

A QUASI-UNIVERSAL REALIZATION OF AUTOMATA (*)

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SOMMARIO. - *Si considera un approccio categoriale alla teoria degli automi nel quale la realizzazione di automi non deterministici risulta essere universale in un senso 2-categoriale (lax).*

SUMMARY. - *We consider a categorical approach to the theory of automata by which the realization of non deterministic automata is universal in a 2-categorical sense (lax).*

Introduction

A known theorem by Goguen [4] shows that «minimal realization» of automata is a functor right adjoint to «external behaviour». On the other hand, a remark by Trnková asserts that such an adjunction holds «*only* in the case considered by Goguen» ([8], p. 341). In this paper we consider an approach to automata theory by which a single (non-deterministic) automaton is itself an enriched category and all automata with their appropriate morphisms can be reflectively embedded in a 2-category [2]. Thus an adjointness relation between realization and behaviour can still be recovered (though not uniquely), but in a lax 2-categorical framework. We tentatively call «local adjunction» such a relation.

Moreover the reduction problem (see e.g. [7], p. 145) finds a natural solution, because the realization we get can be intuitively considered as a «union» of reduced automata. Hence it is univer-

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sally reduced (but not at all a minimal one). Our definition of reduced automata is reminiscent of the classical one, but splits into two notions, taking the duality between automata and their reversed ones into account.

1. The basic machinery

First let us recall the basics of [2], with some natural generalizations. A non-deterministic automaton with inputs from the monoid X can be viewed as an enriched category Q whose homs are taken in the preorder \mathcal{X} of subsets of X . Internal states have to be identified with objects of Q . In \mathcal{X} the subset product acts as a monoidal product and the left quotient provides an internal hom-functor for \mathcal{X} : $A \otimes B = A \cdot B = \{x \in X \mid ab = x \text{ for some } a \in A \text{ and some } b \in B\}$, $\text{Hom}(A, B) = A^{-1}B = \{x \in X \mid ax \in B \text{ for all } a \in A\}$.

Our general reference about enriched categories is provided by [6] and by [3]. The most relevant difference with the base categories usually considered is that \mathcal{X} is not symmetric as a closed category, but it is a biclosed one.

An \mathcal{X} -category Q embodies the whole dynamics of a single automaton, in the sense that the transition relation due to the input $x \in X$ induces an arrow between corresponding objects. Initial and terminal states can be assigned by suitable bimodules $I: Q \dashrightarrow \underline{1}$ and $T: \underline{1} \dashrightarrow Q$ ($\underline{1}$ is the trivial \mathcal{X} -category with only one object).

With respect to these data we have:

DEFINITION. The behaviour of the automaton $\mathbf{Q} = (Q, I, T)$ is the \mathcal{X} -object corresponding to the composite bimodule

$$\underline{1} \begin{array}{c} \dashrightarrow \\ T \\ \dashrightarrow \end{array} Q \begin{array}{c} \dashrightarrow \\ I \\ \dashrightarrow \end{array} \underline{1}$$

It should be remarked that endobimodules of $\underline{1}$ are in bijection with objects of the base category.

DEFINITION. The 2-category of behaviours \mathbf{B} is defined as follows:

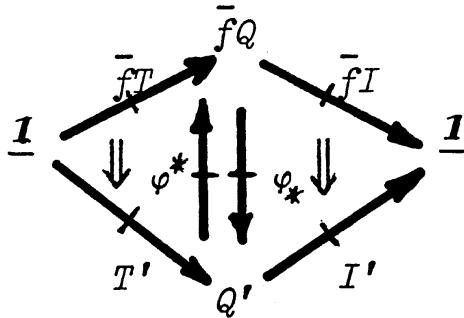
- (i) the objects are those of \mathcal{X} ;
- (ii) the arrows $A \rightarrow B$ are given by monoidal closed functors $f: \mathcal{X} \rightarrow \mathcal{X}$ such that $f(A) \subset B$;
- (iii) 2-cells $f \rightarrow g: A \rightarrow B$ are closed natural transformations (i.e. there is just one 2-cell $\zeta: f \rightarrow g$ exactly when $f(H) \subset g(H)$ for all H in \mathcal{X}).

According to the above definition, the arrows $f: A \rightarrow B$ provide the possibility of re-coding inputs within X itself.

