COHEN'S ITERATION PROCESS FOR BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS (*)

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SOMMARIO. - *Si estende il metodo iterativo introdotto da Cohen per il problema di Dirichlet, a vari tipi di problemi al contorno per certe classi di equazioni differenziali anche funzionali. In questo modo si ottengono, in maniera costruttiva, alcuni risultati noti ed alcuni nuovi sulle soluzioni periodiche di equazioni differenziali funzionali ed altri problemi al contorno. Il metodo appare efficiente anche dal punto di vista numerico.*

SUMMARY. - *The iteration process introduced by Cohen in connection with the Dirichlet problem is extended to various boundary value problems for ordinary and functional differential equations. In this way some known and some new results on periodic solutions of functional differential equations and other b. v. p., can be obtained in a constructive way providing an efficient numerical method for approximating the solutions.*

In [6], Cohen introduced a new iteration process for the approximation of a solution of some elliptic problems in presence of upper and lower solutions. Successively, this process has been extended to other elliptic and parabolic problems by Amann[1] and Sattinger [16]. On the other hand, the existence of solutions in presence of upper and lower solutions has been stated for various other b. v. p. by Deuel-Hess [8], Kaplan-Lasota-Yorke [11], Knobloch [12]-[13], Grim-Schmitt [9] and Schmitt [17], however without a constructive approach.

In particular for elliptic differential equations of second order, many results and a complete bibliography can be found in the survey of Schmitt [18].

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As far as numerical methods for functional differential equations is concerned, they found a growing interest in the last years. Dealing with periodic solutions, various methods can be found in [2], [3], [4], [14]. Other boundary value problems seem more hard to solve numerically, and few papers are avalaible. In particular, the shooting method has been investigated by De Nevers and Schmitt [7], and the convergence of the collocation method with spline functions has been proved by Reddien and Travis [15]. Recently, Chocholaty and Slahor [5] have introduced an iterative method which consists in solving a sequence of non-linear ordinary b. v. p. by quasilinearization.

In this note we observe that Cohen's iteration process can be put in an abstract way in order to unify the above theoretical results. Moreover, it can be successfully used to state some known and some new existence results and to provide an efficient numerical method for various b. v. p. for differential equations with deviating argument of the form

$$L(x, u(x)) = f(x, u, (x), u(g(x)))$$

where L is a first or second order linear differential operator.

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1. The differential operator and boundary conditions considered in this paper.

In this paper we shall consider linear functional differential operators L and linear boundary operator B acting on suitable functions defined on $\Omega \subseteq R^n$ and $\partial\Omega$ respectively, and having the following properties. There exist a dense subset $\Lambda \subseteq]0, +\infty[$ and a Banach space $\mathcal{B} \subseteq C(\bar{\Omega})$ such that:

(i) The conditions $\lambda \in \Lambda$, $L(x, u(x)) - \lambda u(x) \leqq 0$, $B(x, u) \geqq 0$ imply $u \geqq 0$.

(ii) For each $h \in \mathcal{B}$ and $\lambda \in \Lambda$, the boundary value problem

$$L(x, u(x)) - \lambda u(x) = h$$

$$B(x, u) = 0$$

has exactly one solution $u \in \mathcal{B}$.

(iii) If $(u_n)_n$ and $(L(x, u_n))_n$ are uniformly bounded and $B(x, u_n) = 0$ for all n, then there is a subsequence $(u_{n_k})_k$ converging uniformly.

(iv) The conditions $\lim_n u_n = u$ and $\lim_n h_n = h$ uniformly, $L(x, u_n) = h_n$ and $B(x, u_n) = 0$ for $n \geqq 1$ imply $L(x, u) = h$, $B(x, u) = 0$.

