UNIFORM REGULARITY OF MEASURES ON COMPLETELY REGULAR SPACES (*)

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SOMMARIO. - *In questa nota ci si occupa della nozione di uniforme regolarità della misura di Baire e Borel su un arbitrario spazio di Hausdorff completamente regolare.*

SUMMARY. - *In this note we deal with the notion of uniform regularity of Baire and Borel measures on arbitrary completely regular Hausdorff spaces.*

§ 1. INTRODUCTION

The methods of topological measure theory, as opposed to those of classical abstract measure theory, rely on continuity rather than on countability. The measure is linked to the topology of the underlying space by the regularity condition which makes possible the identification of the (finitely additive) Baire measures on a completely regular Hausdorff space X with the positive (norm) continuous linear functionals on $C^*(X)$, the space of all bounded continuous functions on X. Completely regular spaces are precisely those topological spaces admitting uniform structures inducing the given topology. The idea of linking measures on X to the admissible uniform structures on X is not new. Zakon, in his work on measures on uniform spaces [15] and [16] has done this through the notion of essential nestedness of an admissible uniformity relative to a given measure. It has turned out that, when applied to measures on compact spaces, the essential nestedness of the admissible

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uniformity implies the metrization of the support of the measure [1]. Another notion which performs this link is that of uniform regularity of measures, which is both a continuity and a countability property of the associated topological measure space. This notion was introduced and discussed in [2] and [4] for compact and locally compact spaces respectively, and it was shown that uniformly regular measures look in many ways like measures on metric spaces.

In this note we deal with the notion of uniform regularity of Baire and Borel measures on arbitrary completely regular Hausdorff spaces. In the absence of compactness, there is no unique admissible uniform structure and hence the notion of uniform regularity is no longer purely topological but depends on the choice of a specific admissible uniformity. Two versions of uniform regularity are of special interest here. Universal uniform regularity — i.e. uniform regularity relative to all admissible uniformities of X — and $\mathcal{C}^*$-uniform regularity, where $\mathcal{C}^*$ is the uniformity generated by all the bounded continuous functions on X. In § 3 we discuss the relation between uniform regularity and additivity. We show that universally uniformly regular measures are τ-additive and we characterize various versions of uniform regularity of measures on P-spaces. $\mathcal{C}^*$-uniformly regular measures are discussed in § 4 where it is shown that they are characterized by the fact that the corresponding linear functionals can be approximated from above on non-negative functions by their values on a separable subspace of $C^*(X)$. This is used to prove the uniform regularity of the induced measures on βX, the Stone-Čech compactification of X and to obtain representations of the associated topological measure spaces along the lines of those given for compact spaces in [3].

§ 2. PRELIMINARY MATERIAL

Throughout, and unless otherwise explicitly stated, X will denote a completely regular Hausdorff topological space and $C^*(X)$ the real Banach space of all bounded realvalued continuous functions on X. All unexplained notions concerning the topology and the uniform structures of X are those of [7] and [8]. A Baire (Borel) measure μ on X is a finite, non-negative, σ-additive realvalued set function defined on all the Baire (Borel) subsets of X. We further assume that μ is regular in the sense of inner approximation by zero sets in Baire case and closed sets in the Borel case, and we shall always use the word 'measure' to mean either a Baire or a Borel measure.

We shall adopt the well known terminology of topological measure theory as in [14]. Thus we shall sometimes identify a Baire measure μ with the corresponding σ-additive linear functional on $C^* (X)$ and we write $< f, \mu >$ for $\int_X f \, d\mu$. Let us recall that a Baire (Borel) measure μ is said to be:

(a) τ-addive if for any decreasing net $\{C_\alpha\}$ of zero (closed) sets with $\bigcap_\alpha C_\alpha = \emptyset$ we have $\mu (C_\alpha) \to 0$;

(b) tight if for any $\varepsilon > 0$ there is a compact set $K \subset X$ such that $\mu^* (K) + \varepsilon > \mu (X)$.

