

# A NOTE ON CONVERGENCES IN COMMUTATIVE GROUPS (\*)

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**SOMMARIO.** — *Ogni struttura di convergenza sequenziale  $\mathcal{L}$ , definita su un sottoinsieme  $X$  di un gruppo commutativo  $(G, +)$ , viene estesa ad una convergenza  $\tilde{\mathcal{L}}$  su  $G$  che sia compatibile con la addizione, in modo tale che se  $G$  è il gruppo commutativo libero su  $X$ , allora il passaggio da  $\mathcal{L}$  a  $\tilde{\mathcal{L}}$  conserva l'unicità del limite e la proprietà universale dei gruppi liberi.*

**SUMMARY.** — *Given a star multivalued sequential convergence structure  $\mathcal{L}$  on a subset  $X$  of a commutative group  $(G, +)$ , we extend  $\mathcal{L}$  to a convergence  $\tilde{\mathcal{L}}$  on  $G$  which is compatible with respect to  $+$ , in such a way that the passage from  $\mathcal{L}$  to  $\tilde{\mathcal{L}}$  maintains the uniqueness of the limit and the universal property for free groups, whenever  $G$  is the free commutative group on  $X$ .*

## INTRODUCTION

In 1968, J. Novák [6, problem 12], posed the problem of the existence of a convergence group which is not separated. The existence of a non separated sequential convergence structure with uniqueness of the limit was already known (cf. M. Dolcher [2, page 81]) and Novák had also proved in [5, Lemma 10] that there exists a convergence group which it not regular.

In [9] we solved the problem of Novák, modelling a free group

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over the non separated sequential space of Dolcher. The construction of the example was based on the following steps: (i) consider a suitable (sequential) convergence structure  $(X, \mathcal{L}, \lambda)$ ; (ii) take the free commutative group  $(G, +)$  over the set  $X$ ; (iii) define a convergence structure on  $G$  which extends  $\mathcal{L}$  and is compatible with  $+$  and prove that this new convergence preserves some significant properties of  $\mathcal{L}$ . The key point of the proof was based on a result (Proposition II, below) which provides the way how to define the finest convergence group structure on a commutative group such that some preassigned sequences do converge to 0. This technique was usefully employed also for the construction of counterexamples to some conjectures concerning convergence groups and rings (cf. [8] and [1]).

In this work we present a result (Lemma 1) which improves [9] because the crucial point in the discussion of the example in [9] can be obtained as a consequence of the fact (proved here) that in the step (iii) above, the uniqueness of the limit is preserved in the passage from the set  $X$  to the group  $G$ . Therefore many of the above cited results can be seen as an application of this general principle, which proof is nevertheless simpler.

## RESULTS

Let  $(G, +)$  be a commutative group, let  $X$  be a non empty subset of  $G$  and let  $\mathcal{C}$  be a set of sequences of elements of  $X$ . We consider here the following two laws:

- ( $\delta$ ).  $(x_n)_n \in \delta \mathcal{C}$  if and only if  $(x_n)_n$  is a subsequence of some sequence which belongs to  $\mathcal{C}$ .
- ( $\zeta$ ).  $(x_n)_n \in \zeta \mathcal{C}$  if and only if for every subsequence  $(x_{k_n})_n$  of  $(x_n)_n$  there exists a subsequence  $(x_{i_k})_n$  of  $(x_{k_n})_n$  which belongs to  $\mathcal{C}$ .

Then the following two propositions can be proved:

- (I).  $\zeta(\delta \mathcal{C})$  is the least set of sequences of elements of  $X$ , which contains  $\mathcal{C}$  and which is closed with respect to  $\delta$  and  $\zeta$ .
- (II).  $\zeta \langle \delta \mathcal{C} \rangle$  (where  $\langle \mathcal{A} \rangle$  is the subgroup of  $G^{\mathbb{N}}$  which is generated by  $\mathcal{A}$ ) is the least subgroup of  $G^{\mathbb{N}}$  which contains  $\mathcal{C}$  and which is closed with respect to  $\delta$  and  $\zeta$ . (Cf. [8, Lemma 4]).