These hypotheses are satisfied by many b. v. p. for elliptic and parabolic partial differential equations, as well as for ordinary differential equations of first and second order. This follows from the well known maximum principles and Schauder estimates for elliptic and parabolic problems, while for ordinary differential equations ($L(x, u) \equiv a(x)\, u'$ and $L(x, u) \equiv a(x)\, u''$) we may proceed as follows. Hypothesis (i) follows from a simple direct argument in case of periodic solutions for first and second order differential equation, as well as for the most common (two points, Nicoletti, etc.) linear b. v. p. of second order equations. Hypothesis (ii) is obvious, while (iii) follows from Taylor's formula and (iv) from Taylor's formula and the well known results on the differentiability of limits of sequences of differentiable functions.

2. Cohen's iteration scheme for operators satisfying (i)-(iv).

We call *upper solution* of the boundary value problem

$$L(x, u(x)) = f(x, u(x), u(g(x))) \qquad B(x, u) = 0$$

any function $\beta(x)$ such that

$$L(x, \beta(x)) \leqq f(x, \beta(x), \beta(g(x))) \qquad B(x, \beta) \geqq 0$$

and is so smooth that $L(x, \beta)$ and $B(x, \beta)$ makes sense. Substituing $\leqq$, $\geqq$ with the strong sign $<$, $>$ we have a *strict upper solution*. Reversing the above inequalities, we have the notion of *lower solution* and *strict lower solution*. The following theorem states the abstract version of Cohen's iteration process and, in view of the remarks at the end of the § 1, it implies the claims made in the introduction on the known results concerning upper and lower solutions.

LEMMA 1. *Let α and β be lower and upper solutions of the b. v. p.*

$$L(x, u(x)) = f(x, u(x), u(g(x))), \qquad B(x, u) = 0 \tag{1}$$

with $\alpha \leqq \beta$ and $f(x, u, \cdot)$ non increasing. If $\partial f/\partial u$ exists continuous in $K = \{(x, u, v) \mid x \in \bar{\Omega}, \alpha(x) \leqq u(x) \leqq \beta(x), \alpha(g(x)) \leq v(g(x)) \leq \beta(g(x))\}$

and $u \in \mathcal{B} \Rightarrow f(\cdot, u(\cdot), v(g(\cdot))) \in \mathcal{B}$ *then the b. v. p.* (1) *has a solution* u, *with* $\alpha \leqq u \leqq \beta$, *which is the uniform limit of the increasing sequence* $(\alpha_n)_n$ *defined inductively by*

$$\alpha_0(x) = \alpha(x),\ L(x, \alpha_{n+1}(x)) - c\alpha_{n+1}(x) = \tag{2}$$

$$= f(x, \alpha_n(x), \alpha_n(g(x))) - c\,\alpha_n(x), \quad B(x, \alpha_{n+1}(x)) = 0$$

where c *is any fixed positive number such that* $c \geqq \underset{K}{\text{maximum}}\ \partial f(x, u, v)/\partial u$. *If* $\alpha_0(x) = \beta(x)$, *the sequence* $(\alpha_n)_n$ *is decreasing and converges to a solution* v *such that* $u \leq v$.

At the beginning of the proof it is shown that the sequence $(\alpha_n)_n$ is well defined. When L is an ordinary differential operator and $\bar{\Omega} = [a, b]$, then usually we have $\mathcal{B} = C[a, b]$, and when L is an elliptic or parabolic operator, then $\mathcal{B}$ is usually a suitable space of Hölder functions.

Proof of lemma 1. First let's remark that the sequence $(\alpha_n)_n$ is well defined by (ii). Let's show that, if $\alpha_0 = \alpha$

$$\alpha_n \leqq \beta \text{ for all } n. \tag{3}$$

This is known for $n=0$. Assume it true untill n and let's state it for $n+1$. We have

$$L(x, \beta(x) - \alpha_{n+1}(x)) - c(\beta(x) - \alpha_{n+1}(x)) =$$

$$= L(x, \beta(x)) - c\,\beta(x) - f(x, \alpha_n(x), \alpha_n(g(x)) + c\,\alpha_n(x) \leqq$$

