SEMISIMPLENESS, COMPLETENESS, AND DIMENSION OF A BANACH ALGEBRA (*)

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SOMMARIO. - *In questa nota si caratterizzano la semisemplicità, la dimensione e certe proprietà di completezza di un'algebra di Banach combinando e facendo intervenire contemporaneamente proprietà algebriche, topologiche e di teoria della misura le quali sono naturalmente associate ad una tale algebra. La caratterizzazione ottenuta per le algebre semisemplici in termini di normalità, continuità e proprietà* T_2 *completano alcuni risultati precedenti di [3]. La caratterizzazione della completezza e della dimensione estende considerevolmente il lavoro di Cohen e risponde ad alcune delle questioni da lui poste in [1].*

SUMMARY. - *This note characterizes semisimpleness, dimension, and certain completeness properties of a Banach algebra by combining and interlacing algebraic, topological, and measure-theoretic properties naturally associated with such an algebra. Our characterization of semisimpleness in terms of normability, continuity, the* T_2 *property, and denseness nicely rounds out some earlier results in [3]. Our characterizations of completeness and dimension considerably extend Cohen's work and answer some of the questions raised by him in [1].*

1. Preliminaries.

Let $\mathcal{H}$ and $\mathfrak{M}$ respectively denote the non zero homomorphisms and the maximal ideals of a complex, commutative Banach algebra $(X, \| \ \|)$ with identity e and continuous dual X'. If $M \in \mathfrak{M}$, then

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$X/M = \mathbf{C}$ and $x + M = x(M)(e + M)$ for each $x \in X$. Therefore each fixed $x \in X$ determines a mapping $\hat{x}$: $\begin{array}{c} \mathfrak{M} \longrightarrow \mathbf{C} \\ M \rightsquigarrow x(M) \end{array}$, and $\mathfrak{M}$ is assumed to carry the weakest (compact, T_2) topology under which every member of $\hat{X} = \{\hat{x} : x \in X\}$ is continuous. As is well known, the bijection $\begin{array}{c} \mathcal{H} \longrightarrow \mathfrak{M} \\ h \rightsquigarrow h^{-1}(0) \end{array}$ is bicontinuous when $\mathcal{H}$ carries the weakest topology $\sigma(\mathcal{H}, X)$ under which every ξ_x: $\begin{array}{c} \mathcal{H} \longrightarrow \mathbf{C} \\ h \rightsquigarrow h(x) \end{array}$ $(x \in X)$ is continuous.

It is assumed throughout that ν is a probability measure on $\mathfrak{M}$ which is positive on non-empty open subsets of $\mathfrak{M}$. For example, if $\mathfrak{M}$ is separable with $\overline{\{M_n : n \in N\}} = \mathfrak{M}$ [as will be the case ([2], 426) when X is separable or $\mathfrak{M}$ is metrizable], then $\nu = \sum_{n=1}^{\infty} 2^{-n} \chi_{M_n}$ meets our requirements (here χ_{M_n} denotes the characteristic function of $M_n \in \mathfrak{M}$).

2. Semisimpleness Property.

By definition, X is semisimple if $\bigcap_{\mathfrak{M}} M = \{0\}$. One can readily show that the following are also equivalent: (1) X is semisimple; (2) the seminorm $r_\sigma(x) = \sup_{\mathfrak{M}} |\hat{x}(M)| = \sup_{\mathcal{H}} |h(x)|$ is a norm on X; (3) the mapping ψ: $\begin{array}{c} X \longrightarrow \hat{X} \\ \chi \rightsquigarrow \hat{x} \end{array}$ is injective (in fact, an $r_\sigma - \|\ \|_o$ congruence since $r_\sigma(x)$ coincides vith the *sup norm* $\|\hat{x}\|_o$ on $\mathfrak{M}$ for each $x \in X$).

Semisimpleness may be characterized in terms of normability, continuity, the T_2 property, and denseness once we prove the following.