Let us suppose now to assign to each point  $x$  in  $X$ , a set  $\mathfrak{B}_x$  of sequences of elements of  $X$ , such that the constant sequence  $(x)$  belongs to  $\mathfrak{B}_x$ . If we pose:

$$(x_n)_n \xrightarrow{\mathfrak{L}} x \text{ if, and only if } (x_n)_n \in \mathfrak{S}_x = \zeta(\delta \mathfrak{B}_x),$$

we have a sequential convergence structure according to Dolcher [2], that is, a multivalued convergence space according to Novák [4], (cf. Proposition I) and we say that the convergence  $\mathfrak{L}$  on  $X$  is generated by the basis  $(\mathfrak{B}_x)_{x \in X}$ . If, moreover, the condition

$$(0) \quad x \neq y \Rightarrow \delta \mathfrak{B}_x \cap \delta \mathfrak{B}_y = \emptyset$$

holds, then the convergence is onevalued (with uniqueness of the limit) and we have a *star convergence* according to Novák [4] (see also Fréchet [3] and Urysohn [7]).

Now we want to extend the convergence  $\mathfrak{L}$  to  $G$  in such a way to maintain convergent to  $x \in X$  the sequences of  $\mathfrak{S}_x$  and by requiring a compatibility between the convergence and the group structure, namely:

$$(x_n)_n \rightarrow x, (y_n)_n \rightarrow y \Rightarrow (x_n - y_n)_n \rightarrow x - y.$$

Then the finest (according to Dolcher [2]) of the convergences which satisfy these requirements is given in the following manner (cf. Proposition II):

Let  $\tilde{\mathfrak{B}}_0 = \bigcup_{x \in X} (\mathfrak{S}_x - (x))$  and let  $\tilde{\mathfrak{S}}_0 = \zeta < \delta \tilde{\mathfrak{B}}_0 >$ .

Set  $\tilde{\mathfrak{S}}_x = \tilde{\mathfrak{S}}_0 + (x)$ , for every  $x \in G$  and let  $\tilde{\mathfrak{L}}$  be the convergence defined by the system  $(\tilde{\mathfrak{S}}_x)_{x \in G}$ :

$$(x_n)_n \xrightarrow{\tilde{\mathfrak{L}}} x \text{ if, and only if } (x_n)_n \in \tilde{\mathfrak{S}}_x.$$

$\tilde{\mathfrak{L}}$  is characterized by the following six properties:

- (1)  $(x) \xrightarrow{\tilde{\mathfrak{L}}} x$ , for every  $x \in G$ .
- (2)  $(x_n)_n \xrightarrow{\tilde{\mathfrak{L}}} x \Rightarrow (x_{k_n})_n \xrightarrow{\tilde{\mathfrak{L}}} x$ , for every subsequence  $(x_{k_n})_n$  of  $(x_n)_n$ .
- (3)  $(x_n)_n \not\xrightarrow{\tilde{\mathfrak{L}}} x \Rightarrow$  there exists a subsequence  $(x_{k_n})_n$  of  $(x_n)_n$  such that no subsequence of  $(x_{k_n})_n$  converges to  $x$ .
- (4)  $(x_n)_n \xrightarrow{\tilde{\mathfrak{L}}} x, (y_n)_n \xrightarrow{\tilde{\mathfrak{L}}} y \Rightarrow (x_n - y_n)_n \xrightarrow{\tilde{\mathfrak{L}}} x - y$ .

$$(5) \quad (x_n)_n \xrightarrow{\mathcal{L}} x \Rightarrow (x_n)_n \xrightarrow{\tilde{\mathcal{L}}} x.$$

(6) Let  $\mathcal{L}'$  be a convergence on  $G$  satisfying all the properties from (1) to (5), then:

$$(x_n)_n \xrightarrow{\tilde{\mathcal{L}}} x \Rightarrow (x_n)_n \xrightarrow{\mathcal{L}'} x.$$

We say that  $\tilde{\mathcal{L}}$  on  $G$  is *generated by*  $\mathcal{L}$  on  $X$ .

Let  $\lambda$  (respectively  $\tilde{\lambda}$ ) be the closure operator in  $X$  (in  $G$ ) which is deduced in the usual way from  $\mathcal{L}$  (resp.  $\tilde{\mathcal{L}}$ ), (cf. [4]). From (5) it follows that  $\lambda A \subseteq \tilde{\lambda} A$ , for every subset  $A$  of  $X$ , and therefore every  $\tilde{\lambda}$ -neighbourhood of  $x \in X$  contains a  $\lambda$ -neighbourhood of the same point  $x$ . (For definitions, see [4]).

