

CHARACTERIZATION OF CERTAIN DIFFERENTIAL OPERATORS IN THE REPRESENTATION OF PSEUDO-ANALYTIC FUNCTIONS (*)

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SOMMARIO. - *Per le funzioni pseudo-analitiche esistono delle rappresentazioni per mezzo di speciali operatori integrali e differenziali. Si deduce una condizione sufficiente per gli operatori differenziali usati, di modo che tutte le soluzioni della*

$$w_{\bar{z}} = aw + b\bar{w}$$

siano contenute in questa rappresentazione per le soluzioni.

SUMMARY. - *For pseudo-analytic functions we have representations by means of special integral operators and differential operators. Here we deduce a sufficient condition on the differential operators used for all solutions of*

$$w_{\bar{z}} = aw + b\bar{w}$$

to be representable in that form.

1. Introduction

After L. Bers (cf. [6]) the solutions of the differential equation

$$W_{\bar{z}} = aW + b\bar{W} \quad (1)$$

are called pseudo-analytic functions. Transforming (1) by $W = w \exp(A)$ with $a = A_{\bar{z}}$, we obtain

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$$w_z = B\bar{w} \text{ with } B = b \exp(\bar{A}-A). \quad (2)$$

In [10] I. N. Vekua gives a representation of the solutions of (2) using certain integral operators. Up to now the explicit determination of the resolvents needed was possible only for very few equations as for example in [2]. Besides it is possible to give solutions of (2) by means of special differential operators (cf. e.g. [1, 4]). The differential operator representation for solutions of partial differential equations is of great importance in the investigation of the function theoretic properties of the solutions, specially of the pseudo-analytic functions' properties (cf. [5], where many results are collected and further sources are given). Furthermore, this method was also successful in connection with the representation of pseudo-holomorphic functions of several variables [3, 8]. If solutions of (2) can be represented making use of differential operators, it is of interest whether all solutions of the considered equation are contained in this representation. Here we deduce a sufficient condition on the operators used for all solutions of (2) to be representable by that differential operators.

2. The differential operators used

First we prove a lemma which gives a necessary condition on the form of the differential operators used in the representation of the pseudo-analytic functions. In the following we denote by \mathfrak{D} a simply connected domain of the complex plane.

LEMMA 1. *Let*

$$K_1^m = \sum_{k=0}^m \alpha_k(z, \bar{z}) \frac{\partial^k}{\partial z^k}, \quad K_2^n = \sum_{k=0}^n \beta_k(z, \bar{z}) \frac{\partial^k}{\partial \bar{z}^k}, \quad m, n \in \mathbf{N},$$

be given differential operators, where $\alpha_k, k = 0, 1, \dots, m$, and $\beta_k, k = 0, 1, \dots, n$, are continuously differentiable in \mathfrak{D} . Let

$$w = K_1^m g(z) + K_2^n \overline{g(z)} \quad (3)$$

be a solution of (2) in \mathfrak{D} for all $g(z)$ holomorphic in \mathfrak{D} . Then $n = m-1$ follows.

PROOF. If we put Eq. (3) into (2) and collect corresponding terms, we have to consider the three cases $m > n$, $m = n$ and $m < n$. For $m > n$ we get $\alpha_k \equiv 0$ for all $k \geq n+2$, for $m = n$ $\beta_m \equiv 0$ follows and for $m < n$ we have to require $\beta_k \equiv 0$ for all $k \geq m$.

3. Condition on the operators

Every solution of (2) satisfies also the elliptic differential equation

$$w_{z\bar{z}} - \frac{B_z}{B} w_z - B\bar{B}w = 0, \quad (4)$$

provided that $B \neq 0$ and that B is differentiable in the considered domain. Let v be a solution of (4). Then after I. N. Vekua [9], p. 140, every solution of (2) can be represented in the form

$$w_1 = \frac{1}{2} \left(v + \frac{1}{B} \bar{v}_z \right) \quad \text{or} \quad w_2 = \frac{1}{2i} \left(v - \frac{1}{B} \bar{v}_z \right), \quad (5)$$

at which one of the two formulas (5) is sufficient.

With respect to Lemma 1 we consider for a solution of (4) defined in \mathfrak{D} with B analytic in \mathfrak{D} the representation

$$v = H_1^m g(z) + H_2^{m-1} \overline{h(z)}, \quad (6)$$

where $g(z)$ and $h(z)$ are holomorphic functions in \mathfrak{D} and H_1^m and H_2^{m-1} respectively are the differential operators

$$H_1^m = \sum_{k=0}^m a_k(z, \bar{z}) \frac{\partial^k}{\partial z^k}, \quad H_2^{m-1} = \sum_{k=0}^{m-1} b_k(z, \bar{z}) \frac{\partial^k}{\partial \bar{z}^k},$$

$m \in \mathbf{N}$, with $a_k, k = 0, \dots, m, b_k, k = 0, \dots, m-1$, twice continuously differentiable and $a_m \neq 0, b_{m-1} \neq 0$ in \mathfrak{D} . Applying the results of R. Heersink [7] we can state that all solutions of (4) defined in \mathfrak{D} can be represented in this form. Putting Eq. (6) into (4) we obtain for the coefficients a_k and b_k respectively the following conditions:

$$a_{k-1, \bar{z}} + a_{k, z\bar{z}} - \frac{B_z}{B} a_{k, \bar{z}} - B\bar{B}a_k = 0, \quad (7a)$$

$$b_{k-1, z} - \frac{B_z}{B} b_{k-1} + b_{k, z\bar{z}} - \frac{B_z}{B} b_{k, \bar{z}} - B\bar{B}b_k = 0, \quad (7b)$$

$0 \leq k \leq m+1$, with $a_{-1} = a_{m+1} = b_{-1} = b_m = b_{m+1} = 0$.