$$\leqq f(x, \beta(x), \beta(g(x))) - c\,\beta(x) - (f(x, \alpha_n(x), \alpha_n(g(x))) - c\,\alpha_n(x)) \leqq 0$$

since $u \hookrightarrow f(x, u(x), u(g(x))) - cu$ is decreasing in $[\alpha(x), \beta(x)]$ in view of the choice of the constant c. Moreover $B(x, \beta - \alpha_{n+1}) \geqq 0$, and therefore (i) implies $\beta - \alpha_{n+1} \geqq 0$ as desired. Now let's show that

$$\alpha_n \leqq \alpha_{n+1} \text{ for all } n. \tag{4}$$

For $n=0$ we have

$$L(x, \alpha_1(x) - \alpha_0(x)) - c(\alpha_1(x) - \alpha_0(x)) =$$

$$= f(x, \alpha_0(x), \alpha_0(g(x))) - L(x, \alpha_0(x)) + c\,\alpha_0(x) =$$

$$= f(x, \alpha_0(x), \alpha_0(g(x))) - L(x, \alpha_0(x)) \leqq 0$$

because $\alpha_0 = \alpha$. Moreover $B(x, \alpha_1 - \alpha_0) = B(x, \alpha_1) - B(x, \alpha_0) \geqq 0$, then (i) applied to $u = \alpha_1 - \alpha_0$ implies $\alpha_1 \geqq \alpha_0$. Now let $n > 0$ and $\alpha_n \geqq \alpha_{n-1}$ by inductive hypothesis. Then

$$L(x, \alpha_{n+1}(x)-\alpha_n(x))-c(\alpha_{n+1}(x)-\alpha_n(x))= \tag{5}$$

$$=f(x, \alpha_n(x), \alpha_n(g(x)))-c\,\alpha_n(x)-(f(x, \alpha_{n-1}(x), \alpha_{n-1}(g(x)))-c\,\alpha_{n-1}(x)).$$

By the inductive hypothesis and the decreasing behavior of the function $u \hookrightarrow f(x, u(x), u(g(x)))-cu$ in $[\alpha(x), \beta(x)]$, the right hand side of (5) is $\leqq 0$. Since we have also $B(x, \alpha_{n+1}-\alpha_n)=0$, from (i) it follows that $\alpha_{n+1}-\alpha_n \geqq 0$, hence (4). From (3) and (4) we have that $(\alpha_n)_n$ is uniformly bounded. From the equation (2) it follows that the sequence $(L(x, \alpha_n))_n$ is also uniformly bounded. Then, by (iii), for every subsequence $(\alpha_{n_k})_k$ of $(\alpha_n)_n$ there exists a uniformly convergent subsequence $(\alpha_{n_{k_i}})_i$. Let $u_0(x)=\lim_i \alpha_{n_{k_i}}(x)$. By (4) $u_0(x)=\sup_n \alpha_n(x)$, therefore we have shown that every subsequence of $(\alpha_n)_n$ has a subsequence which converges uniformly to a function $u=\sup_n \alpha_n(x)$. A well known property on limits implies that $\lim_n \alpha_n=u$ uniformly. Then we apply (iv) and we get $L(x, u(x))=f(x, u(x), u(g(x)))$, $B(x, u)=0$ q. e. d.

For the case $\alpha_0=\beta$ it is sufficient to reverse the inequalities and to invert the role of α and β in the proof.

3. Application to periodic solutions of functional differential equations.

Let's consider the problem of periodic solutions of the equation

$$L(x, u(x))=f(x, u(x), u(g(x))) \tag{6}$$

where $L(x, u(x)) \equiv u'(x)$ or $L(x, u(x)) \equiv u''(x)$, $f(\cdot, s, t)$ is a T-periodic function $R \to R$ for each $s, t \in R$, and $g(x)$ is T-periodic or satisfies the more general condition $g(x+T)=g(x)+T$ which is fulfilled for the class of deviating argument of the form $x+\Delta(x)$ with $\Delta(x)$ T-periodic.

