OSCILLATIONS OF *n-th* ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PERTURBATIONS (*)

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SOMMARIO. - *Di recente, si è riscontrato un crescente interesse per lo studio di equazioni differenziali di ordine n in cui figura l'operatore differenziale di ordine n*

$$L_0\, x(t) = x(t), \;\; L_i\, x(t) = \frac{1}{r_i(t)} \frac{d}{dt} L_{i-1}\, x(t), \quad 1 \leq i \leq n,$$

$$r_n(t) = 1,$$

che da luogo a termini smorzati.

In questo lavoro, vengono studiati criteri oscillatori per le soluzioni limitate di equazioni funzionali di ordine n, con argomenti devianti di tipo generale, aventi la forma

$$(E) \qquad L_n\, x(t) + H(t, x[g_1(t)]) = Q(t, x[g_2(t)]), \;\; n \;\; \text{even}$$

e vengono date condizioni sufficienti per H e Q, tali da assicurare che tutte le soluzioni limitate di (E) siano oscillatorie.

SUMMARY. - *Recently, there is an increasing interest in studying the n-th order differential equations involving the so called n-th order r-derivative of x*

$$L_0\, x(t) = x(t), \;\; L_i\, x(t) = \frac{1}{r_i(t)} \frac{d}{dt} L_{i-1}\, x(t), \quad 1 \leq i \leq n,$$

$$r_n(t) = 1,$$

which causes damped terms.

Here, are studied the oscillatory criteria of bounded solutions of n-th

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order functional differential equations with general deviating arguments of the form

$$(E) \qquad L_n x(t) + H(t, x[g_1(t)]) = Q(t, x[g_2(t)]), \quad n \text{ even}$$

and are given the sufficient conditions on H and Q, which guarantee that all bounded solutions of (E) are oscillatory.

1. Introduction.

In this paper we consider the *n-th* order functional differential equations with general deviating arguments of the form

$$(E) \qquad L_n x(t) + H(t, x[g_1(t)]) = Q(t, x[g_2(t)]), \ n \text{ even},$$

where the differential operator L_n is recursively defined by

$$L_0 x(t) = x(t), \ L_i x(t) = \frac{1}{r_i(t)} \frac{d}{dt} L_{i-1} x(t), \quad 1 \le i \le n,$$

$$r_n(t) = 1.$$

We note that $g(t)$ is a general deviating argument, that is, it is allowed to be advanced ($g(t) \ge t$), or retarded ($g(t) \le t$) or otherwise. A solution $x(t)$ of (E) is said to be continuable if the exists on some ray $[a, \infty), a > 0$. A nontrivial solution of (E) is *oscillatory* if it is continuable and has arbitrary large zeros. By a nonoscillatory solution we mean a continuable solution which is not oscillatory. The term « solution » for the remainder of this work will mean a nontrivial continuable solution.

The motivation for this study comes from a recent article by Kartsatos [4]. In [4], Kartsatos considers a special class of *n-th* order functional differential equation of the form

$$x^{(n)}(t) + H(t, x[g_1(t)]) = Q(t, x[g_2(t)]), \ n \text{ even},$$

and derives some oscillation criteria. We also refer to the works of Chen-Yeh [1] - [2], Kartsatos [3] and Lovelady [5].

The following assumptions are made without further mention:

(*a*) $r_i \in C(R_+ \equiv [0, \infty), R_+ \backslash \{0\})$, *and*

$$\int^{\infty} r_i(t)\,dt=\infty, \quad 1\leq i\leq n-1;$$

(b) $g_i \in C(R_+, R=(-\infty, \infty))$

and

$$\lim_{t\to\infty} g_j(t)=\infty \quad \text{for } j=1,2;$$

(c) *H, $Q \in C(R_+ \times R, R)$, $H(t,u)$ is increasing in u and $uH(t,u)\geq 0$ for $u \neq 0$.*

Our purpose here is to give the sufficient conditions on H and Q under which all bounded solutions of (E) are oscillatory.