Every Baire measure μ on X induces, in a natural way, a Baire measure $\tilde{\mu}$ on βX, the Stone-Čech compactification of X. If $\tilde{\nu}$ is the Borel extension of $\tilde{\mu}$ then μ is τ-additive if and only if $\tilde{\nu}^* (X) = \tilde{\nu} (\beta X)$ [9].

Now suppose that X is compact and Hausdorff and μ a measure on X. Following [2] we say that μ is uniformly regular if there is a sequence $\{ U_n \}$ of neighbourhoods of the diagonal $\Delta = \{ (x,x) : x \in X\}$ of $X \times X$ such that for any closed μ-measurable set K, $U_n (K) = \{y : (x, y) \in U_n$ for some $x \in K\}$ is μ-measurable for all n, and $\mu (K) = \lim_n \mu [U_n (K)]$. Such measures share a number of properties with measures on compact metric spaces. They have separable supports and separable L^p-spaces, $1 \leqslant p < \infty$ [2]. Furthermore, the associated topological measure space of a uniformly regular measure admits the following representation proved in [3].

2.1. Theorem

Let μ be Baire measure on a compact Hausdorff space X. Then μ is uniformly regular if and only if there is a compact metric space T and a continuous surjection p: $X \to T$ such that for any μ-measurable set E we have

$$\mu (E) = p (\mu) \; [p (E)],$$

where $p (\mu)$ is the image measure in the sense of [5].

The $p (\mu)$ measurability of $p (E)$ in (3.1) follows from the fact that the image of any Baire subset of X is analytic and hence absolutely Borel measurable. The Borel version of (3.1) gives $\mu (E) = p (\mu) \; [p (E)]$ when E is either compact or Baire-measurable (i.e. measurable relative to the Baire restriction of μ).

§ 3. UNIFORM REGULARITY AND ADDITIVITY

Let $\mathcal{U}$ be an admissible uniformity of the completely regular space X. (i.e. $\mathcal{U}$ is a uniformity inducing the given topology on X). We say that a Baire (Borel) measure μ on X is uniformly regular relative to $\mathcal{U}$ (or simply $\mathcal{U}$-uniformly regular) if there exists a decreasing sequence $\{U_n\}$ of elements of $\mathcal{U}$ such that for any zero (closed) set C, $U_n(C)$ is a Baire (Borel) set for all n and,

$$\mu(C) = \lim_n \mu\,[U_n(C)].$$

When X is compact, there is only one admissible uniformity for X and the uniform regularity of a Baire measure was shown to be equivalent to that of its Borel extension [2], prop. (2.1). For an arbitrary completely regular Hausdorff space not every Baire measure μ has a regular Borel extension. However if μ is τ-additive then there is a unique Borel measure ν extending μ [9]. The extension ν satisfies the condition:

$$\nu(Q) = \inf\{\mu(Z) : Z \text{ is a zero set and } Z \supset Q\},$$

for all closed sets $Q \subset X$. For such measures we have,

3.1. PROPOSITION.

Let $\mathcal{U}$ be an admissible uniformity of X. Then a τ-additive Baire measure μ is $\mathcal{U}$-uniformly regular if and only if its Borel extension ν is $\mathcal{U}$-uniformly regular.

Proof.

Let $\mathfrak{D}$ be a family of pseudometrics on X generating $\mathcal{U}$, and let $U = \{(x, y) : d(x, y) < \varepsilon\}$ for a given $d \in \mathfrak{D}$ and $\varepsilon > 0$. Then for any $E \subset X$, the function $f_E : \to I\mathbf{R}$ defined by $f(x) = d(x, E) = \inf\{d(x, y) : y \in E\}$ is continuous. Thus $U(E) = \{x \in X : d(x, E) < \varepsilon\}$ is a cozero set. It follows that $\mathcal{U}$ has a base $\mathcal{V}$ such that $V(E)$ is a Baire set for all $E \subset X$, $V \in \mathcal{V}$. Therefore if the Borel measure ν is $\mathcal{U}$-uniformly regular, its Baire restriction μ is $\mathcal{U}$-uniformly regular.