The problem of periodic solutions of (6) in presence of upper and lower solutions has been already investigated by Schmitt [17] for $L(x, u(x)) \equiv u''(x)$ and g depending also on u, in case that $f(x, u, \cdot)$ is non-increasing as well as under stronger conditions on the upper and lower solution α and β, that is:

$$\alpha'' \geqq f(x, \alpha(x), y), \quad \beta'' \leqq f(x, \beta(x), y) \quad \forall y \in [\alpha(x), \beta(x)].$$

In order to apply the Cohen's iterations process to equation (6), we need to prove that properties (i)-(iv) hold for the operator:

$$L'(x, u(x)) \equiv L(x, u(x)) - ku(g(x)) \quad k \in]0, +\infty[\tag{7}$$

with $\Omega = R$ and $\mathcal{B}$ the space of T-periodic functions of class C^0 or C^1.

LEMMA 2. *The operator* (7) *satisfies the properties* (i)-(iv) *in the space* $\mathcal{B}$ *for any* $\lambda \in \Lambda$ *such that* $(k+\lambda) T \leqq 1$ *in case of first order equation, and such that* $(k+\lambda) T^2 \leqq 8$ *in case of second order equation.*

PROOF. (1^{st} case: $L(x, u(x)) = u'(x)$). Property (i) holds by a simple application of the Taylor's formula $u(x) - u(x_0) = (x - x_0)\, u'(\xi)$. In fact let's suppose $u'(x) - ku(g(x)) - \lambda u(x) \leqq 0$. If u has both positive and negative values, it has a minimum value $u(x_0) = \sigma < 0$ and a maximum value $u(x) = \varepsilon > 0$. Since, by the periodicity of u, we may suppose $x_0 < x$, we have

$$\varepsilon - \sigma \leqq T \max (u'(x)) \leqq T(k+\lambda)\, \varepsilon.$$

If $T(k+\lambda) < 1$, the last relation should imply $\sigma \geq 0$ or $\varepsilon \leq 0$, therefore u has constant sign. It cannot be $u \leqq 0$ because $u'(x) \leqq ku(x-g)(x)) + \lambda u(x) \leqq$ $\leqq 0 \Rightarrow u'(x) = 0 \Rightarrow u(x) = 0$, hence $u(x) \geqq 0$. Property (ii) holds too, by the Fredholm alternative principle for periodic solutions of linear delay differential equations (see [10] lemma 2.2) which states the existence and uniqueness of periodic solutions of the system $u' + Au = h$, with A a linear operator of $C^0_{\#}(T, R^n)$[1] into itself, and $h \in C^0_{\#}(T, R^n)$, provided the homogeneous system has, in $C^0_{\#}(T, R^n)$, only the trivial solution.

For $n = 1$ and $Au(x) = -ku(g(x)) - \lambda u(x)$, the homogeneous equation $u' + Au = u'(x) - ku(g(x)) - \lambda(x) = 0$ has, by (i), only the trivial periodic solution and therefore (ii) holds.

Properties (iii) and (iv) can be verified by the same arguments in case $g(x) = x$.

(2^{nd} case: $L(x, u(x)) = u''(x)$). Property (i) is easily verified by analogous considerations on Taylor's formula $u(x) - u(x_0) =$ $= (x - x_0)\, u'(x_0) + (x - x_0)^2\, \tfrac{1}{2}\, u''(\xi)$ as in the previous case, provided $(k+\lambda) T^2 \leqq 8$.

Since the equation $u''(x) - ku(g(x)) - \lambda u(x) = h$ is equivalent to

(1) $C^i_{\#}(T, R^n)$ is the space of T-periodic function $R \to R^n$ of class C^i.

the first order system:

$$\begin{cases} z'(x) - y(x) = 0 \\ y'(x) - kz(g(x)) - \lambda z(x) = h \end{cases} \tag{8}$$

with $z = u$ and $y = u'$, choosing the linear operator A, acting on $C^0_\#(T, R^2)$, as

$$A(z, y) = (-y(x), -kz(g(x)) - \lambda z(x)),$$

the omogeneous system associated to (8) has, by (i), only the trivial solution in $C^0_\#(T, R^2)$ and then (ii) holds.

