A NOTE ON THE METHOD OF ORTHOGONAL INVARIANTS (*)

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SOMMARIO. - *Il metodo degli invarianti ortogonali sviluppato da Fichera (vedi [4, 5]) è applicato a problemi di autovalori del tipo:*

(1) $$Tu = \lambda\, Su$$

dove T *ed* S *sono operatori* K*-positivi in uno spazio* H*, di Hilbert, complesso e separabile.*

SUMMARY. - *The method of orthogonal invariants as developed by Fichera (see [4, 5]) is applied to eigenvalue problems of type*

(1) $$Tu = \lambda\, Su$$

where T *and* S *are* K*-positive operators in a complex separable Hilbert space* H.

1. Preliminaries

First we describe the method of orthogonal invariants which uses orthogonal invariants of degree one and order one following Fichera [4] which we apply in the next section to some linear eigenvalue problem.

Let a complex separable Hilbert space H be given with inner product $(.,.)$ and let K be a compact positive definite operator in H where

$$0 < \ldots \leqslant \lambda_2 \leqslant \lambda_1$$

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are the eigenvalues of K repeated according to their multiplicity. For a arbitrary complete orthonormal system $\{v_i\}_{i=1}^{\infty}$ in H we consider the expression

$$J_1^1\,(K) := \sum_{i=1}^{\infty} (Kv_i, v_i).$$

$J_1^1\,(K)$ is called the orthogonal invariant of degree one and order one and its value is independent of the special complete orthonormal system $\{v_i\}_{i=1}^{\infty}$.

Now let $\{w_i\}_{i=1}^{\infty}$ a complete system of linear independent vectors in H and denote by P_m the orthogonal projection from H on span $(w_1, \ldots, w_m)$. It follows that the positive eigenvalues

$$\lambda_m^{(m)} \leqslant \ldots \leqslant \lambda_2^{(m)} \leqslant \lambda_1^{(m)}$$

of the operator $P_m K P_m$ are the Rayleigh-Ritz lower bounds for the eigenvalues of K which can be determined as the positive roots of the equation

$$\det \{(Kw_i, w_j) - \lambda\,(w_i, w_j)\}_{i,j=1}^{m} = 0.$$

Thus we have

THEOREM 1.

Let $J_1^1\,(K) < \infty$ *and define*

$$\sigma_k^{(m)} := J_1^1\,(K) - \sum_{i=1}^{m} \lambda_i^{(m)} + \lambda_k^{(m)}, \quad k, m = 1, 2, \cdots.$$

Then

$$\sigma_k^{(m)} \leqslant \sigma_k^{(m+1)} \text{ and } \lim_{m\to\infty} \sigma_k^{(m)} = \lambda_k,$$

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2. Eigenvalue Problems of Type $Tu - \lambda Su = 0$

We denote by $D\,(A)$ and $R\,(A)$, respectively, the domain and the range in H of a given linear operator A. In this section we consider the eigenvalue problem

$$Tu = \lambda\, Su, \quad u \in D\,(T),$$

where T and S are linear operators such that $D(T)$ is dense in H and $D(T) \subseteq D(S)$, $S \neq 0$.

We assume that the following conditions are satisfied:

(K 1) $$R(S) \subseteq R(T);$$

(K 2) there exists a linear operator K with $D(K) \supseteq D(T)$ such that

$$\begin{aligned} (Tu, Kv) &= (Ku, Tv), \ u, \ v \in D(T), \\ (Su, Kv) &= (Ku, Sv), \ u, \ v \in D(T), \end{aligned}$$

(3) $$\begin{aligned} (Tu, Ku) &> 0, \ u \in D(T), \ u \neq 0, \\ (Su, Ku) &> 0, \ u \in D(T), \ u \neq 0; \end{aligned}$$

(K 3) if λ is not an eigenvalue of equation (1), then

$$Tu - \lambda Su = f$$

has a solution for each $f \in R(T)$.

Let $D[T]$ denote the set $D(T)$ with the new metric

$$[u, v]_T := (Tu, Kv), \| u \|_T := [u, u]_T^{1/2}, \qquad u, \ v \in D(T).$$

Formula (3) implies the existence of $T^{-1} : R(T) \to D(T)$.

Thus, the operator $N := T^{-1} S : D(T) \to D(T)$ is defined. We suppose, that the following condition is satisfied: (K 4) the operator $N : D[T] \to D[T]$ is precompact.

In [3] (see also [1], [6], [7], [8]) there are given several conditions which imply the validity of (K 4).