Conversely suppose that μ is $\mathcal{U}$-uniformly regular, and let $\{U_n\}$ be a sequence of elements of $\mathcal{U}$ such that for any zero set Z, $U_n(Z)$ is a Baire set and $\mu(Z) = \lim_n \mu\,[U_n(Z)]$. Let Q be any closed subset of X, and $\{Z_\alpha\}$ a decreasing family of zero sets such that $\bigcap_\alpha Z_\alpha = Q$. The τ-additivity of μ, gives, for each $\varepsilon > 0$ a zero set Z_{α_j} such that $\mu(Z_{\alpha_j}) < \nu(Q) + \varepsilon$. Hence,

$$\nu(Q) + \varepsilon > \mu(Z_{\alpha_j}) = \lim_n \mu[U_n(Z_{\alpha_j})] \geqslant \lim_n \nu[U_n(Q)].$$

So $\nu(Q) = \lim_n \nu[U_n(Q)]$

i.e. ν is $\mathcal{U}$-uniformly regular.

We now examine the relation between the notion of uniform regularity and additivity properties of the measure. Let us recall that a cardinal number P is said to be non-measurable if the only measure on the discrete space of cardinal P and which vanishes on singleton sets is a trivial measure. i.e. identically zero. Otherwise P is called a measurable cardinal ([12] and [13]). Any measurable cardinal is necessarily weakly inaccessible in the sense of [12], and so it is consistent with standard axiom systems for set theory to assume that all cardinal numbers are non-measurable.

Every measure on a discrete space is trivially uniformly regular relative to the discrete uniformity (i.e. the uniformity consisting of all subsets of $X \times X$ containing the diagonal). Assuming the existence of measurable cardinals, there are discrete spaces admitting measures which are not τ-additive, and so uniform regularity relative to an admissible uniformity does not imply τ-additivity. In the following theorem we generalize theorem (3.3) of [4] and show that τ-additivity follows from a stronger version of uniform regularity. We say that a measure μ on X is *universally uniformly regular* if μ is uniformly regular relative to any admissible uniformity of X.

3.2. THEOREM.

Any universally uniformly regular measure on the completely regular space X is τ-additive.

Proof.

Since the "if" part of proposition (3.1) clearly follows without the assumption of τ-additivity, it is sufficient to prove the theorem in the case when μ is a Baire measure. Embed X in its Stone-Čech compactification βX and let μ be the induced Baire measure on βX and $\tilde{\nu}$ the Borel extension of $\tilde{\mu}$. Suppose that μ is universally uniformly regular and not τ-additive. It can be easily shown that any measure μ' with $\mu' \leqslant \mu$, is universally uniformly regular. So by considering the purely σ-additive part of μ if necessary, we may assume that μ is purely σ-additive (i.e. μ has no nontrival τ-additive minorant). In particular $\mu^* (\{x\}) = 0$ for all $x \in X$. In terms of the induced measures on βX, the pure σ-additivity of μ is equivalent to the condition $\tilde{\nu}^* (X) = 0$ (c.f. [9]).

Let $K \subset \beta X \setminus X$ be a compact set with $\tilde{\nu} (K) = \delta > 0$. The uniformity $\mathcal{U}$ generated by all bounded continuous functions f with $\tilde{f} (K) = 0$ ($\tilde{f}$ being the Stone extension of f to βX) is clearly admissible. $\mathcal{U}$ has a base consisting of $U (f_1, \ldots. f_n; \varepsilon)$, $f_1 \in C^* (X)$, $\tilde{f}_i (K) = 0$, $i = 1, 2, \ldots. n$; $\varepsilon > 0$ where

$$U (f_1, \ldots. f_n; \varepsilon) = \{ (x, y) \in X \times X : | f_i (x) - f_i (y) | < \varepsilon; \quad i = 1, 2, \ldots, n\}$$

