# UNCONDITIONAL CONVERGENT SERIES AND SUBALGEBRAS OF $C_o(X)$ (*)

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Sommario. - *In questo lavoro si dimostra che una subalgebra di $C_o(X)$ non contenente un sottospazio isomorfo allo spazio $c_o$ di Banach è di dimensione finita. Si dà inoltre una nuova dimostrazione di certi risultati di analisi numerica (teoremi di Helson e di Segal).*

Summary. - *In this paper we prove that subalgebras of $C_o(X)$ not containing a subspace isomorphic to the Banach space $c_o$ is finite dimensional. Also we give new proofs for certain results in harmonic analysis (Helsons and Segals theorems).*

**1. Introduction.** Let $X$ be locally compact Hausdorff space. Denote by $C_b(X)$ the Banach algebra of bounded continuous functions defined on $X$, and $C_o(X)$ the subalgebra of functions vanishing at infinity.

In this paper we give the proofs of the results announced in [1] and continue the study of subspaces of $C_b(X)$ not containing any subspace isomorphic to the Banach space $c_o$ of convergent to zero complex sequences. As a consequence we improve some results of C.F. Dunkl and D.E. Ramirez [4] and gave new proofs of certain results in harmonic analysis (theorem of H. Helson [3], [6] and theorem of L.E. Segal [5], [7], [8], [10]).

The paper is divided into four sections. In the second section we prove a technical lemma concerning certain subspaces of $C_b(X)$.

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In section 3 we extend the results of [4] to the case of subalgebras of $C_o(X)$ not containing a subspace isomorphic to $c_o$. Finally, in section 4 we give applications to harmonic analysis.

We conclude this section by recalling some needed notions. Let $B$ be a Banach space, and $B'$ its dual Banach space. We write $B \supset c_o$ ($B \not\supset c_o$) if $B$ contains (does not contain) a subspace isomorphic to $c_o$. The series $\sum_{n \in N} x_n$ of elements of $B$ is said to be weakly unconditionally convergent (w.u.c.) iff $\sum_{n \in N} |x'(x_n)| < \infty$ for each $x' \in B'$. The series $\sum_{n \in N} x_n$ is unconditionally convergent (u.c.) iff for each permutation $\sigma$ of $N$ the series $\sum_{n \in N} x_{(\sigma)n}$ is convergent. Our proofs are based only on the fact that w.u.c. series are u.c. iff $B \not\supset c_o$ [2].

In what follows subspaces of a Banach space are closed.

**2. Technical lemma.** Let $K$ be compact Hausdorff space and $C(K)$ be the Banach algebra of complex valued continuous functions on $K$. We prove

LEMMA 2.1. *Let $B$ be a subspace of $C(K)$ satisfying the following condition $(P_1)$: for each open subset $U$ of $K$ and each $\varepsilon > o$ there exists $f \in B$ such that $\|f\| = 1$ and $|f(x)| < \varepsilon$, $x \notin U$. Let $K$ be infinite set. Then $B \supset c_o$.*

*Proof.* Since $K$ is infinite, there exists a sequence $U_n$ of open subsets of $K$ such that $U_i \cap U_j = \emptyset$, $i \neq j$. Using $(P_1)$ for each $U_i$ one can find $f_i \in B$ such that $\|f_i\| = 1$, $|f_i(x)| < \frac{1}{2^i}$, $x \notin U_i$. We have $\sup_{x \in K} \sum_{i=1}^{\infty} |f_i(x)| < 2$, hence the series $\sum_{i=1}^{\infty} f_i$ is w.u.c. Since $\sum_{i=1}^{\infty} f_i(x)$ is not uniformly convergent on $K$, the series $\sum_{i=1}^{\infty} f_i$ is not u.c.. Hence $B \supset c_o$.