2. Main results.

THEOREM 1. *Let*

(i) *for each $\alpha>0$ there exists a function $Q_\alpha \in C(R_+, R_+)$ such that*

$$|Q(t,u)| \leq Q_\alpha(t)$$

for each $u \in R$ with $|u| \leq \alpha$;

(ii) *for each $c>0$, $\alpha>0$*

$$\int^{\infty} \omega_{n-1}(t)\{H(t, \pm c) \mp Q_\alpha(t)\}\,dt=\pm\infty,$$

where $\omega_{n-1}(t)$ is defined by

$$\omega_1(t)=\int_0^t r_1(s)\,ds, \quad \omega_\varkappa(t)=\int_0^t r_\varkappa(s)\,\omega_{\varkappa-1}(s)\,ds,$$

$$\varkappa=2,3,\dots,n-1.$$

Then if $x(t)$ is a bounded eventually positive (negative) nonoscillatory solution of (E), there exists a sequence $\{\bar{t}_n\}$, $n=1,2,\dots$ such that

$\lim_{n\to\infty} \bar{t}_n = \infty$, *and* $H(\bar{t}_n, x[g_1(\bar{t}_n)] \leq Q(\bar{t}_n, x[g_2(\bar{t}_n)])$, $(H(\bar{t}_n, x[g_1(\bar{t}_n)]) \geq$ $\geq Q(\bar{t}_n, x[g_2(\bar{t}_n)]))$. *Now, in addition to the above assume that* $g_1(t) \equiv$ $\equiv g_2(t)$ *and that the inequality*

$$H(t_n, x_n) \leq Q(t_n, x_n) \; (H(t_n, x_n) \geq Q(t_n, x_n))$$

for a sequence $\{t_n\}$ *with* $\lim_{n\to\infty} t_n = \infty$ *and a sequence* $\{x_n\}$ *which is positive (negative) and bounded, is impossible; then every bounded solution of (E) is oscillatory.*

PROOF. Let $x(t)$ be a bounded eventually positive nonoscillatory solution of (E). Then there exist $t_1 \geq 0$ and t_2 such that $x(t) > 0$ for $t \geq t_1$ and $g_j(t) \geq t_1$ for $t \geq t_2$ and $j = 1, 2$. Thus $x[g_j(t)] > 0$ for $t \geq t_2$ and $j = 1, 2$. Since $x(t)$ is bounded, there is a $\alpha > 0$ such that $|x(t)| < \alpha$ for each $t \geq t_1$. Thus, by (i),

$$|Q(t, x[g_2(t)])| \leq Q_\alpha(t) \tag{1}$$

for $t \geq t_2$. Assume that

$$H(t, x[g_1(t)]) - Q(t, x[g_2(t)]) > 0$$

for $t \geq T$, where $T \geq t_2$ is a fixed number. Then, by (E), $L_n x(t) < 0$ for $t \geq T$. It follows from Lemma of [1] (or, cf. [4]) that

$$(-1)^{\varkappa+1} L_\varkappa x(t) > 0, \quad \varkappa = 1, 2, \dots, n-1 \tag{2}$$

for $t \geq T$. It T is large enough, then, for $t \geq T$

$$(-1)^{\varkappa+1} L_\varkappa x[g_j(t)] > 0, \quad \varkappa = 1, 2, \dots, n-1, \quad j = 1, 2.$$

It follows from $x'[g_1(t)] > 0$ for $t \geq T$ that

$$x[g_1(t)] > x[g_1(T)] = c.$$

Thus

$$H(t, x[g_1(t)]) \geq H(t, c). \tag{3}$$

From (E), (1) and (3), we have

$$\int_T^t \omega_{n-1}(s)\, L_n x(s)\, ds +$$

$$+\int_T^t \omega_{n-1}(s)\{H(s, x[g_1(s)]) - Q(s, x[g_2(s)])\}\, ds = 0$$

i. e.

$$x(t) \geq x(T) + D(t) - D(T) + \int_T^t \omega_{n-1}(s)\{H(s, c) - Q_\alpha(s)\}\, ds \tag{4}$$

where $D(t) = \sum_{i=1}^{n-1} (-1)^{i+1} \omega_i(t)\, L_i x(t)$.