Since μ is $\mathcal{U}$-uniformly regular, there exists a sequence $\{U_n\}$ of elements of $\mathcal{U}$ approximating μ uniformly on all zero subsets of X. $\{U_n\}$ can be chosen in such a way that,

$$U_n = \{ (x, y) : | f_i (x) - f_i (y) | < \varepsilon_n, \ i = 1, 2, \ldots, n\}$$

where $\{f_m\}$ is a fixed sequence of bounded continuous functions on X with $\tilde{f}_m (K) = 0$ for all m, and $\{\varepsilon_m\}$ is a sequence of positive numbers decreasing to zero. Write

$$\tilde{Z}_m = \{p \in \beta X; \ \tilde{f}_m (p) = 0\},$$

$$\tilde{Z} = \bigcap_m \tilde{Z}_m$$

$$Z = \tilde{Z} \cap X = \{x \in X : f_m (x) = 0, \ m = 1, 2, \ldots.\}$$

Then $\mu (Z) = \tilde{\mu} (\tilde{Z}) \geqslant \tilde{\nu} (K) = \delta$.

Now let $x \in Z$ and C a zero subset of X such that $x \in C$.

Then,

$$\mu (C) \geqslant \mu (C \cap Z) = \lim_n \mu [U_n (C \cap Z)] =$$
$$= \mu [\bigcap_n U_n (C \cap Z)] \geqslant \mu [\bigcap_n U_n (x)]$$

As $\bigcap_n U_n (x) = Z$, it follows that

$$\mu (C) \geqslant \mu (Z) \geqslant \delta$$

Thus $\mu^* (\{x\}) \geqslant \delta > 0$ — a contradiction.
Therefore μ must be τ-additive.

One natural uniformity to consider for completely regular spaces is the uniformity $C^*(X)$ (or just C^*) generated by all the bounded continuous functions on X. I do not know whether a $\mathcal{C}^*$-uniformly regular measure is necessarily τ-additive. For a certain class of spaces which includes all discrete spaces, the following theorem shows that under a mild set theoretic assumption (viz. all cardinals are non-measurable) uniform regularity relative to some admissible uniformity ensures τ-additivity. The remark preceding theorem (3.2) shows that this assumption is necessary. We also show under an even milder assumption, which is implied by the continuum hypothesis, that τ-additivity follows from $\mathcal{C}^*$-uniform regularity for measures on such spaces. Let us recall that $x \in X$ is said to be a P-point if every $f \in C^*(X)$ is constant in a neighbourhood of x. X is a P-space if every $x \in X$ in a P-point.

3.3. THEOREM.

Let X be a P space and μ a non-zero measure on X. Consider the following conditions:

(a) *μ is universally uniformly regular*

(b) *There is a sequence $\{x_n\}$ of isolated points of X such that $\mu = \sum_n \alpha_n \delta_{x_n}$, $\alpha_n \geqslant 0$ for all n, where for each $x \in X$, δ_x is the measure induced by a unit mass at x.*

(c) *μ is C^*-uniformly regular*

(d) *There is an admissible uniformity $\mathcal{U}$ such that μ is $\mathcal{U}$-uniformly regular.*

Then

(i) $(a) \Leftrightarrow (b)$

(ii) *If the continuum cardinal number c is non-measurable then $(b) \Leftrightarrow (c)$*

(iii) *If the cardinal of X is non-measurable then all the four conditions are equivalent.*

Proof

(i) Suppose that μ is universally uniformly regular. By (3.2), μ is τ-additive and so μ has support S with $\mu^*(S) = \mu(X)$. For any $x \in S$ there is a zero set Z such that $\mu(Z) = \mu^*(\{x\})$. Z being a G_δ is open in X and so $0 < \mu(Z) = \mu^*(\{x\})$. Thus S is at most countable and so μ has the form $\mu = \sum_n \alpha_n \delta_{x_n}$ where $S = \{x_1, x_2, \ldots\}$

and $\alpha_n = \mu^* (\{x_n\})$. Now, let $\{U_m\}$ be any sequence of simmetric open neighbourhoods of the diagonal Δ approximating μ uniformly on zero sets. Then clearly, for each n we have,

$$0 < \alpha_n = \mu [\bigcap_m U_m (x_n)] = \mu [\bigcap_m U_m (x)] = \mu^* (\{x\}),$$

for all $x \in \bigcap_m U_m (x_n)$.