COROLLARY 2.1. *Let $A$ be a subalgebra of $C(K)$ satisfying the condition $(P_2)$: for each open subset $U \subset K$ there exists $f \in A$ such that $\|f\| = 1$, $|f(x)| < 1$, $x \notin U$. Let $K$ be infinite. Then $A \supset c_0$.*

The proof follows from the fact that $(P_2)$ implies $(P_1)$.

COROLLARY 2.2. *Let $E$ be a subspace of $C_0(X)$ satisfying the condition $(P)$: for each compact $K$ of $X$ and each $\varepsilon > 0$ there exists $f \in E$ such that $\|f\| = 1$, and $|f(x)| < \varepsilon$, $x \in K$. Then $E \supset c_0$.*

*Proof.* Condition $P$ implies that $X$ is not compact, hence it is infinite. Take the one point compactification $X_0$ *of* $X$. Then $E$ is isometric to a subspace of $C(X_0)$ satisfying the condition $(P_1)$ of Lemma 2.1. Hence $E \supset c_0$.

COROLLARY 2.3. *Let $Y$ be locally compact group which is not compact. Denote by $E$ the subspace of $C_0(Y)$ generated by the left translations of a function $f$ of $C_0(Y)$, $\|f\| = 1$. Then $E \supset c_0$.*

*Proof.* Since $f \in C_0(Y)$, for each $\varepsilon > 0$ there exists a compact subset $K_0$ of $Y$ such that $|f(x)| < \varepsilon$, $x \notin K_0$. Let $K$ be a compact subset of $Y$. Then there exists an element $a \in Y$ such that $aK \cap K_0 = \varnothing$. The function $f(ax)$ satisfies the condition $(P)$ of corollary 2.2. This proves the needed result.

**3. Subalgebras of $C_0(X)$.** In this section we prove the following

THEORFM 3.1. *Let $A$ be a subalgebra of $C_0(X)$. If $A \not\supset c_0$ then $A$ is finite dimensional.*

*Proof.* Noticing that each positive function of $C_0(X)$ attains its maximum on $X$, we show that $A$ satisfies the condition $(P')$: there exists a compact subset $K_s$ of $X$ such that $\{y : f(y) = \|f\|\} \cap K_s \neq \varnothing$, $f \in A$. Indeed, assuming the contrary, for each compact subset $K$ of $X$ there exists an element $f \in A$ such that $\{y : \|f\| = f(y)\} \cap K = \varnothing$. This condition is equivalent to $(P)$ of corollary 2.2. Hence $A \supset c_0$. This is a contradiction which proves $(P')$. Now using transcendental induction we can assume that $K_s$ is a minimal subset satisfying $(P')$ (i.e. $K_s$ contains no proper subset satisfying $(P')$). Hence for each point $x \in K_s$ and each open subset $U(x)$ containing $x$, there exists $f \in A$ such that

$$\max_{K_s \setminus U(x)} |f(x)| < \|f\| = \max_{K_s} |f(x)| \qquad (P'')$$

Otherwise, there exists $x_0 \in K_s$ and an open subset $U(x_0)$ containing $x_0$ such that $\max_{x \in K_s \setminus U(x_0)} |f(x)| = \|f\|$, $f \in A$. This contradicts the fact that $K_s$ is a minimal subset satisfying $(P')$ and proves $(P'')$. Now consider the linear isometric mapping $f \to f|_{K_s}$ of $A$ onto the closed subalgebra $A|_{K_s}$ consisting of the restrictions of elements of $A$ on $K_s$. Noticing that $A | K_s$ satisfies $(P_2)$ of corollary 2.1., we conclude that $K_s$ is finite. This implies that $A|_{K_s}$ is finite dimensional and hence $A$ is finite dimensional.

Let $B$ be a Banach space and $B \not\supset c_0$, let $A$ be infinite dimensional subalgebra of $C_0(X)$. We prove the following.

THEOREM 3.2. *Let* $\rho \in L(B, A)$. *There does not exist a section* $\pi : A \rightarrow B$, *that is* $\pi \in L(A, B)$ *for which* $\rho_0 \pi = id$.

