PURE SUBMODULES OF INJECTIVE MODULES (*)

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SOMMARIO. - *Si dà una nuova caratterizzazione degli anelli semiprimi di Goldie.*

SUMMARY. - *We assigne a new characterization of semiprime Goldie rings.*

1. Introduction.

If G is an abelian group, then a subgroup H of G is pure if $h \varepsilon H$ and $h = ny$ (n an integer, $y \varepsilon G$) imply $h = nh_1$ with $h_1 \varepsilon H$. When R is an integral domain and M is a torsionfree left R-module, this definition has been generalized to: a submodule P of M is pure if $rm \varepsilon P$ and $r \neq 0$ imply $m \varepsilon P$ ([8]). However, when R is not an integral domain, then there are several ways of defining purity. One of the most widely used definitions is the one given by P. M. Cohn in [1], and, with this definition, one may characterize left Noetherian rings as those rings with the property that pure submodules of injective modules are injective (cf. [2] p. 133).

In this note, we consider the case when R is a semiprime left Goldie ring, define purity in terms of the regular elements of R and, in analogy to the preceding result, characterize semiprime left Goldie rings as those rings with the property that pure submodules of torsionfree injective modules are injective.

More precisely, we show:

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THEOREM 1: *If* R *is a ring with a left quotient ring* S, *then* S *is semisimple artinian if and only if every pure submodule of an injective, torsionfree left* R*-module is injective.*

In connection with the existence of the left quotient ring S, the following result is proved:

THEOREM 2: *Let* R *be an integral domain. If* R *has a torsionfree divisible left module, then* R *possesses a left quotient ring.*

2. **Preliminaries.**

In what follows, R is an associative ring with unit and M is a left R-module. A *regular* element of R is one which is not a zero-divisor. A ring S is said to be a (classical) *left quotient ring* of R if S contains R, every regular element of R has a two-sided inverse in S, and every element of S has the form $d^{-1}r$ for properly chosen r, d in R. It is known that R has a left quotient ring if and only if, for every r, d in R with d regular, there exist r_1, d_1 in R, with d_1 regular, such that $d_1 r = r_1 d$. We recall the definition of a *left Goldie ring* as a ring which satisfies the ascending chain condition on annihilator left ideals and has no infinite direct sums of nonzero left ideals. We also recall the well-known result of A. W. Goldie: a ring R has a semisimple artinian left quotient ring if and only if R is a semiprime left Goldie ring ([3]).

An element m of M is a *torsion element* of M if $dm = 0$ for some regular d in R. M is said to be *torsionfree* if it contains no nonzero torsion elements. M is said to be *divisible* if $dM = M$ for every regular d in R. If R has a left quotient ring S, then it is known that M is an R-submodule of some S-module if and only if M is torsionfree; when the condition holds, every element of SM has the form $d^{-1}m$ $(m \in M,\ d \in R)$ and $SM \cong S \otimes_R M$ under the correspondence $sm \rightarrow s \otimes m$ (cf. e.g. Proposition 1.5 of [6]). Because of this, we shall use the phrase «consider M to be a submodule of $S \otimes_R M$» to mean «identify m and $1 \otimes m$». When this identification is permissible, we shall then have $SM = S \otimes_R M$.

3. **Pure Submodules.**

Since our main result is concerned with semiprime left Goldie rings, in which regular elements play a prominent role, and in view of the definition — for R an integral domain —: P pure in M if $rm \in P$ and $r \neq 0$ imply $m \in P$, the following definition seems the natural one for us to use:

Definition: A submodule P of M is said to be *pure in M* if, for any $m \in M$ and regular $d \in R$, $dm \in P$ implies $m \in P$.

With this definition of purity, the pure submodules of a torsionfree M over a semiprime left Goldie R form a complete lattice which is lattice-isomorphic to the lattice of submodules of the completely reducible S-module SM; in fact, in this case, the pure submodules of M coincide with the complements of M (cf. [4], Theorem 2.2 and Lemma 2.4). For our purposes here, however, we shall only need the following two simple lemmas.

LEMMA *1: Let* R *be a semiprime left Goldie ring with left quotient ring* S, M *a torsionfree left* R*-module and consider* M *to be a submodule of* S $_R$M. *Then:*

(i) P *is a pure submodule of* M *if and only if* P *is equal to the intersection of* SP *and* M.

(ii) *If* P' *is any* S*-submodule of* SM *and* P *is the intersection of* P' *and* M, *then* P' = SP *and* P *is a pure submodule of M.*

Proof: (i) Clearly, P is contained in the intersection of SP and M, for any submodule P of M. To show the reverse inclusion, suppose P pure and let $m = d^{-1}p$ be in SP and in M. Then $dm = p$ and therefore $m \varepsilon P$, since P is pure. Hence P equals the intersection of SP and M.