It follows that $\bigcap_m U_m (x_n) = \{x_n\}$ a G_δ in X and hence x is isolated in X.

Conversely, suppose that $\mu = \sum_n \alpha_n \delta_{x_n}$ where x_n is an isolated point of X. Let $\mathcal{U}$ be any admissible uniform structure of X. For each n let V_n be a symmetric element of $\mathcal{U}$ such that $V_n (x_n) = \{x_n\}$ and define $U_n = \bigcap_{j=1}^{n} V_j$. It is easy to see that for any μ-measurable set E, $\mu (E) = \lim_n \mu [U_n (E)]$. Therefore μ is $\mathcal{U}$-uniformly regular.

(ii) Suppose that μ is $\mathcal{C}^*$-uniformly regular. Then we can find a sequence $\{f_n\}$ of bounded continuous functions on X and a sequence $\{\varepsilon_n\}$ of positive numbers decreasing to zero such that if

$$U_n = \{ (x, y) : | f_i (x) - f_i (y) | < \varepsilon_n , \ i = 1, 2, \ldots, n\}$$

then $\{U_n\}$ approximates μ uniformly on zero sets. Define the relation R on X by: $x R y$ if $f_n (x) = f_n (y)$ for all n. Clearly R is an equivalence relation. Let $\hat{X} = X/R$ be the quotient space and $p : X \to \hat{X}$ the quotient map. For each $x \in X$ we have,

$$p^{-1} (p (x)) = \bigcap_n U_n (x) \text{ is clopen in } X,$$

and hence $\hat{X}$ is discrete. Moreover the cardinal of $\hat{X}$ is at most c — a non-measurable cardinal by assumption. It follows that the image measure $\lambda = p (\mu)$ has the form $\lambda = \sum_n \alpha_n \delta_{x_n}$. The $\mathcal{C}^*$-uniform regularity can be used to show that $p^{-1} (\hat{x}_n)$ consists of a single isolated point.

Hence $\mu = \sum_n \alpha_n \delta_{x_n}$ for some sequence $\{x_n\}$ of isolated points of X. i.e. $(c) \Rightarrow (b)$. The converse follows from (i).

(iii) We only need to prove that if X has a non-measurable cardinal then $(d) \Rightarrow (b)$, and the latter follows by an argument similar to the one used in proving (ii). This completes the proof.

Under the assumption that the continuum cardinal c is non-measurable, and in particular under the continuum hypothesis, there exists a completely regular space with the property that for no non-zero measure μ and for no admissible uniformity $\mathcal{U}$ is μ uniformly regular relative to $\mathcal{U}$.

3.4. Example.

Let X be any totally ordered set of cardinal c with the property that for any countable subsets A and B with $a < b$ for all $a \in A$, $b \in B$, there exists $x \in X$ such that $a < x < b$ for all $a \in A$, $b \in B$. For the existence of such a set, called an η_1-set, we refer to [7], Ch. 13. Under the order topology X is a P-space without isolated points. It follows from (3.3) that no non-zero measure on X is universally uniformly regular and that if c is non measurable no non-zero measure on X is uniformly regular relative to any admissible uniformity.

§ 4. $\mathcal{C}^*$-UNIFORMLY REGULAR MEASURES

In this section we give a functional criterion for $\mathcal{C}^*$-uniform regularity. This is used to prove the uniform regularity of the induced measures on the Stone-Čech compactification and to obtain representation theorems along the lines of (2.1). We shall use the following version of the Gelfand theorem.