Properties (iii) and (iv) are fulfilled as in case $g(x) = x$.

THEOREM 1. *Let α and β a lower and upper solution of the equation* (6) *with $\alpha \leqq \beta$. If $\partial f(x, y, z)/\partial y$ and $\partial f(x, y, z)/\partial z$ exist continuous on $K' = \{(x, y, z) \mid x \in R;\ \alpha(x) \leqq y \leqq \beta(x);\ \ \alpha(g(x)) \leqq z \leqq \beta(g(x))\}$, then the equation* (6) *has a T-periodic solution $u(x) \in [\alpha(x), \beta(x)]$ if $(c+k)T \leqq 1$, in case of first order equation, and if $(c+k)T^2 \leqq 8$ in case of second order equation, where c and k are positive reals numbers such that $c \geqq \max\limits_{K'} \partial f/\partial y$ and $k \geqq \max\limits_{K'} \partial f/\partial z$.*

PROOF. The problem (6) is equivalent to

$$L'(x, u(x)) = h(x, u(x), u(g(x))) \tag{9}$$

where $L'(x, u(x))$ is the operator (7) and

$$h(x, u(x), u(g(x))) = f(x, u(x), u(g(x))) - ku(g(x))$$

which, by the choice of k, is non-increasing with respect to $u(g(x))$ on the set K'. On the other hand, by the choice of c and by lemma 2, the operator L' satisfies the properties (i)-(iv). Then, by lemma 1, we find an increasing and uniformly convergent sequence $(\alpha_n)_n$ defined inductively by

$$\alpha_0(x) = \alpha(x)$$

$$L'(x, \alpha_{n+1}(x)) - c\,\alpha_{n+1}(x) = h(x, \alpha_n(x), \alpha_n((g(x))) - c\,\alpha_n(x), \tag{10}$$

whose limit is a solution of (6).

If $\alpha_0(x) = \beta(x)$, the sequence $\{\alpha_n\}_n$ is decreasing and converges to a solution $v \geq u$.

COROLLARY 1. If $f(x, u, \cdot)$ is non-increasing, the process of theorem 1 can be applied for $k=0$, therefore the left hand operator does not depend on deviating argument and then the thesis holds with no limitation on the derivative of f. This constructive result is the same of theorem 1 in [17] when g *does* not depend on u.

COROLLARY 2. *If* $f=f(x, u(g(x)))$, theorem 1 holds under conditions $kT\leqq 1$ or $kT^2\leqq 8$, respectively for first or second order equations. This can be proved either by a direct argument on equation $L(x, u(x))= =f(x, u(g(x)))$ or noting that the theorem 1 holds for arbitrary $c>0$.

COROLLARY 3. If $f(x, \cdot, u)$ is non-increasing the theorem 1 holds again for $kT\leqq 1$ or $kT^2\leqq 8$ because of the arbitrariness of $c>0$.

4. Some remarks.

REMARK 1.

In case of periodic solutions of first and second order ordinary and functional differential equations, it is easy to prove that, under the conditions $(\lambda+k)T\leqq 1$ and $(\lambda+k)T^2\leqq 8$, the following properties hold for the operator $L'(x, u)=u^{(i)}(x)+ku(g(x))$ $i=1, 2$; $k>0$ and $\mathcal{B}$ as above.

(i)^ The conditions $\lambda\in\Lambda$, $L'(x, u(x))+\lambda u(x)\leqq 0$, $u\in\mathcal{B}$ imply $u(x)\leqq 0$

(ii)^ For each $h\in\mathcal{B}$ and $\lambda\in\Lambda$, the problem

$$L'(x, u(x))+\lambda u(x)=h$$

has exactly one periodic solution $u\in\mathcal{B}$.