Recall now that when f is a morphism in \mathbf{B} , as an endofunctor of \mathcal{X} , it induces a functor $\bar{f}: \mathcal{X}\text{-cat} \rightarrow \mathcal{X}\text{-cat}$ which transfers \mathcal{X} -categories, functors and bimodules to corresponding new \mathcal{X} -categorical notions (see [3], p. 449). Analogously a 2-cell $\zeta: f \rightarrow g$ gives rise to a natural transformation $\bar{\zeta}: \bar{f} \rightarrow \bar{g}$. So, given the automaton $\mathbf{Q} = (Q, I, T)$, we denote by $\bar{f}\mathbf{Q}$ the triple $(\bar{f}Q, \bar{f}I, \bar{f}T)$.

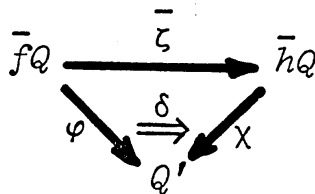
DEFINITION. The 2-category $X\text{-aut}$ of X -automata is defined as follows:

- (i) the objects are triples (Q, I, T) , where the bimodules I and T are suitably induced by initial and terminal states, while Q is an \mathcal{X} -category;
- (ii) the arrows $\mathbf{Q} = (Q, I, T) \rightarrow \mathbf{Q}' = (Q', I', T')$ are pairs (f, φ) where $f: I \circ T \rightarrow I' \circ T'$ is an arrow in \mathbf{B} , and φ is an \mathcal{X} -functor $\bar{f}Q \rightarrow Q'$ which satisfies the following inclusions of bimodules



i.e. $\bar{f}I \subset I' \circ \varphi_*$ and $\bar{f}T \subset \varphi^* \circ T'$ (the bimodules φ_* and φ^* are canonically defined by $\varphi_*(q', q) = Q'(q', \varphi q)$ and $\varphi^*(q, q') = Q'(\varphi q, q')$);

- (iii) a 2-cell $(f, \varphi) \rightarrow (h, \chi): \mathbf{Q} \rightarrow \mathbf{Q}'$ is a pair (ζ, δ) where ζ is a closed natural transformation $f \rightarrow h$, and δ is an \mathcal{X} -natural transformation of the form $\varphi \rightarrow \bar{\zeta}\chi$:



A few remarks on the above definition: the first is that ar-

rows in $X\text{-aut } \mathbf{Q} \xrightarrow{(f, \varphi)} \mathbf{Q}' \xrightarrow{(g, \psi)} \mathbf{Q}''$ compose in the obvious way:
 $(f, \varphi) (g, \psi) = (fg, \bar{g}(\varphi) \psi) : \bar{g}(\bar{f} \mathbf{Q}) \xrightarrow{\bar{g}\varphi} \bar{g} \mathbf{Q}' \xrightarrow{\psi} \mathbf{Q}''.$

The second concerns inclusions (1): their meaning is that, after the coding f , initial and terminal states are preserved by morphism. Moreover the condition (1) can be equivalently stated by: $\bar{f} I \circ \varphi^* \subset I'$ and $\varphi_* \circ \bar{f} T \subset T'$, by taking account of the fact that compositions with φ_* and φ^* provide an adjoint pair of bimodules (see [6], p. 163).

Finally recall that, in the particular case when the base category is a preorder, as is our \tilde{X} , to give a 2-cell $(\zeta, \delta) : (f, \varphi) \rightarrow (h, \chi)$ means that for each pair of objects q and q' , the following inclusions hold (provided by δ):

$$Q'(q, q') \subset Q'(\varphi q, \bar{\zeta}(\chi q'))$$

$$Q'(q, q') \subset Q'(\bar{\zeta}(\chi q), q').$$

It is now straightforward to verify that the assignment of the behaviour $I \circ T$ to each automaton $\mathbf{Q} = (Q, I, T)$ amounts to a functor $\Xi : X\text{-aut} \rightarrow \mathbf{B}$ of 2-categories, which forgets the second component of morphisms and 2-cells.

2. Local adjunctions

We introduce now a notion which amounts to a family of adjoint pairs or, by using the 2-categorical terminology, to a pair of lax functors, suitably adjoint at each level. Our general reference for 2-categories is provided by [1] and [5].