LEMMA 2. Let the functions a_k and b_k respectively satisfy the system (7). Hence it follows for $0 \leq l \leq m+1$

$$\bar{B}a_l = \overline{b_{l-1}} + \overline{b_{l,z}} \quad (8a)$$

$$\bar{B}b_{l-1} = \overline{a_{l-1,z}} \quad (8b)$$

PROOF. First we consider (7a) with $k = m+1$ and (7b) with $k = m$ and obtain $a_m = \Theta(z), b_{m-1} = \overline{\Psi(z)B}, \Theta(z), \Psi(z)$ holomorphic in \mathfrak{D} . With this we have

$$a_m - \frac{\overline{b_{m-1}}}{B} = \Theta(z) - \psi(z) = \Phi(z). \quad (9)$$

With respect to (9)

$$b_{m-1} - \frac{\overline{a_{m-1,z}}}{B} = B \overline{(a_m - \Phi)} - \frac{\overline{a_{m-1,z}}}{B}$$

follows. From (7a) with $k = m$ and with the relation $a_{m,\bar{z}} = 0$ we obtain

$$b_{m-1} - \frac{\overline{a_{m-1,z}}}{B} = -\bar{\Phi}B.$$

The normalization $\Phi(z) \equiv 0$ is allowed.

Proceeding from $k = m, m-1, \dots$ we can prove Eq. (8) successively. Namely if we put $k = r-1$ in (7b), $0 < r < m$, r fixed, we obtain the relation

$$\left[\frac{b_{r-2} + b_{r-1,\bar{z}}}{B} \right]_z - \bar{B}b_{r-1} = 0.$$

On the premises that (8) holds for $l = r$ we can integrate and get

$$\frac{b_{r-2} + b_{r-1,\bar{z}}}{B} - \overline{a_{r-1}} = \overline{\Phi_r(z)}, \Phi_r(z) \text{ holomorphic in } \mathfrak{D}. \quad (10)$$

If we differentiate Eq. (8b) with $l = r$ with respect to \bar{z} and put the complex conjugate of the result into (7a) with $k = r-1$ we get with (8b)

$$\frac{\overline{a_{r-2,z}}}{B} = B\overline{a_{r-1}} - b_{r-1,\bar{z}}.$$

By it and with (10) it follows

$$b_{r-2} - \frac{\overline{a_{r-2,z}}}{B} = \overline{\Phi_r(z)} B.$$

With regard to system (7) we set $\Phi_r(z) \equiv 0$.

Putting the representation (6) for the function ν into the first of the two formulas (5) we obtain

$$\begin{aligned} 2\bar{B}w = & \sum_{k=0}^m [\bar{B}a_k g^{(k)}(z) + (\overline{b_{k-1}} + \overline{b_{k,z}}) h^{(k)}(z)] + \\ & + \sum_{k=0}^{m-1} [\overline{a_{k,z}} \overline{g^{(k)}(z)} + \bar{B}b_k \overline{h^{(k)}(z)}]. \end{aligned}$$

With $a_k = 2\hat{a}_k$, $b_k = 2\hat{b}_k$ and $g(z) + h(z) = f(z)$, $f(z)$ holomorphic in \mathfrak{D} , we have in connection with Lemma 2

$$w = \sum_{k=0}^m \hat{a}_k f^{(k)}(z) + \sum_{k=0}^{m-1} \hat{b}_k \overline{f^{(k)}(z)}.$$

From these results we can easily deduce the following theorem.

THEOREM. Let H_1^m and H_2^{m-1} be differential operators of the form

$$H_1^m = \sum_{k=0}^m \hat{a}_k \frac{\partial^k}{\partial z^k}, \quad H_2^{m-1} = \sum_{k=0}^{m-1} \hat{b}_k \frac{\partial^k}{\partial \bar{z}^k},$$

where the functions $\hat{a}_k(z, \bar{z})$, $k = 0, 1, \dots, m$, and $\hat{b}_k(z, \bar{z})$, $k = 0, 1, \dots, m-1$, respectively are twice continuously differentiable in \mathfrak{D} with $\hat{a}_m \neq 0$, $\hat{b}_{m-1} \neq 0$ in \mathfrak{D} . In \mathfrak{D} let B be analytic, $f(z)$ holomorphic and $H_1^m f + H_2^{m-1} \bar{f}$ a solution of (2)

$$w_{\bar{z}} = B\bar{w}.$$

Then for every solution w of (2) defined in \mathfrak{D} there exists a holomorphic function $f(z)$ in \mathfrak{D} such that

$$w = H_1^m f + H_2^{m-1} \bar{f}.$$

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