4.1. Lemma.

Let B be a real Banach algebra of bounded realvalued functions on a set S under the pointwise algebraic operations and the supremum norm. Then B is isometrically isomorphic to the algebra $C(T)$ of all realvalued continuous function on a compact Hausdorff space T.

Proof

By taking the appropriate quotient we may assume that B separates points of S so that the weakest topology on S making every $f \in B$ continuous is completely regular and Hausdorff. Let T be the quotient space obtained by identifying points of βS, the Stone-Čech compactification of S, not distinguished by the algebra $\tilde{B} = \{\tilde{f} : f \in B\}$. An application of the Stone-Weierstrass' theorem gives an isometric isomorphism of B onto $C(T)$.

4.2. THEOREM.

A Baire measure μ *on* X *is* $\mathcal{C}^*$*-uniformly regular if and only if there is a norm separable subalgebra* $B \subset C^*(X)$ *such that for any non-negative* $f \in C^*(X)$ *we have,*

$$<f, \mu> = \inf\{<g, \mu> : g \in B \text{ and } g \geqslant f\}.$$

Proof.

Suppose that μ is $\mathcal{C}^*$-uniformly regular and let $\{U_n\}$ be a sequence of elements of $\mathcal{C}^*(X)$ such that for any zero set $Z \subset X$, we have, $\mu(Z) = \lim_n \mu[U_n(Z)]$. The sequence $\{U_n\}$ can be chosen so that,

$$U_n = \{(x, y) \in X \times X : |f_i(x) - f_i(y)| \leqslant \varepsilon_n,\ i = 1, 2, \ldots, n\},$$

where $\{f_i\}$ is a fixed sequence of bounded continuous functions with $f_1 = 1$, and $\{\varepsilon_i\}$ is a sequence of positive numbers decreasing to zero.

By considering, if necessary, the set of all finite products of functions in $F = \{f_1, f_2, \ldots\}$ we may assume that F is closed under (pointwise) multiplication. It follows that the closed linear hull B of F is a Banach subalgebra of $C^*(X)$ which is clearly norm separable. By (4.1) there is a compact Hausdorff space T such that B is isometrically isomorphic to $C(T)$ the Banach algebra of all continuous functions on T. For $g \in B$, denote by $\tilde{g}$ the image of g under this isomorphism. For $x \in X$ let $p(x)$ be the unique point of T satisfying $\tilde{f}(p(x)) = 0$ for all $\tilde{f}$ such that $f(x) = 0$. Then $p : X \to T$ is continuous. Let $\hat{X} = p(X) \subset T$. Points of $\hat{X}$ may be identified with the equivalence classes induced by the equivalence relation R on X defined by $x R y$ if $g(x) = g(y)$ for all $g \in B$.

Furthermore, for any $x \in X$, $f \in B$ we have

$$f(x) = \tilde{f}(p(x)).$$

For each positive integer n let,

$$\tilde{U}_n = \{(s, t) \in T \times T : |\tilde{f}_i(s) - \tilde{f}_i(t)| \leqslant \varepsilon_n,\ i = 1, 2, \ldots, n\}$$

$$\hat{U}_n = \tilde{U}_n \cap (\hat{X} \times \hat{X}).$$

A routine argument can be used to show that for any $E \subset X$, $p(U_n(E))$ is a zero subset of $\hat{X}$ and

(a) $p\,(U_n\,(E)) = \hat{U}_n\,(p\,(E))$;

(b) $p^{-1}\,p\,(U_n\,(E)) = U_n\,(E)$.

We define the Baire measures $\hat{\mu}$ and λ on $\hat{X}$ and T respectively by:

$$< h, \hat{\mu} > \; = \; < h_o p, \mu >, \text{ for all } h \in C^*(\hat{X});$$
$$< \tilde{g}, \lambda > \; = \; < g, \mu >, \quad \text{for all } \tilde{g} \in C\,(T).$$

It is easy to verify that $\lambda\,(K) = \hat{\mu}\,(K \cap \hat{X})$ for any compact G_δ $-$set $K \subset T$.