DEFINITION. Let $C \xrightleftharpoons[\Phi]{\Gamma} D$ be a diagram of 2-categories and functors between them. Φ and Γ are said to be *locally adjoint functors* (Φ on the left, Γ on the right) when for each pair of objects (c in C and d in D) an adjoint pair is given

$$D(\Phi c, d) \xleftarrow[\Phi_{cd}]{\gamma_{cd}} C(c, \Gamma d)$$

subject to the following naturality conditions: for all $h : d \rightarrow d'$ and all $k : c' \rightarrow c$ in their respective categories, the following coherent morphism are assigned:

$$\varphi(fh) \leftarrow \varphi(f) \cdot \Gamma h, \quad \varphi(\Phi k \cdot f) \leftarrow k \cdot \varphi(f) \text{ and}$$

$$\gamma(kg) \rightarrow \Phi k \cdot \gamma(g), \quad \gamma(g \cdot \Gamma h) \rightarrow \gamma(g) h$$

where f and g are arbitrary objects in the categories $D(\Phi c, d)$ and, respectively, $C(c, \Gamma d)$.

Of course there are two distinct notions of locally adjoint functors $(\Phi \dashv \Gamma)$, depending on the side of γ with respect to ϕ . In both the realizations considered below we will deal only with one of these notions, namely $\gamma \dashv \phi$.

3. The realization functor

Let us now define a realization built up in a purely syntactical way, which assigns to each object A of the category of behaviours \mathbf{B} the triple $X_A = (\mathcal{X}, I, \mathbf{T}_A)$ where, for each object L of \mathcal{X} , the bi-modules I and \mathbf{T}_A are defined by:

$$I(L) = \sum_{k \subset H} \mathcal{X}(H, L),$$

$$T_A(L) = \sum_{H \subset A} \mathcal{X}(L, H),$$

(here k denotes the unit of the tensor product in \mathcal{X} , i.e. the X -subset with only the unit element).

The above definition of I and \mathbf{T}_A means that in X_A , objects containing k are taken as initial states, and objects contained in A as terminal ones. X_A is of course neither reachable nor observable.

But let us notice that to each $\mathbf{Q} = (Q, I, T)$ such that $\Xi \mathbf{Q} = A$ we can canonically associate an automaton with the same behaviour which is reduced in the following sense:

DEFINITION. An automaton $\mathbf{Q} = (Q, I, T)$ is said to be *initial-reduced* (resp. *terminal-reduced*) if $I(q) \neq I(q')$ implies $q \neq q'$ (resp. $T(q) \neq T(q')$ implies $q \neq q'$).

For each automaton $\mathbf{Q} = (Q, I, T)$ let us consider the canonical morphism $\eta: \mathbf{Q} \rightarrow X_A$ provided by $\eta = (id, \bar{\eta})$ and $\bar{\eta}(q) = I(q)$. If \mathbf{Q}_A denotes the full \mathcal{X} -subcategory of \mathcal{X} which is the image of $\bar{\eta}$ we get easily the following:

PROPOSITION. \mathbf{Q}_A is initial-reduced.

There is actually a duality between initial and terminal notions. So we could indifferently define the realization $X^A = (\mathcal{X}, \mathbf{T}_A, I)$ by the dynamics $\mathcal{X}(R, S) = RS^{-1} = \{x \in X \mid xS \subset R\}$ and interchanging initial and terminal states. Suitably defining an X -aut morphism $\Theta: \mathbf{Q} \rightarrow X^A$ for each automaton $\mathbf{Q} = (Q, I, T)$ by $\Theta(q) = T(q)$ we get an automaton \mathbf{Q}^A which exhibits the same behaviour as \mathbf{Q} and which is terminal-reduced.

From now on, we shall deal only with the «initial» side, but the results we are going to prove are easily dualizable.

The realization just performed acts also on morphisms of \mathbf{B} : for each $f: A \rightarrow B$ in \mathbf{B} , $\mathbf{P}f$ is the pair $(f, \hat{f}): X_A \rightarrow X_B$, where \hat{f} is the \mathcal{X} -functor defined by $\hat{f}L = fL$ and \hat{f}_{LM} is the inclusion $f(\mathcal{X}(L, M)) \subset \mathcal{X}(fL, fM)$ for each pair of \mathcal{X} objects L and M (see [3], p. 449).

It is an easy calculation to prove that the assignment $\mathbf{P}: A \mapsto X_A$ works also on 2-cells of \mathbf{B} and amounts to a 2-functor $\mathbf{B} \rightarrow X\text{-aut}$.

Finally it is straightforward to check the following:

PROPOSITION. For each behaviour A , it holds $\Xi(\mathbf{P}(A)) = A$.

4. The main result

THEOREM. Realization is a 2-functor locally right adjoint to behaviour.