LEMMA. *Distinct members of $\mathcal{H}$ are linearly independent.*

Proof. By induction. The result is trivially true for $n = 1$. If $\alpha_1 h_1 + \alpha_2 h_2 = 0$ and $h_2(x_o) \neq h_1(x_o)$ for some $x_o \in X$, subtract $h_2(x_o)\{\alpha_1 h_1(x) + \alpha_2 h_2(x)\} = 0$ from $\alpha_1 h_1(x) h_1(x_o) + \alpha_2 h_2(x) h_2(x_o) = 0$ to obtain $\alpha_1\{h_1(x_o) - h_2(x_o)\} h_1(x) = 0$ for each $x \in X$. Thus, $\alpha_1 = \alpha_2 = 0$. If every set of n distinct members of $\mathcal{H}$ is linearly independent and $\sum_{i=1}^{n+1} \alpha_i h_i = 0$, then $h_{n+1}(x_o) \neq h_1(x_o)$ for some $x_o \in X$ and $0 = \sum_{i=1}^{n+1} \alpha_1 h_1(x) h_i(x_o) - h_{n+1}(x_o) \sum_{i=1}^{n+1} \alpha_i h_i(x) =$ $= \sum_{i=1}^{n} \alpha_i \{h_i(x_o) - h_{n-1}(x_o)\} h_i(x)$ for all $x \in X$. By hypothesis, $\alpha_1 = 0$

and $\sum_{i=2}^{n+1} \alpha_i h_i = 0$ implies that $\alpha_i = 0$ for all $i = 2, 3, \ldots, n + 1$.

THEOREM 1. *The following are equivalent for* X:

(i) X *is semisimple*

(ii) *The seminorm* $p_\alpha(x) = \left\{ \int_{\mathfrak{M}} |\hat{x}|^\alpha d\nu \right\}^{1/\alpha}$ $(\alpha \geq 1)$ *is a norm*

(iii) *There is a Hausdorff* TVS Y *and a collection* $\mathfrak{A}$: X ⟶ Y *of* p_α*— continuous linear mappings satisfying* $\bigcap_{\mathfrak{A}} A^{-1}(0) = \{0\}$

(iv) $\sigma(X, \mathcal{H})$ *is* T_2

(v) $\overline{[\mathcal{H}]}^{\sigma(X', X)} = X'$

Proof. By definition, $x \in M$ iff $x(M) = 0$ and $x \in \bigcap_{\mathfrak{M}} M$ iff $\hat{x} \equiv 0$. If X is semisimple and $p_\alpha(x) = 0$, then $\hat{x} = 0$ a.e. relative ν. Since $u = \{M \in \mathfrak{M}: x(M) \neq 0\}$ is open, $u = \varnothing$ and $\hat{x} \equiv 0$ yields $x = 0$. Conversely, if p_α is a norm and $x \in \bigcap_{\mathfrak{M}} M$, then $\hat{x} \equiv 0$ and $p_\alpha(x) = 0$ gives $x = 0$.
Thus, (i) ⇔ (ii). Since (ii) is precisely the requirement that the p_α topology be T_2, one has (i) ⇒ (iii) by taking $Y = (X, p_\alpha)$ and $\mathfrak{A} = \{\mathbf{1}\}$. If $\mathfrak{A}$ satisfies (iii), the weakest topology on X making all $A \in \mathfrak{A}$ continuous is T_2 and (by definition) weaker than p_α. Therefore, p_α is T_2 and (ii) holds. Surely (i) ⇔ (iv) since (i) means $\bigcap_{\mathcal{H}} h^{-1}(0) = \{0\}$ and $\sigma(X, \mathcal{H})$ is determined by the seminorms $\{p_h(x) = |h(x)| : h \in \mathcal{H}\}$. Finally, the polar $[\mathcal{H}]^\circ$ always contains $\{0\} \subset X$. The reverse inclusion holds iff $\bigcap_{\mathcal{H}} h^{-1}(0) = \{0\}$. Therefore, X is semisimple iff $\{0\} = [\mathcal{H}]^\circ$ which is equivalent to $\{0\}^\circ = X'$ being equal to $[\mathcal{H}]^{\circ\circ} = \overline{[\mathcal{H}]}^{\sigma(X', X)}$ [See [5], 274, for example].