In order that  $(G, +, \tilde{\mathcal{L}}, \tilde{\lambda})$  be a convergence commutative group with star convergence (according to Novák [5]), we must have that  $\tilde{\mathcal{L}}$  be onevalued. Obviously, a necessary condition for this fact is that  $\mathcal{L}$  be onevalued. The condition is also sufficient if  $G$  is a free commutative group whose basis is  $X$ , as stated in the following.

**LEMMA 1.** *If  $G$  is the free  $\mathbf{Z}$ -module generated by  $X$ , then  $\tilde{\mathcal{L}}$  is onevalued if (and only if)  $\mathcal{L}$  is onevalued.*

*Proof of Lemma 1:* Since  $\tilde{\mathcal{L}}$  is onevalued if and only if there is not any  $x \neq 0$  such that  $(x) \xrightarrow{\tilde{\mathcal{L}}} 0$  (cf. [5, Theorem 2] or [8, Lemma 2]) we shall suppose that there exists some  $\bar{x} \in G$  such that  $(\bar{x}) \in \tilde{\mathcal{S}}_0 = \zeta < \delta ( \bigcup_{x \in X} (\mathcal{S}_x - (x)) ) >$ , and we shall prove that  $\bar{x}$  must be equal to 0. Let  $(\bar{x}) \in \zeta < \delta ( \bigcup_{x \in X} (\mathcal{S}_x - (x)) ) >$ , then we have that  $(\bar{x}) \in < \delta ( \bigcup_{x \in X} (\mathcal{S}_x - (x)) ) > = < \bigcup_{x \in X} (\delta \mathcal{S}_x - (x)) > = < \bigcup_{x \in X} (\mathcal{S}_x - (x)) >$ . (Because every  $\mathcal{S}_x$  is closed with respect to  $\delta$ ). This means that there exist points  $x_1, \dots, x_r$  of  $X$  and sequences  $S_1 = (x_n^1)_n, \dots, S_r = (x_n^r)_n$  of elements of  $X$ , such that, for every  $i = 1, \dots, r$ ,  $S_i \xrightarrow{\mathcal{L}} x_i$  and there exist integers  $t_1, \dots, t_r$  such that the equality

$$(7) \quad (\bar{x}) = \sum_{i=1}^r t_i (S_i - (x_i)) = \sum_{i=1}^r t_i (x_n^i - x_i)_n$$

holds. Without loss of generality (passing to a subsequence) we can suppose that for  $i \neq j$ , either  $x_n^i = x_n^j$  for every  $n$ , or  $x_n^i \neq x_n^j$  for every  $n$  (in (7)). In fact, it can be easily proved, using an inductive argument, that, given a finite set  $\{S_i\}$  of sequences, there exists an increasing sequence  $R$  of natural numbers (indices) such that for  $i \neq j$  either  $S_i \circ R$  and  $S_j \circ R$  are different term by term, either they are equal. Moreover we note that if  $S_i = S_j$  in (7) then we must have  $x_i = x_j$  in virtue of the uniqueness of the limit in  $(X, \mathcal{L})$ .

Thanks to the above remarks we can assume that there exist  $k + 1$  numbers,  $1 = i_1 < i_2 < \dots < i_{k+1} = r + 1$  such that  $S_i = S_j$  if and only if both indices  $i$  and  $j$  belong to a same interval  $[i_h, i_{h+1}[$  (recall also that  $G$  is commutative).

Let us set now:

$$\begin{aligned} \tilde{S}_1 &= S_1 = \dots = S_{i_2-1} = (y_n^1)_n; & x_1 &= \dots = x_{i_2-1} = y_1; \\ \dots & \dots \dots \dots \dots \dots & \dots & \dots \dots \dots \dots \dots \\ \tilde{S}_k &= S_{i_k} = \dots = S_r = (y_n^k)_n; & x_{i_k} &= \dots = x_r = y_k; \end{aligned}$$

$$z_j = t_{i_j} + \dots + t_{i_{j+1}-1} \quad (j = 1, \dots, k), \quad (z_j \in \mathbf{Z}, \text{ for every } j).$$

Hence we conclude that equality (7) can be written as follows:

$$(8) \quad (\bar{x}) = \sum_{j=1}^k z_j (\tilde{S}_j - (y_j)) = \sum_{j=1}^k z_j (y_n^j - y_j)_n$$

where  $\tilde{S}_j \xrightarrow{\mathcal{L}} y_j$ , for every index  $j$ .