Now let Z be any zero subset of X and $\varepsilon > 0$. The uniform regularity of μ gives a positive integer n such that $\mu\,(Z) + \varepsilon >$ $> \mu\,[U_n\,(Z)]$. Thus, using (a) and (b) and denoting the indicator function of a set E by I_E, we have,

$$\begin{aligned}
\mu\,(Z) + \varepsilon > \hat{\mu}\;[\hat{U}_n\,(p\,(Z))]& \\
&= \lambda\;[\tilde{U}_n\,(p\,(Z))] \\
&= \inf\{< \tilde{g}, \lambda > : \tilde{g} \in C\,(T),\; \tilde{g} \geqslant I_{\hat{U}_n(p(Z))}\} \\
&\geqslant \inf\{< \tilde{g}, \lambda > : \tilde{g} \in C\,(T),\; \tilde{g} \geqslant I_{p\,(U_n(Z))}\} \\
&= \inf\{< g, \mu > : g \in B,\, g \geqslant I_{U_n(Z)}\} \\
&\geqslant \inf\{< g, \mu > : g \in B,\, g \geqslant I_Z\} \\
&\geqslant \mu\,(Z).
\end{aligned}$$

Thus,

$$\mu\,(Z) = \inf\{\, <g, \mu > : g \in B,\, g \geqslant I_Z\}$$

This can be easily extended to obtain

$$< f, \mu > \; = \inf\{< g, \mu > : g \in B,\, g \geqslant f\} \qquad (*)$$

for all f of the form $f = \sum_{j=1}^{K} \alpha_j I_{Z_j}$ where for each j, $Z_j \subset X$ is a zero set and $\alpha_j \geqslant 0$. Since any non-negative $f \in C^*(X)$ can be approximated from above in $L^1(\mu)$ by such function, (*) holds for all non-negative $f \in C^*(X)$.

Conversely, suppose that there is a separable subalgebra $B \subset C^*(X)$ such that for any non-negative $f \in C^*(X)$ we have

$$< f, \mu > = \inf \{ < g, \mu > : g \in B, g \geqslant f \}.$$

Let $\tilde{\mu}$ be the induced Baire measure on βX the Stone-Čech compactification of X. If $\tilde{f}$ is the Stone extension of $f \in C^*(X)$, we have

$$< \tilde{f}, \tilde{\mu} > = \inf \{ < \tilde{g}, \tilde{\mu} > : g \in B, \tilde{g} \geqslant \tilde{f} \}$$

for all non-negative $f \in C(\beta X)$. It follows from [3], th. 4.1, that $\tilde{\mu}$ is uniformly regular on βX. Let $\{\tilde{U}_n\}$ be a sequence of neighbourhoods of the diagonal of $\beta X \times \beta X$ approximating $\tilde{\mu}$ uniformly on zero subsets of βX, and write $U_n = \tilde{U}_n \cap (X \times X)$. Then $U_n \in C^*(X)$. For any zero subset Z of X, let $\tilde{Z}$ be any zero subset of βX such that $Z = \tilde{Z} \cap X$. Then,

$$\mu(Z) = \tilde{\mu}(\tilde{Z}) = \lim_n \tilde{\mu}[\tilde{U}_n(Z)] = \lim_n \mu[\tilde{U}_n(\tilde{Z}) \cap X] \geqslant$$

$$\geqslant \lim_n \mu[U_n(Z)] \geqslant \mu(Z).$$

i.e. μ is $\mathcal{C}^*$-uniformly regular on X. This completes the proof.

The following corollary follows from (4.2) and th. (4.1) of [3].