(iii)^ = (iii).

(iv)^ = (iv).

We can verify that, if α and β are a lower and upper solution of (6) with the reverse relation $\alpha\geqq\beta$, then the problem has still a solution u such that $\beta\leqq u\leqq\alpha$. The proof need the prove of the analogues of the theorems 1 and lemma 1, where c and k are exchanged by $-c$ and $-k$, with $-c\leqq\min_{K'}\partial f(x, y, z)/\partial y$ and $-k\leqq\min_{K'}\partial f(x, y, z)/\partial z$, $K'= =\{(x, y, z)|x\in R;\ \beta(x)\leqq y\leqq\alpha(x);\ \beta(g(x))\leqq z\leqq\alpha(g(x))\}$. The sequence $(\alpha_n)_n$ turns out to be non-increasing and bounded by β, hence it will converge uniformly to a solution of (6). However, the existence of periodic solutions of small period, when $\alpha \geqq \beta$, has been noted earlier by Schmitt [19] in ordinary case.

So we can state the following theorem concerning the number of solutions.

THEOREM 2. *If the equation* (6) *admits n lower solutions* $v_1, \dots, v_n$ *and n upper solution* $w_1, \dots, w_n$ *in* $C^0_{\#}$ (T, R) *such that*

$$v_1 \leqq w_1 \leqq v_2 \leqq w_2 \leqq \dots \leqq v_n \leqq w_n$$

and $\partial f(x, y, z)/\partial y$ *and* $\partial f(x, y, z)/\partial z$ *exist continuous on* $K'' = \{(x, y, z) \mid x \in R;\ v_1(x) \leqq y \leqq w_n(x);\ v_1(g(x)) \leqq z \leqq w_n(g(x))\}$ *with* $c \geqq \max |\partial f(x, y, z)/\partial y|$ *and* $k \geqq \max |\partial f(x, y, z)/\partial z|$, *then the equation* (6) *has* $2n-1$ *T-periodic solution* u_i *such that*

$$v_i \leqq u_{2i-1} \leqq w_i \quad \text{for } i = 1, \dots, n$$

$$w_i \leqq u_{2i} \leqq v_{i+1} \quad \text{for } i = 1, \dots, n-1$$

if $(c+k)\,T \leqq 1$, *in case of first order equation, and* $(c+k)\,T^2 \leqq 8$, *in case of second order equation.*

REMARK 2. In view of numerical applications, it is useful to notice that, for periodic boundary conditions, the property (i) in 1. holds with strict inequalities. Therefore lemma 1 and theorem 1 state the existence of a periodic solution u such that $\alpha < u < \beta$, provided α and β are strictly lower and upper solutions satisfying $\alpha < \beta$. In this case the sequence $\{\alpha_n\}_n$ is strictly monotone.

REMARK 3. For the boundary value problem

$$u''(x) = f(x, u(x), u(g(x))) \qquad a \leq x \leq b$$

$$u(x) = \varphi(x) \qquad x < a$$

$$u(x) = \psi(x) \qquad x > b$$

with $f(x, u, \cdot)$ non increasing for fixed (x, u) and g, φ, ψ continuous on $[a, b]$, lemma 1 gives a sufficient condition for the existence of a solution which can be computed by means of the iteration 2. The result, already stated in [9], can be improved by dropping the monotonicity condition on f and adding a further condition obtained by similar arguments like in lemma 2 and theorem 1, that is $(c+k)\,T^2 < 2$.

