilon$ for all Baire sets $E \subset X \setminus K$ and all $\mu \in H$.

It can be easily verified that M_p is a strongly closed vector subspace of $M(X)$ and that $M_t(X) \subset M_p(X) \subset M(X)$. But neither $M_\tau(X) \subset M_p(X)$ nor $M_p(X) \subset M_\tau(X)$ holds in general, as witnessed by a metrizable non-measure compact X and a pseudo-compact non-compact X respectively.

3. The pseudostrict topology on $C_b(X)$. We define β' to be the finest locally convex topology on $C_b(X)$ which agrees with the to-

pology s of uniform convergence on pseudocompact subsets of X on all uniformly bounded subsets of $C_b(X)$.

The proof of the following proposition is straightforward and we omit it.

3.1. PROPOSITION. *The topology β' has a base of neighborhoods at 0 consisting of the closed convex hulls of all sets of the form*

$$U_n \{f \in C_b(X) : \|f\| < n \text{ and } \|f1_{K_n}\| < \varepsilon_n\},$$

where $(K_n)_n$ is an increasing sequence of pseudocompact subsets of X and $\varepsilon_n \downarrow 0$.

From this proposition we clearly have $\beta_0 \leq \beta' \leq \beta_1$ and so

$$M_t(X) \subset (C_b(X), \beta')' \subset M(X).$$

3.2. PROPOSITION. *A subset H of $(C_b(X), \beta')'$ is β' -equicontinuous if and only if H is uniformly pseudotight.*

PROOF. First suppose that H is β' -equicontinuous. Then clearly H is uniformly bounded. If $\varepsilon > 0$ let V be a β' -neighborhood of 0 such that $|\mu(f)| < \varepsilon/4$ for all $f \in V$ and all $\mu \in H$. There exists a pseudocompact set K and a $\delta > 0$ such that $f \in V$ for all $f \in C_b(X)$ with $\|f\| < 1$ and $\|1_K f\| < \delta$. If $Z \subset X \setminus K$ is a zero set, let $X = P \cup N$ be a Hahn decomposition for $\mu \in H$. We can find zero sets Z_1, Z_2 with $Z_1 \subset Z \cap P, Z_2 \subset N$ such that

$$\mu^+(Z) = \mu(Z \cap P) < \mu(Z_1) + \varepsilon/8$$

and

$$\mu(Z_2) = \mu^-(Z_2) > \mu^-(X) - \varepsilon/8.$$

Since $Z_1 \cap (K \cup Z_2) = \phi$ and K is pseudocompact, there exists an $f \in C_b(X)$ with $0 \leq f \leq 1, f = 1$ on Z_1 , and $f = 0$ on $K \cup Z_2$, so that $|\mu(f)| < \varepsilon/4$ for all $\mu \in H$. But

$$\mu^-(f) = \mu^-(f1_N) = \mu^-(f1_{Z_2}) + \mu^-(f1_{N \setminus Z_2}) \leq \mu^-(N \setminus Z_2) \|f\| \leq \varepsilon/8,$$

thus

$$\begin{aligned} \mu^+(Z) &< \mu^+(Z_1) + \varepsilon/8 \leq \mu^+(f) + \varepsilon/8 = \\ &= \mu(f) + \mu^-(f) + \varepsilon/8 \leq \mu(f) + \varepsilon/4 \leq \varepsilon/2. \end{aligned}$$

Similarly $\mu^-(Z) < \varepsilon/2$ so that $|\mu|(Z) < \varepsilon$, i.e. H is uniformly pseudotight.

Conversely suppose that H is uniformly pseudotight. Let L be such that $|\mu|(X) < L$ for all $\mu \in H$, and let K_n be a pseudocompact

set such that $|\mu|(X \setminus K_n) < 1/2n$ for all $\mu \in H$. If $\|f\| < n$ and $\|1_{K_n} f\| < 1/2L$, set $U_n = \{x \in X : |f(x)| < 1/2L\}$. Then,

$$\begin{aligned} |\mu(f)| &< |\mu|(|f|) = |\mu|(|f| 1_{U_n}) + |\mu|(|f| 1_{X \setminus U_n}) < \\ &< |\mu|(U_n) / 2L + \|f\| |\mu|(X \setminus U_n) < 1. \end{aligned}$$

Therefore the polar H^0 of H contains the closed convex hull of

$$U_n \{f \in C_b(X) : \|f\| < n \text{ and } \|f 1_{K_n}\| < 1/2L\},$$

so it is a β' -neighborhood of 0.

3.3 COROLLARY. *The space $M_p(X)$ of all pseudotight Baire measures is the topological dual of $C_b(X)$ under the pseudistrict topology β' .*

3.4. REMARKS.

- (a) The uniform norm topology on $C_b(X)$ coincides with β' if and only if X is pseudocompact.
- (b) If X is either metrizable or realcompact then $\beta' = \beta_0$.
- (c) $M_p(X) = M(X)$ only if νX , the Hewitt realcompactification is strongly measure compact.
- (d) It follows from proposition (2.2) that every β' -equicontinuous set $H \subset M_p(X)$ is relatively weakly compact. However not every relatively weakly compact $H \subset M_p(X)$ is β' -equicontinuous.