Observe now that it is not restrictive to suppose (passing to a subsequence) that for any index  $j$ , either  $y_n^j = y_j$  for every  $n$ , either  $y_n^j \neq y_m^j$  for every pair  $(n, m)$  with  $n \neq m$ . In fact, it can be easily proved, using an inductive argument, that, given a finite set  $\{\tilde{S}_j\}$  of sequences, there exists an increasing sequence  $R$  of natural numbers (indices) such that, for every  $j$ , either the terms of  $\tilde{S}_j \circ R$  are all different one to each other, either  $\tilde{S}_j \circ R$  is constant; moreover, note that, in our case, if  $\tilde{S}_j \circ R$  is constant, then it must be  $\tilde{S}_j \circ R = (y_j)$ , because of the uniqueness of the limit in  $(X, \mathcal{L})$ . Therefore, without loss of generality, we can assume that there exists  $h$  ( $0 \leq h \leq k$ ) such that  $\tilde{S}_j$  is constant if and only if  $j > h$  and delete from (8) the terms  $y_n^j - y_j = 0$ , for every  $n$  ( $j > h$ ).

If  $h = 0$ , clearly  $\bar{x} = 0$  and the thesis follows, otherwise from (8), using the above remarks, we have:

$$(9) \quad \bar{x} + \sum_{j=1}^h z_j y_j = \sum_{j=1}^h z_j y_1^j = \dots = \sum_{j=1}^h z_j y_n^j = \dots$$

Remember that in (9),  $y_n^i \in X$ ,  $y_n^i \neq y_n^j$  (for every  $n$ ) for  $i \neq j$ ,  $y_n^j \neq y_m^j$  (for every  $j$ ) for  $n \neq m$ .

Hence, since  $G$  is a free  $\mathbf{Z}$ -module over  $X$ , we easily conclude that  $z_1 = \dots = z_h = 0$ , and so  $\bar{x} = 0$ .

In fact, if  $z_{j^0} \neq 0$  for some  $j^0$ , then the element  $y_1^{j^0}$  appears with non null coefficient in the second sum of (9) while it does not appear in the  $n$ -th sum of (9) for  $n$  sufficiently large

$$(n > \max \{n \mid y_n^j = y_1^{j^0}, \text{ for some } j\})$$

and we have a contradiction with the uniqueness of the representation of the elements in a free  $\mathbf{Z}$ -module. The proof is complete.  $\square$

**COROLLARY:** *There exists a convergence commutative group which is not separated.* (Cf. [6, problem 12] and [9]).

*Proof:* There exists a star sequential (onevalued) convergence space  $(X, \mathcal{L}, \lambda)$  which is not separated (see [2, p. 81]). Then, consider the free commutative group  $(G, +)$  over  $X$ , endowed with  $\tilde{\mathcal{L}}$ . According to Lemma 1,  $(G, +, \tilde{\mathcal{L}}, \tilde{\lambda})$  is a convergence commutative group with star convergence, and finally recall that every  $\tilde{\lambda}$ -neighbourhood of  $x \in X$  contains a  $\lambda$ -neighbourhood.  $\square$

Let  $(X, \mathcal{L})$  and  $(Y, \mathcal{M})$  be two sequential convergence structures (i.e.  $\mathcal{L}$  and  $\mathcal{M}$  satisfy to (1), (2) and (3)); a map  $f: X \longrightarrow Y$  is *sequentially continuous* if the inference

$$(x_n)_n \xrightarrow{\mathcal{L}} x \Rightarrow (f(x_n))_n \xrightarrow{\mathcal{M}} f(x)$$

holds (cf. [2, p. 70, § 4]).