PROOF. For all \mathbf{Q} in $X\text{-aut}$ and for all B in \mathbf{B} we have to exhibit the adjunction

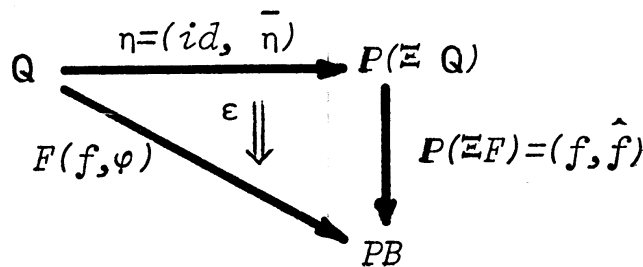
$$X\text{-aut}(\mathbf{Q}, \mathbf{PB}) \begin{matrix} \xleftarrow{\rho} \\ \xrightarrow{\xi} \end{matrix} \mathbf{B}(\Xi \mathbf{Q}, B)$$

natural in \mathbf{Q} and B .

Let us suppose $\Xi \mathbf{Q} = A$. ξ can be viewed as the restriction of Ξ to the category $X\text{-aut}(\mathbf{Q}, \mathbf{PB})$, while ρ is given by the composition on the left of the image of \mathbf{P} with η :

$$\rho(-) = \eta \cdot \mathbf{P}(-).$$

We shall describe the counit 2-cell ε of the promised adjunction $\rho \dashv \xi$: ε occurs in the following diagram and, for the closed functors in the first component coincide, it remains to define an \mathcal{X} -natural transformation $\bar{\varepsilon}: \bar{f} \bar{\eta} \cdot \hat{f} \rightarrow \varphi$:



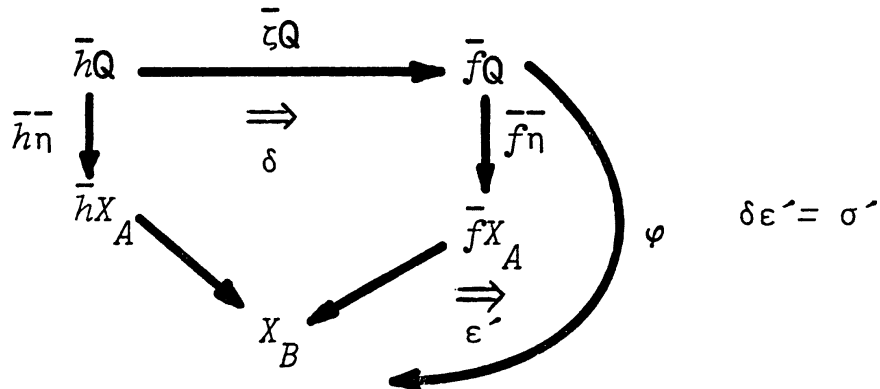
The required ε is (uniquely) determined by observing that for each q in Q the inclusion $\hat{f}(\bar{f}\bar{\eta}(q)) \subset \varphi(q)$ holds and hence the requirements (2) in the definition of the 2-category $X\text{-aut}$ are fulfilled:

$$\bar{X}(\hat{f}(\bar{f}\bar{\eta}q), \hat{f}(\bar{f}\bar{\eta}q')) \subset \bar{X}(\hat{f}(\bar{f}\bar{\eta}q), \varphi q'),$$

$$\bar{X}(\varphi q, \varphi q') \subset \bar{X}(\hat{f}(\bar{f}\bar{\eta}q), \varphi q').$$

It remains to show that for each 2-cell $\sigma: \rho h \rightarrow F$ in \mathbf{B} there exists a (necessarily unique) 2-cell $\zeta: h \rightarrow f$ such that $\sigma = \rho\zeta \cdot \varepsilon$. But $\sigma: \rho h = (h, \bar{h}\bar{\eta} \cdot \hat{h}) \rightarrow F = (f, \varphi)$ has two components, and we shall take the first one as the required closed natural transformation.

Since the other component is $\sigma': \bar{h}\bar{\eta} \cdot \hat{h} \rightarrow \bar{\zeta}Q \cdot \varphi$, to show that $\sigma = \rho\zeta \cdot \varepsilon$, it suffices to show $\sigma' = \delta \cdot \varepsilon'$. This follows by inspection of the diagram (δ and ε' are the second components of $\rho\zeta$ and ε respectively). Naturality of the adjunction comes from an easy calculation.



COROLLARY. X_A is quasi-terminal in the full subcategory of $X\text{-aut}$ consisting of objects of behaviour A , in the sense that for each object Q in this subcategory, there exists an arrow $Q \rightarrow X_A$ unique up to a (non-invertible, uniquely determined) 1-cell.

PROOF. Put $A = B$ and $f = id$ in the previous proof.

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