3.5. PROPOSITION. *The following conditions are equivalent.*

- (a) $\beta' = \beta_0$.
- (b) $M_p(X) = M_t(X)$.
- (c) $M_p(X) = M_\tau(X)$.
- (d) *Every 2-valued pseudotight Baire measure on X is τ -additive.*

PROOF. It is obvious that (a) implies (b), (b) implies (c), and (c) implies (d). So it is sufficient to demonstrate that (d) implies (a). For this let K be any closed pseudocompact subset of X . Let λ be a Baire measure on K which only has values 0 and 1. Define μ on any Baire set E in X by $\mu(E) = \lambda(E \cap K)$. Then clearly λ is a Baire measure attaining only values 0 and 1. From (d) it follows that $\mu = \delta_x$, the unit point mass at a point $x \in X$. It is also obvious that $x \in K$. So K must be compact. Hence $\beta' = \beta_0$.

3.6 REMARK. The Hewitt realcompactification νX of X can be identified with the space of all 0-1-valued pseudotight Baire measures on X , so that $X \subset \rho X \subset \nu X$. More explicitly

$$\rho X = \bigcup \{cl_{\cup X} K : K \subset X \text{ and } K \text{ is pseudocompact}\}.$$

It can be verified that:

- (a) The space $(C_b(X), \beta')$ is isomorphic to $(C_b(\rho X), \beta_0)$. From this and [4], theorem 10 follows that $(C_b(X), \beta')$ has the approximation property.
- (b) $\beta' = \beta_0$ if and only if $X = \rho X$.
- (c) Obviously $M(X) = M_p(X)$ implies $\rho X = \cup X$, while the converse is false as can be seen from any realcompact space X which is not strongly measure compact.
- (d) Further $\rho X = \beta X$ if and only if X is pseudocompact.

4. A disintegration theorem. We call a Baire measure μ on X disintegrable if for any measurable space (Y, \mathfrak{B}) and any map g from X into Y such that $g^{-1}(B)$ is μ -measurable for all $B \in \mathfrak{B}$, there exists a map λ from Y into the space $P(X)$ of all probability Baire measures on X such that $\lambda_y(E)$ is $g(\mu)$ -measurable in Y for all Baire sets E of X , and

$$\int_B \lambda_y(E) dg(\mu) = \mu(E \cap g^{-1}(B)).$$

Any tight Baire measure on X is disintegrable [3].

We note that any Baire measure μ has a disintegration relative to the injection $i: X \rightarrow \cup X$, when \mathfrak{B} is the Baire σ -algebra of X (take $\lambda_y = \delta_y$ for all $y \in \cup X$). If μ is a pseudotight Baire measure, Y a completely regular Hausdorff space, and \mathfrak{B} its Baire σ -algebra then a Baire measurable $g: X \rightarrow Y$ can be extended to $\tilde{g}: \cup X \rightarrow Y$ and $g = \tilde{g} \circ i$. Let $\tilde{\mu}$ be the induced Baire measure on $\cup X$, i. e. $\tilde{\mu} = i(\mu)$. We have seen that μ has a disintegration with respect to i . By [3] exists a disintegration of $\tilde{\mu}$ with respect to \tilde{g} . These can be composed into a disintegration of μ with respect to g . But for general measurable space (Y, \mathfrak{B}) and measurable map g from X into Y such an extension procedure is not available. But we can prove the following more general result.

4.1. THEOREM. *Every pseudotight Baire measure is disintegrable.*

PROOF. Let μ be a pseudotight Baire measure. First assume $\mu^*(K) = 1$ for a pseudocompact set K . For a measurable space (Y, \mathfrak{B}) and Baire measurable $p: X \rightarrow Y$ denote by ν the image

measure $p(\mu)$ on \mathfrak{B} , and assume that (Y, \mathfrak{B}) is complete. Now $V(f)(g) = \int (g \circ p) f d\mu$ for $g \in L^1(Y, \nu)$ and $f \in C_b(X)$ defines a linear map from $C_b(X)$ into $L^1(Y, \nu)' = L^\infty(Y, \nu)$ with $\|V(f)\|_\infty = \|f\|_\infty$. If the net (f_α) in $C_b(X)$ converges to 0 for the topology β' then it converges uniformly to 0 on the pseudocompact set K . So for given $\varepsilon > 0$ choose α_0 with $\|f_\alpha 1_K\| < \varepsilon$ for $\alpha \geq \alpha_0$. Then $P_\alpha = \{f_\alpha > \varepsilon\}$ is a cozero set with $P_\alpha \cap K = \emptyset$ for $\alpha \geq \alpha_0$. Hence $\mu(P_\alpha) = 0$ for $\alpha \geq \alpha_0$, i.e. $\|f_\alpha\|_\infty \leq \varepsilon$ for $\alpha \geq \alpha_0$. It follows that $\|V(f_\alpha)\|_\infty = \|f_\alpha\|_\infty \rightarrow 0$, i.e. V is β' -continuous.