Let  $(X, \mathcal{L})$  be a sequential convergence structure and let  $(G, +, \tilde{\mathcal{L}})$  be the free commutative group over  $X$ , which convergence is generated by  $\mathcal{L}$ . From (5) it is obvious that the natural embedding  $j: (X, \mathcal{L}) \longrightarrow (G, \tilde{\mathcal{L}})$ ,  $j(x) = x$ , is sequentially continuous; much more we can prove the following universal property:

LEMMA 2. For any multivalued convergence commutative group  $(Y, +, \mathfrak{N})$  <sup>(1)</sup> and for any sequentially continuous mapping

$$f: (X, \mathfrak{L}) \longrightarrow (Y, \mathfrak{N}),$$

there exists a unique sequentially continuous homomorphism

$$g: (G, + \tilde{\mathfrak{L}}) \longrightarrow (Y, +, \mathfrak{N})$$

such that  $g \circ j = f$ .

*Proof of Lemma 2:* It is sufficient to prove that the homomorphism  $g: (G, +) \longrightarrow (Y, +)$  which is defined by

$$g(z) = \sum_{i=1}^n t_i f(x_i), \text{ where } z = \sum_{i=1}^n t_i x_i \text{ (} t_i \in \mathbf{Z}, x_i \in X \text{)}$$

is sequentially continuous. Since  $\tilde{\mathfrak{L}}$  and  $\mathfrak{N}$  are compatible with respect their group operations and  $g$  is a homomorphism, we have only to prove that  $g(\tilde{\mathfrak{S}}_0) \subseteq M_0$ , where  $g(\tilde{\mathfrak{S}}_0) = \{(g(z_n))_n \mid (z_n)_n \in \tilde{\mathfrak{S}}_0\}$  and  $\tilde{\mathfrak{S}}_0, M_0$  are the sets of 0-sequences of  $\tilde{\mathfrak{L}}$  and  $\mathfrak{N}$  respectively. Recall that  $\tilde{\mathfrak{S}}_0 = \zeta \langle \delta \tilde{\mathfrak{B}}_0 \rangle$ .

Let  $S = (x_n)_n - (x) = (x_n - x)_n \in \tilde{\mathfrak{B}}_0$ , then  $g \circ S = (g(x_n - x))_n = (g(x_n) - g(x))_n = (f(x_n) - f(x))_n = (f(x_n))_n - (f(x)) \in M_0$ , because  $f$  is sequentially continuous and for [8, Lemma 1]; hence  $g(\tilde{\mathfrak{B}}_0) \subseteq M_0$ .

Let us observe now that  $g(\delta \tilde{\mathfrak{B}}_0) = \delta(g(\tilde{\mathfrak{B}}_0))$  (for trivial reasons),  $g(\langle \delta \tilde{\mathfrak{B}}_0 \rangle) = \langle g(\delta \tilde{\mathfrak{B}}_0) \rangle$  (since  $g$  is a homomorphism),  $g(\zeta \langle \delta \tilde{\mathfrak{B}}_0 \rangle) = \zeta(g(\langle \delta \tilde{\mathfrak{B}}_0 \rangle))$  (for the same reasons as for  $\delta$ ). Then we conclude that  $g(\tilde{\mathfrak{S}}_0) = g(\zeta \langle \delta \tilde{\mathfrak{B}}_0 \rangle) = \zeta \langle \delta(g(\tilde{\mathfrak{B}}_0)) \rangle \subseteq M_0$ , since  $M_0$  is a subgroup of  $Y^{\mathbf{N}}$  which is closed with respect to  $\delta$  and  $\zeta$  (see [8, Lemma 1]).

□

At the end, we remark that for convergence commutative rings with star convergence (cf. [5, p. 374]) it is possible to obtain the same results we stated here, with the obvious modifications which are necessary for the different setting. A result which is similar

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(1) i.e.  $\mathfrak{N}$  satisfies properties from (1) to (4). It is not necessary that  $\mathfrak{L}, \tilde{\mathfrak{L}}$  and  $\mathfrak{N}$  be onevalued, as considered in [5].

to Lemma 1 is implicitly used in the proof of the uniqueness of the limit in the example in [1]: the interested reader can easily develop analogous properties for commutative convergence rings. For instance, combining the arguments in [1], [9] and Lemma 1 we could prove the existence of a (unitary) convergence commutative ring which is not separated. We omit here the proofs since they do not involve any new idea.

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