Remark. If X is semisimple, $(x, y) = \int_{\mathfrak{M}} \hat{x}\bar{\hat{y}}\, d\nu$ defines an inner product on X.

COROLLARY 1.1. *A finite dim Banach algebra* X *is semisimple iff* $\aleph(\mathcal{H}) = \dim X$.

Proof. Dim $X < \infty$ assures that dim $X' = \dim X$ and $[\mathcal{H}] = \overline{[\mathcal{H}]}^{\sigma(X', X)}$.

Therefore, X is semisimple iff $[\mathcal{H}] = X'$.

3. **Dimension and Completeness.**

Throughout this section, X is assumed to be semisimple. For convenience, we let X'_τ abbreviate the continuous dual of $(X, \mathcal{T})$ and we let $\| \|_1 \sim \| \|_2$ denote norm equivalence.

THEOREM 2. *The following are equivalent for* X:

(i)	*Dim* $\mathrm{X} < \infty$	(i)'	*Dim* $\mathrm{L}(\mathfrak{M}, \nu) < \infty$
(ii)	$(\mathrm{X}, \mathrm{p}_\alpha)$ *is complete*	(ii)'	$(\hat{\mathrm{X}}, \| \|_\alpha)$ *is complete*
(iii)	$\mathrm{p}_\alpha \sim \mathrm{r}_\sigma$ *on* X	(iii)'	$\| \|_\alpha \sim \| \|_o$ *on* $\hat{\mathrm{X}}$

Proof. Clearly (i) $\Leftrightarrow$ (i)' by virtue of $\hat{X} \subset C(\mathfrak{M}) \subset L_\alpha(\mathfrak{M}, \nu) \subset \pi_{\mathfrak{M}} \mathbf{C}$, Corollary 1.1 and the bijective nature of $\mathcal{H} \rightarrow \mathfrak{M}$. Furthermore, (ii)' $\Leftrightarrow$ (ii) and (iii)' $\Leftrightarrow$ (iii) since $\psi\colon (X, p_\alpha) \rightarrow (\hat{X}, \| \|_\alpha)$ and $\psi\colon (X, r_\sigma) \rightarrow (\hat{X}, \| \|_o)$ are congruences. It suffices therefore to prove (i) – (iii) equivalent. First, (i) $\Rightarrow$ (ii) since a finite dim vector space has exactly one T_2 topology compatible with its linear structure. Next, (ii) $\Rightarrow$ (iii) by the uniqueness of complete norm on the semisimple algebra X ([5], 262). Since $p_\alpha(x) \leqslant r_\sigma(x)\ \forall\, x \in X$, condition (iii) is equivalent to the existence of some $\lambda > 0$ satisfying $r_\sigma(x) \leqslant \lambda\, p_\alpha(x)\ \forall\, x \in X$. Therefore $\{p_\alpha(x)\}^\alpha \leqslant \{r_o(x)\}^{\alpha-1} p_1(x) \leqslant \lambda^{\alpha-1} p_1(x)\ \forall\, x \in X$ and the norm topologies on X satisfy $r_\sigma = p_\alpha \subset p_1$. At the same time, (Riesz-Markoff Theorem) there is a $\Phi_\nu \in C'(\mathfrak{M})$ satisfying $\Phi_\nu(f) = \int_{\mathfrak{M}} f\, d\nu\ \forall\, f \in C(\mathfrak{M})$. Since $\underset{f \rightsquigarrow |f|(x) = |f(x)|}{C(\mathfrak{M}) \longrightarrow C(\mathfrak{M})}$ is continuous, $p_1\colon x \rightsquigarrow \{\Phi_\nu |\psi|\}(x) = \int_{\mathfrak{M}} |\hat{x}|\, d\nu$ is r_σ-continuous. In particular, p_1 is $\sigma(X, X'_{r_\sigma})$-continuous and $p_1 \subset \sigma(X, X'_{r_\sigma}) \subset r_\sigma$. Therefore, $p_1 = \sigma(X, X'_{r_\sigma})$ and dim $X < \infty$ ([5], 167).