Hence, by (2), (4) and (ii),

$$x(t) > x(T) - D(T) + \int_T^t \omega_{n-1}(s)\{H(s, c) - Q_\alpha(s)\}\, ds \to \infty$$

as $t \to \infty$, a contradiction. Consequently, there exists at least $\bar{t}_1 \geq T$ such that

$$H(\bar{t}_1, x[g_1(\bar{t}_1)]) - Q(\bar{t}_1, x[g_2(\bar{t}_1)]) \leq 0.$$

Since T was arbitrary, it follows that there exists a sequence $\{\bar{t}_n\}$, $n = 1, 2, \dots,$ such that $\bar{t}_n \geq t_2$ and

$$H(\bar{t}_n, x[g_1(\bar{t}_n)]) - Q(\bar{t}_n, x[g_2(\bar{t}_n)]) \leq 0, \quad n = 1, 2, \dots,, \tag{5}$$

Similarly, we can prove the case for an eventually negative bounded nonoscillatory solution $x(t)$. As for as the second part is concerned, it is obvious from (5) with $g_1(t_n) = g_2(t_n)$ and the corresponding inequality for a negative solution $x(t)$. This completes our proof.

REMARK 1. Taking $r_i(t) = 1$ for $i = 1, 2, \dots, n-1$, a result of Kartsatos (Theorem 1, [3]) is a special case of our theorem.

We now state an auxiliary lemma, which is due to Chen-Yeh [2].

LEMMA. *Consider the equation*

$$L_n x(t)+H(t, x(t))=0, \tag{6}$$

and the inequality

$$L_n x(t)+H(t, x(t))\leq 0, \tag{7}$$

where $H\in C(R\times R, R)$ *is increasing with respect to* x.

If there exists a solution (bounded solution) $x(t)$ *of* (7) *with* $x(t_1)=u_1>0$ *for some* $t_1>0$ *and* $x'(t)\geq 0$, $t\in[t_1, \infty)$, *then there is a solutoin (bounded solution)* $y(t)$ *of* (6) *with* $y(t_2)=u_1$ *and* $y'(t)\geq 0$, $t\in[t_2, \infty)$ *for some* $t_2\geq t_1$.

THEOREM 2. *Let* $H(t, u)$ *be strictly increasing in* u *and such that for every* $t\in R_+, \mu_1, \mu_2\in R$ *with* $\mu_1\mu_2>0, |\mu_1| < |\mu_2|$,

$$|H(t, \mu_1)| \leq k(\mu_1, \mu_2)\, |H(t, \mu_2)|,$$

where k *is a constant depending on* μ_1, μ_2 *with* $k(\mu_1, \mu_2)\in(0, 1)$.

Let every solution (bounded solution) of

$$L_n x(t)+\mu H(t, x(t))=0 \tag{8}$$

be oscillatory for every $\mu>0$. *Let the inequality*

$$L_n x(t)+H(t, x(t))\leq Q(t) \tag*{$(9)_1$}$$

have an eventually positive solution $x_1(t)$, *and the inequality*

$$L_n x(t)+H(t, x(t))\geq Q(t) \tag*{$(9)_2$}$$

have an eventually negative solution $x_2(t)$ *such that*

$$\lim_{t\to\infty} x_j(t)=0, \quad j=1, 2. \tag{10}$$

Then every solution (bounded solution) of

$$L_n x(t)+H(t, x(t))=Q(t) \tag{11}$$

is oscillatory.