The eigenvalue problem (1) which satisfies (K 1) - (K 4) has countably many positive eigenvalues of finite multiplicity

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots$$

with eigenvectors u_1, u_2, ... orthonormalized by

$$(Tu_i, Ku_j) = \delta_{ij}, \quad i, j = 1, 2, \cdots.$$

These eigenvalues can be characterized through the following variational principle

$$\frac{1}{\lambda_n} = \sup_{V_n \subseteq D(T)} \ \inf_{0 \neq u \in V_n} \frac{(Su, Ku)}{(Tu, Ku)}, \qquad n = 1, 2, \cdots.$$

Eigenvalue problems of type (1) satisfying some or all of the conditions (K 1) - (K 4) have been considered by Abramov [1, 2], Abramov and Harazov [3], Harazov [6] and Petryshyn [8] which give besides existence theorems also numerical methods for the approximative computation of the eigenvalues of (1).

We describe now an application of the method of orthogonal invariants (see Theorem 1) to the eigenvalue problem (1). The operator $N = T^{-1} S : D[T] \to D[T]$ is symmetric and positive in $D[T]$:

$$[Nu, v]_T = [T^{-1} S\, u, v]_T = (Su, Kv) = (Ku, Sv) = [u, Nv]_T,$$

$$[Nu, u]_T = (Su, Ku) > 0, \quad u,\ v \in D[T].$$

Thus, the extension by continuity of N to the completion $\overline{D[T]}$ of $D[T]$ is a compact, selfadjoint and positive operator denoted by $\overline{N}$.

Applying Theorem 1 to the operator $\overline{N}$ let $\{w_i\}_{i=1}^{\infty}$ be a system of linearly independent vectors in $D[T]$ which is complete in $\overline{D[T]}$ and let

$$\mu_1^{(m)} \geq \mu_2^{(m)} \geq \dots \geq \mu_m^{(m)}$$

the positive roots of the equation

$$\det \{ (Sw_i, Kw_j) - \mu\, (Tw_i, Kw_j) \}_{i,j=1}^{m} = 0.$$

We define the orthogonal invariant $J_1^1(S, T, K)$ of the triplet (S, T, K) of degree one and order one through

$$J_1^1(S, T, K) := \sum_{i=1}^{\infty} (Sv_i, Kv_i)$$

where the system $\{v_i\}_{i=1}^{\infty}$ satisfies

$$(Tv_i, Kv_j) = \delta_{ij}, \quad i, j = 1, 2, \cdots,$$

and is complete in $\overline{D[T]}$.

From Theorem 1 then it follows

THEOREM 2.

Let $J_1^1(S, T, K) < \infty$ *and define*

$$\sigma_k^{(m)} := J_1^1(S, T, K) - \sum_{i=1}^{m} \mu_i^{(m)} + \mu_k^{(m)}, \qquad k, m = 1, 2, \cdots.$$

Then

$$\sigma_k^{(m)} \geq \sigma_k^{(m+1)} \text{ and } \lim_{m\to\infty} \sigma_k^{(m)} = \mu_k = \frac{1}{\lambda_k},$$

$$\mu_k^{(m)} \leq \mu_k^{(m+1)} \text{ and } \lim_{m\to\infty} \mu_k^{(m)} = \mu_k = \frac{1}{\lambda_k}, \qquad k = 1, 2, \cdots.$$

Orthogonal invariants of a higher order and a higher degree of the operator $\overline{N}$ can also be considered, but in the second case there occur higher powers of the operator T^{-1}.

3. **Example.**

We consider the following ordinary differential eigenvalue problem arising in problems of elastic stability (cf. Petryshyn [8]):

$$- u''' = \lambda\ (1 - x^2)\ u' = 0,\ u\ (0) = u'\ (0) = u''\ (1) = 0.$$

In this case we have $H = L_2\ (0,1)$ and

$$D\ (T) = \{u \in C_3\ (0,1) \mid u\ (0) = u'\ (0) = u''\ (1) = 0\},$$

$$D\ (S) = D\ (K) = \{u \in C_1\ (0,1) \mid u\ (0) = 0\},$$

$$Ku = u', Tu = - u''', (Su)\ (x) = (1 - x^2)\ u'\ (x).$$

The system

$$u_i\ (x) = \frac{4\sqrt{2}}{\pi^2\ (2i-1)^2}\ (1 - cos\ (2i-1)\frac{\pi}{2}\ x), \qquad i = 1, 2, ...,$$

is a complete orthonormal system in $D\,[T]$. Therefore we have

$$J_1^1\ (S, T, K) = \sum_{i=1}^{\infty}\ (Su_i, Ku_i) =$$

$$= \sum_{i=1}^{\infty} \left(\frac{8}{3\,\pi^2\ (2i-1)^2} - \frac{8}{\pi^4\ (2i-1)^4} \right)$$

$$= \frac{1}{4}.$$

The obtained bounds for $m = 3$ are

$$4.598 \leq \lambda_1 \leq 5.122$$

$$21.072 \leq \lambda_2 \leq 39.679$$

$$33.818 \leq \lambda_3 \leq 136.699$$

(where u_1, u_2, u_3 have been used in the Rayleigh-Ritz method).

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