Conversely, assume P is equal to the intersection of SP and M and let $rm \varepsilon P$, with r regular in R. Then $r^{-1}(rm) \varepsilon SP$ and $r^{-1}(rm) = m \varepsilon M$; hence $m \varepsilon P$ and P is pure. (ii) Clearly, SP is contained in P'. Conversely, if $d^{-1}m \varepsilon P'$, then $d\,(d^{1-}m) = m \varepsilon P'$, hence, since $m \varepsilon M$, $m \varepsilon P$. Then $d^{-1}m \varepsilon SP$, i.e. P' is contained in SP.

Finally, since $P' = SP$, and P is the intersection of P' and M, P is pure by (i).

Remark: By (ii) of Lemma 1, one can write any S-submodule P' of SM in the form SP, where P is the intersection of P' and M and is a pure submodule of M.

LEMMA 2: *Let* R *and* M *be as in Lemma* 1. *Then, if* M *is injective, every pure submodule of* M *is a direct summand.*

Proof: Since M is injective, it is divisible ([6], Theorem 3.1), and therefore $SM = M$ and SQ is contained in M for any submodule Q of M. Hence, for P a pure submodule of M, $P = SP$. By the preceding Remark, any S-submodule, Q', of SM may be written as SQ, with Q pure in M. Hence, since M is completely reducible as an S-module, for any pure P in M, $M = SM = SP \otimes SQ = P \otimes Q$.

COROLLARY 1: *If* M *is a torsionfree, injective module ove a semiprime left Goldie ring, then every pure submodule of* M *is injective.*

COROLLARY 2: *If* M *is a torsionfree, divisible module over a semi-*

prime left Goldie ring, then every pure submodule of M *is divisible and a direct summand.*

Proof of Theorem 1: If S is semisimple artinian, then R is semiprime left Goldie, and by Corollary 1, every pure submodule of a torsionfree injective left R-module is injective.

Conversely, assume that every pure submodule of an injective, torsionfree ${}_RM$ is injective. By Theorem 3.3 of [6], if R has a left quotient ring, S, then S is semisimple artinian if and only if every torsionfree divisible left R-module is injective. Therefore, to show S is semisimple artinian, let M be torsionfree divisible and show M is injective.

Since M is divisible, $Hom_Z(R, M)$ is an injective left R-module (cf. e.g. [7], p. 28) and we have a monomorphism f from M into $Hom_Z(R, M)$ given by:

$$[f(m)](r) = rm, \text{ for } m \in M \text{ and } r \in R.$$

To show $Hom_Z(R, M)$ is torsionfree, let $g \in Hom_Z(R, M)$ and d, regular, in R, be such that $dg = 0$. Then, for each $r \in R$, $0 = (dg)(r) = d[g(r)] = dm$, for some $m \in M$, and, since M is torsionfree, this implies $m = g(r) = 0$. Hence, $g = 0$ and $Hom_Z(R, M)$ is torsionfree.

Set $M_1 = Hom_Z(R, M)$ and consider M as a submodule of M_1 via the monomorphism f. Since M is divisible, $SM = M$ and hence M is equal to the intersection of SM and M_1, which, by Lemma 1 (i), implies that M is pure as a submodule of M_1. Now, by hypothesis, since M is a pure submodule of the torsionfree injective module M_1, M is injective and the proof is complete.

Proof of Theorem 2: Let R be an integral domain which has a torsionfree divisible left module D. As in the preceding proof, $M = Hom_Z(R, D)$ is a torsionfree injective left R-module. Let r, s be any two elements of R and fix m in M. Let $N = Rsm$ and let P be an injective hull of N contained in M. P exists since M is injective (cf. e.g. [5], Lemma 4, p. 91).

Since P is injective, it is divisible; hence if $dm_1 = p \in P$, for some $m_1 \in M$ and $d \in R$, then $p = dp_1$, with $p_1 \in P$, and therefore $dm_1 = dp_1$ and $m_1 = p_1 \in P$, i.e. P is a pure submodule of M. Since P is pure and $sm \in P$, m must be in P and therefore $rm \in P$ and Rrm is contained in P. Since P is an injective hull of N, N is essential in P and therefore N intersects Rrm nontrivially. Hence there exist u_1, u_2 in R such that $u_1rm = u_2sm$, or $(u_1r - u_2s)m = 0$. Since M is torsionfree and all elements of R are regular, this gives $u_1r = u_2s$, a common left multiple of r and s. By the remarks in Section 2, this is sufficient for R to have a left quotient ring.

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