Therefore the adjoint V' of V transforms the unit ball of $L^1(Y, \nu)$ into an equicontinuous subset of $(C_b(X), \beta)'$. An application of the Dunford Pettis theorem for locally convex spaces (see [6], p. 40) gives a disintegration map $\lambda: Y \rightarrow (C_b(X), \beta)'$ such that for all functions $f \in C_b(X)$ and for all classes of functions $\hat{g} \in L^1(Y, \nu)$ we have $\lambda_y(f) \in \mathcal{L}^\infty(Y, \nu)$ and $\int_Y \lambda_y(f) d\nu = \int f(g \circ p) d\mu$. This implies the assertion of the theorem in this special case.

For the general case choose an increasing sequence (K_n) of pseudocompact subsets such that $\mu^*(K_n) > 1 - 1/n$ for all natural n . Now define $\psi_n(E) = \mu^*(E \cap K_n)$ for Baire sets E in X and each n , and write $\mu_1 = \psi_1$ and $\mu_n = \psi_n - \psi_{n-1}$ for $n > 1$. Then $\mu_n^*(K_n) = \mu_n(X)$, and $\mu = \sum_n \mu_n$. Let $\nu_n = p(\mu_n)$. Then $\nu = \sum_n \nu_n$. By the special case above there exists a disintegration $\lambda^n: Y \rightarrow M_p(X)$ of μ_n with respect to p . Let $h_n \leq 1$ be a Radon Nikodym derivative of ν_n with respect to ν . Then define

$$\lambda_y = \sum_n h_n(y) \lambda_y^n.$$

Since $\sum_n \int h_n(y) \lambda_y^n(X) d\nu = \sum_n \int \lambda_y^n(X) d\nu_n = \sum_n \mu_n(X) = \mu(X) = 1$, it follows $\lambda_y(X) < \infty$. Clearly λ_y is a measure for which all properties of a disintegration are satisfied.

5. Further properties of β' . Call X a pk^* -space if any bounded real valued function is continuous whenever its restriction to each pseudocompact subspace is continuous.

5.1. THEOREM. *If X is pk^* -space then $(C_b(X), \beta')$ is complete.*

PROOF. Suppose that X is a pk^* -space and let $F \in M_p(X)^*$, the algebraic dual of $M_p(X)$, such that F is $\sigma(M_p(X), C_b(X))$ -continuous on each β' -equicontinuous subset of $M_p(X)$. Define $f: X \rightarrow \mathbf{R}$ by: $f(x) = F(\delta_x)$. It follows from (2.2) that f is continuous on each

pseudocompact subset of $M_p(X)$, and so $f \in C_b(X)$. Also $\mu(f) = F(\mu)$ for all $\mu \in D$, the set of all linear combinations of point masses. Each $\mu \in M_p(X)$ being pseudotight is clearly a weak limit of a uniformly pseudotight net (μ_α) of elements of D . Hence $F(\mu) = \lim_\alpha F(\mu_\alpha) = \lim_\alpha \mu_\alpha(f) = \mu(f)$. So μ is $\sigma(M_p(X), C_b(X))$ -continuous. Hence, by Grothendieck's completeness theorem, $(C_b(X), \beta')$ is complete.

We remark that (5.1) can be proved using the fact that $(C_b(X), \beta')$ is isomorphic to $(C_b(\rho X), \beta_0)$ by showing that ρX is a k^* -space if X is a pk^* -space and then applying the completeness theorem of β_0 (see [2], [4], and [7]).

A locally convex topology T on $C_b(X)$ is said to have the Stone-Weierstrass property if any subalgebra A of $C_b(X)$ is T -dense whenever it separates points and is such that for all $x \in X$ exists a function $f \in A$ such that $f(x) \neq 0$. It is known that β_0 has the Stone-Weierstrass property. Also if X is strongly measure compact then β_1 has the Stone-Weierstrass property, and if β_1 has the Stone-Weierstrass property then X is measure compact [5].

5.2. THEOREM. *The topology β' has the Stone-Weierstrass property if and only if $\beta' = \beta_0$.*

PROOF. Suppose that β' has the Stone-Weierstrass property. Then $X = \rho X$. Otherwise let $x \in \rho X \setminus X$ and put

$$A = \{f \in C_b(X) : f(x) = 0\}.$$

A separates points in X , and for any $x \in X$ there exists $f \in A$ with $f(x) \neq 0$. However $A \subset \{f \in C_b(X) : \delta_x(f) = 0\}$. As $\delta_x \in M_p(X)$ for all $x \in \rho X$, the β' -closure of A differs from $C_b(X)$. Hence $\rho X = X$, and so $\beta' = \beta_0$. The converse is plain.

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