*Proof.* Assume that $\pi$ of $L(A, B)$ is a section. Let $\sum_{n=1}^{\infty} a_n$ be w.u.c. series of $A$. Then $\sum_{n=1}^{\infty} \pi a_n$ is also w.u.c. series of $B$, therefore it is u.c. [2]. Consequently, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (\rho_0 \pi) a_n$ is also u.c. Again using [2] we conclude that $A \not\supset c_0$. By theorem 3.1. the subalgebra $A$ is finite dimensional, a contradiction which proves the theorem.

REMARK 3.1. *Weakly sequentially Banach spaces are not containing* $c_0$. This shows that theorem 3.1. and 3.2. are the improvement of [4].

**4. Application to harmonic analysis.** Let $P$ be a Helson subset of the locally compact abelian group $G$ and $M(P)$ the Borel measures concentrated on $P$.

THEOREM 4.1. (HELSON). *If* $\mu M(P)$, $\mu \mp 0$, *then* $\mu \notin C_0(\Gamma)$.

*Proof.* We consider only the case when $G$ is nondiscrete, for the proof in the discrete case is trivial. The dual group $\Gamma$ is noncompact locally compact group. The space $M(P)$ is isomorphic to the subspace $E$ of $C_b(\Gamma)$ consisting of the Fourier transforms of $M(P)$, and hence the space $E$ contains its translates ([9] p. 115). If $\mu \in M(P)$, $\hat{\mu} \in E \cap C_0(\Gamma)$, then using corollary 2.3. the subspace generated by $\hat{\mu}$ and its translates contains $c_0$ which means that $E \supset c_0$. This contradicts the fact that $E$ is isomorphic to the weakly sequetially complete Banach space $M(P)$ and proves the theorem.

THEOREM 4.2. (SEGAL). *If* $L^1(G)$ *is isomorphic to* $C_0(\Gamma)$, *then* $G$ *is finite* [8], [10].

The proof follows from corollary 2.3. or theorem 3.1. using the fact that $L^1(G) \not\supset c_0$.

The same result is true for the Banach space $L^p(G)$, $p > 1$.

Also if $L^p(G)$ is isomorphic to the space $AP(\Gamma)$ of almost periodic functions on $\Gamma$, then $G$ is finite [7].

Finally, Let $G$ be compact group and $\Sigma$ be its dual object. If $L^1(G)$ is isomorphic to $C_0(\Sigma)$, then $G$ is finite.

## REFERENCES

[1] B. BASIT, *Series commutativement convergentes et analyse harmonique,* C.R. Acad. Sc. Paris, t. 285, Serie A (1977), 849-850.

[2] C. BESSAGE and A. PELCZENSKI, *On basis and unconditional convergence of series in Banach spaces,* Studia Math., T. 17 (1958), 151-164.

[3] R. DOSS, *Elementary proof of a theorem of Helson,* Proc. Amer. Soc., 27 N⁰ 2 (1971), 418-420.

[4] C. F. DUNKL, and D. E. RAMIREZ, *Sections induced from weakly sequentially complete spaces,* Studia Math. T. 39 (1973), 95-97.

[5] R. E. EDWARDS, *On fuctions which are Fourier transforms,* Proc. Amer. Math. Soc. 5 (1954), 71-78.

[6] H. HELSON, *Fourier transforms on perfect sets,* Studia Math., 14, (1954) 209-213.

[7] E. HEWITT, *Representation of functions as absolutely convergent Fourier-Stieltjes transformations,* Proc. Amer. Math. Soc. 4, (1953), 663-670.

[8] M. RAJAGOPALAN, *Fourier transform in locally compact groups,* Acta Litt. Sco. Szeged 25 (1964), 86-89.

[9] W. RUDIN, *Fourier analysis on groups* (Interscience, N.Y. 1962).

[10] I. E. SEGAL, *The class of functions which are absolutely convergent,* Fourier transforms, Acta Litt. Sci. Szeged 12 (1950), 157-161.