PROOF. Let $x_2(t)<0$, $t\geq t_1>0$, satisfy $(9)_2$, and let every solution of (8) be oscillatory. If $x(t)$ is a positive solution of (11), then $u(t)\equiv$

$\equiv x(t)-x_2(t)$ is a solution of

$$L_n u(t)+H(t, u(t)+x_2(t))-H(t, x_2(t))\leq 0. \tag{12}$$

Assume that $u(t)>0$ for $t\geq t_2\geq t_1$. Then $L_n u(t)<0$ for $t\geq t_2$. Since n is even, $u'(t)>0$ for $t\geq t_2$, where t_2 is large enough. We choose $\varepsilon>0$ so that $\varepsilon<\frac{1}{2}u(t_2)$ and, $|x_2(t)|<\varepsilon$ for every $t\geq t_3$, for some $t_3\geq t_2$. It follows from (12) that

$$\begin{aligned} L_n u(t)+H(t, u(t)-\varepsilon)-H(t, \varepsilon)\leq \\ \leq L_n u(t)+H(t, u(t)+x_2(t))-H(t, x_2(t))\leq 0 \end{aligned} \tag{13}$$

for $t\geq t_3$. Since $u(t)-2\varepsilon>u(t_2)-2\varepsilon>0$ for $t\geq t_3$,

$$H(t, u(t)-\varepsilon)-H(t, \varepsilon)>0, \quad t\geq t_3. \tag{14}$$

Let $v(t)\equiv u(t)-\varepsilon$, then $v(t), t\geq t_3$ is a positive solution of

$$L_n v(t)+H(t, v(t))-H(t, \varepsilon)\leq 0, \quad t\geq t_3, \tag{15}$$

and such that $v'(t)>0$ and $v(t)>\varepsilon$ for $t\geq t_3$. It follows from Lemma that this is also true for the equation

$$L_n z(t)+H(t, z(t))-H(t, \varepsilon)=0. \tag{16}$$

Let $z(t)$ be such a solution of (16) with $z(t), z'(t)>0$ for $t>t_3$ and $z(t_3)>\varepsilon$. Since $H(t, x)$ is strictly increasing in x,

$$\begin{aligned} H(t, \varepsilon)<k(\varepsilon, z(t_3))\, H(t, z(t_3))< \\ <k(\varepsilon, z(t_3))\, H(t, z(t)) \end{aligned}$$

for $t\geq t_3$. Letting $\mu=1-k(\varepsilon, z(t_3))$, we obtain that

$$L_n W(t)+\left\{1-\frac{H(t, \varepsilon)}{H(t, z(t))}\right\}H(t, W(t))=0 \tag{17}$$

have a positive solution $z(t), t\geq t_3$, with the coefficient of $H(t, W(t))$ bounded below by the constant μ. Since every solution of (8) is oscillatory for every $\mu\geq 0$, we obtained by Theorem of [2], that every solution of (17) must also be oscillatory, a contradiction. Hence there exists a sequence $\{t_m\}$, $m=1,2,\dots$, such that $\lim_{m\to\infty} t_m=\infty$ and $x(t_m)<x_2(t_m)<0$, $m=1,2,\dots$, which contradicts the positiveness of $x(t)$. Similarly, we can prove the case where $x_1(t)$ is a positive solution of $(9)_1$ and the case where every solution of (8) is bounded oscillatory.

REMARK 2. Taking $r_i(t)=1$ for $i=1,2,\dots,n-1$, a result of Kartsatos (Theorem 2.1, [4]) is a special case of our theorem.

From the proof of Theorem 2, we can obtain the following two corollary.

COROLLARY 1. *Let the assumption on H, Q of Theorem 2 be satisfied. Let* $x_1(t)$ *is a solution of* $(9)_1$ *with* $\lim_{t\to\infty} x_1(t)=0$. *If every solution of* (8) *is oscillatory (bounded oscillatory) for every* $\mu>0$, *then every eventually positive (negative) [bounded and positive (bounded and negative)] solution* $x(t)$ *of* (11) *satisfies* $\liminf_{t\to\infty} x(t)=0$.

COROLLARY 2. *Let the assumption on H, Q of Theorem 2 be satisfied. Let* $x_1(t)$ *is an eventually positive (negative) solution of* (11) *with* $\lim_{t\to\infty} x_1(t)=0$. *Assume that* $x_2(t)$ *is another solution of* (11) *with the same property. If every bounded solution of* (8) *is oscillatory for every* $\mu>0$, *then* $x_1(t)-x_2(t)$ *is an oscillatory function.*

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