COROLLARY 2.1. X *is finite dim iff* $\mathrm{X}'_{\mathrm{p}_\alpha} = \mathrm{X}'$.

Proof. Dim $X < \infty$ implies that $p_\alpha = \| \|$. Conversely, $p_\alpha = \tau(X, X'_{p_\alpha})$ (the Mackey topology on X) and $X'_{p_\alpha} = X'$ implies that $p_\alpha = \tau(X, X') = \| \| \supset r_\sigma$. Thus, $p_\sigma = r_\sigma$ on X.

As is well known, (X, r_σ) is complete if and only if $\hat{X}$ is $\| \|_o$ – closed in $C(\mathfrak{M})$. By comparison, $(L_\alpha(\mathfrak{M}, \nu), \| \|_\alpha)$ – completeness yields

COROLLARY 2.2. (X, p_α) *is complete if and only if* $\hat{X}$ *is* $\| \|_\alpha$-*closed in* $L_\alpha(\mathfrak{M}, \nu)$. *The completion of* (X, p_α) *is* $\overline{\hat{X}}^{\| \|_\alpha}, \| \|_\alpha) \subset$ $\subset L_\alpha(\mathfrak{M}, \nu)$.

The nature of ν can be exploited to generalize the fact (Theorem 2) that p_α — completeness implies r_σ — completeness.

EXAMPLE 1. If $\mathfrak{T} \subset r_\sigma$ and $x_n \xrightarrow{\mathfrak{T}} x$ implies that $x_n \xrightarrow{\text{measure}} x$, then $\mathfrak{T}$ — sequential completeness implies r_σ — completeness. Verification: If $\mathfrak{T}$ above is sequentially complete and $f \in \overline{\hat{X}}^{\| \|_o} \subset C(\mathfrak{M})$, there exist $x_n \in X$ such that $\| \hat{x}_n - f \|_o < 1/n$. Since $\{\hat{x}_n\}$ is $\| \|_o$-Cauchy in $\hat{X}$, the sequence x_n (being r_σ, therefore $\mathfrak{T}$-Cauchy) is $\mathfrak{T}$-convergent to some $x \in X$. In particular, $\hat{x}_n \xrightarrow{\text{measure}} \hat{x}$ and (since $\hat{x}_n \xrightarrow{\text{mean}} f$) $\hat{x} = f$ a.e. on $\mathfrak{M}$. The reasoning in Theorem 1 confirms that $\hat{x} = f$ and $\hat{X} = \overline{\hat{X}}^{\| \|_o}$ in $C(\mathfrak{M})$.

4. **Additional Comments.**

The preceding results may be specialized [4] as well as extended [3]. In fact, all our results remain valid for complete $LMCT_2$ Q-algebras with identity; that is, topological algebras with a nbd. base of absolutely convex idempotent sets (*LMC* algebras) whose set of units is open (*Q*-algebras). This generalization is non vacuous.

EXAMPLE 2. The algebra, under pointwise operations, of infinitely differentiable functions on $[a, b] \subset \mathbf{R}$ determined by $\{q_n(f) = \sup_{[a,b]} | f^{(n)}(t) | : n \in N\}$ is a semisimple, *LMC*, Frechet *Q*-algebra which ([5], 278) is non normable.

REFERENCES

[1] S. COHEN, «*An Inner product for a Banach Algebra*», Boll. UMI IV (1973), 35-41.

[2] N. DUNFORD and J. T. SCHWARTZ, *Linear Operators*, PartI., Wiley-Interscience, New York, 1958.

[3] W. PAGE, «*Measure-Induced Seminormed Topologies on a Banach Algebra*», Rend. Mat. (2), Vol. 6 (1973), 238-255.

[4] W. PAGE, «*Characterizations of B* and Semisimple A*-Algebras*», Rev. Rom. Mat. Pures Appl (8) XVIII (1973), 1241-1244.

[5] W. PAGE, *Topological Uniform Structures*, Wiley-Interscience, New York, 1978.