4.3. COROLLARY.

Let μ be a Baire measure on X and $\tilde{\mu}$ the induced measure on βX, the Stone-Čech compactification of X. Then μ is $\mathcal{C}^*$-uniformly regular if and only if $\tilde{\mu}$ is uniformly regular.

We now give a representation theorem for $\mathcal{C}^*$-uniformly regular measures on arbitrary completely regular Hausdorff spaces along the line of that given for compact spaces in (3.1).

4.4. THEOREM.

Let μ be a $\mathcal{C}^*$-uniformly regular Baire measure on X. Then there exists a compact metric space T, a measure λ on T and a continuous map $p : X \to T$ such that for any μ-measurable set $E \subset X$, we have,

$$\mu(E) = \lambda^* [p(E)],$$

where λ^* is the outer measure induced by λ on subsets of T.

Proof.

Let $\{U_n\} \subset \mathcal{C}^*$ be a sequence approximating μ uniformly on zero sets, and let T, λ, p, $\hat{X}$, $\hat{\mu}$ and $\{\tilde{U}_n\}$ be as in the proof of the first part of theorem (4.2). Then T is a compact metric space ($C(T)$ is norm separable), λ is a measure on T and $p : X \to T$ is continuous.

For any μ-measurable set $E \subset X$ and any Baire subset F of T with $p(E) \subset F$ we have,

$$\lambda(F) = \hat{\mu}(F \cap \hat{X}) = \mu[p^{-1}(F \cap \hat{X})] \geqslant \mu(E).$$

Thus, $\mu(E) \leqslant \lambda^* [p(E)]$.

Now let G be a cozero subset of X. Then $G = \bigcup_k Z_k$ where, for each positive integer k, Z_k is a zero subset of X and $Z_k \subset Z_{k+1}$. So,

$$\begin{aligned} \mu(G) &= \lim_k \mu(Z_k) = \lim_k [\lim_n \lambda(\tilde{U}_n(Z_k))] \\ &= \lim_k [\lambda(\bigcap_n \tilde{U}_n(Z_k))] \\ &= \lambda[\bigcup_k \bigcap_n \tilde{U}_n(Z_k)] \\ &\geqslant \lambda^* [p(G)]. \end{aligned}$$

i.e. $\mu(G) = \lambda^* [p(G)]$, for any cozero set G.

By the outer regularity of μ we have,

$\mu(E) = \lambda^* [p(E)]$, for any μ-measurable $E \subset X$. This completes the proof.

If in (4.4) we further assume that μ is tight, then μ is carried by a σ-compact subspace Y and hence for any Baire set $E \subset X$, $p(E \cap Y)$ is analytic in T and thus λ-measurable [6], [10] and [11]. It follows that the inner measure $\lambda_*(p(E)) = \mu(E)$. Hence we have the following representation of tight $\mathcal{C}^*$-uniformly regular measures.

4.5. COROLLARY.

For any $\mathcal{C}^*$-uniformly regular tight Baire measure μ on X there exists a compact metric space T, a Baire measure λ on T and a continuous map $p : X \to T$ such that for any μ-measurable set $E \subset X$, $p(E)$ is λ-measurable and $\lambda[p(E)] = \mu(E)$.

We finally give a local topological property of $\mathcal{C}^*$-uniformly regular probability measures on X. We denote by $\mathcal{P}(X)$ the space of all probability measures on X (i.e. Baire measures μ with $\mu(X) = 1$) endowed with the weakest topology making all the functions: $\mu \to \langle f, \mu \rangle$, $f \in C^*(X)$, continuous. The map $\mu \to \tilde{\mu}$, where $\tilde{\mu}$ is the measure induced by μ on βX, establishes a homeomorphism between $\mathcal{P}(X)$ and a subspace of $\mathcal{P}(\beta X)$. An application of corollary (4.3) above and theorem (5.2) of [17] gives the following result.

4.6. COROLLARY.

Any $\mathcal{C}^*$-uniformly regular measure $\mu \in \mathcal{P}(X)$ has a countable base of neighbourhoods.

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