5. Numerical considerations and examples.

The sequence $\{\alpha_n\}_n$ in the theorem 1, is given by the solutions of the linear equations (10). Although many efficient numerical methods are available for such kind of problems (linear equation with constant coefficients), at each step, the computed solution $\bar{\alpha}_{n+1}$ is the exact solution of a perturbed equation, that is

$$(11) \quad L'(x, \bar{\alpha}_{n+1}(x)) - c\,\bar{\alpha}_{n+1}(x) = h(x, \bar{\alpha}_n(x), \bar{\alpha}_n(g(x))) - c\,\bar{\alpha}_n(x) + \varepsilon_n(x).$$

In order to guarantee the monotonicity and the convergence of the computed sequence $\{\bar{\alpha}_n\}_n$ to the solution $u(x)$, it is sufficient to start from a strictly lower solution α_0 (analogous considerations hold for upper solutions) and to prove that:

if, for each n, $\bar{\alpha}_n$ is a strictly lower solution of (9) *such that $\bar{\alpha}_n < u$ then the following conditions hold for $\bar{\alpha}_{n+1}$:*

j) $\bar{\alpha}_n(x) < \bar{\alpha}_{n+1}(x)$

jj) $L'(x, \bar{\alpha}_{n+1}(x)) > h(x, \bar{\alpha}_{n+1}(x), \bar{\alpha}_{n+1}(g(x)))$

jjj) $\bar{\alpha}_{n+1}(x) < u(x)$.

If the numerical method for the equation

$$(12) \quad L'(x, \alpha_{n+1}(x)) - c\,\alpha_{n+1}(x) = h(x, \bar{\alpha}_n(x), \bar{\alpha}_n(g(x))) - c\,\bar{\alpha}_n(x)$$

provides a residual $\varepsilon_n(x)$ which is so small to satisfy

$$(13) \quad \varepsilon_n(x) < L'(x, \bar{\alpha}_n(x)) - h(x, \bar{\alpha}_n(x), \bar{\alpha}_n(g(x))) \qquad (>0),$$

then (j) holds. In fact, (11), the inequality (13) and the condition on $\bar{\alpha}_n$, imply

$$L'(x, (\bar{\alpha}_{n+1}(x) - \bar{\alpha}_n(x))) - c\,(\bar{\alpha}_{n+1}(x) - \bar{\alpha}_n(x)) < 0$$

and therefore, by remark 2, $\bar{\alpha}_{n+1} > \bar{\alpha}_n$.

On the other hand, if $\varepsilon_n(x)$ satisfies

$$(14) \quad \varepsilon_n(x) > h(x, \bar{\alpha}_{n+1}(x), \bar{\alpha}_{n+1}(g(x))) - c\,\bar{\alpha}_{n+1}(x) - \\ - [h(x, \bar{\alpha}_n(x), \bar{\alpha}_n(g(x))) - c\,\bar{\alpha}_n(x)],$$

where the right side term is negative because of j), then (11) implies jj).

Finally, the condition jjj) is fulfilled giving, once againg, a condition on the residual, that is

$$\varepsilon_n(x) > h(x, u(x), u(g(x))) - c\,u(x) - [h(x, \bar{\alpha}_n(x), \bar{\alpha}_n(g(x))) - c\,\bar{\alpha}_n(x)]$$

where the right side term is negative.

Summarizing, if (12) is solved by a method which is so good to verify the conditions j) - jjj), the sequence $\{\bar{\alpha}_n\}_n$ turns out to be monotone and give a lower bound for u. From a practical point of view, while j) may be checked by looking at the sequence $\{\bar{\alpha}_n\}_n$, and jj) by an « a posteriori » test on $\bar{\alpha}_{n+1}$, the condition jjj) is not verifiable since it depend on the solution u. However jj) may be used as a test for jjj), in fact jj) together with $\bar{\alpha}_{n+1} < \beta$ should imply the existence of a solution v, $\alpha_{n+1} < v < \beta$. Therefore, in case of a unique solution u in (α, β), if jj) doesn't hold, jjj) doesn't too.

EXAMPLE 1. Let consider the T-periodic solutions of

$$u''(x) - 2u(x) + u^3(x-1) = \sin\frac{4}{3}\pi x \qquad T = 1.5. \tag{15}$$

Since the functions $\alpha = -1$ and $\beta = 1$ are, respectively, lower and upper periodic solutions of (15), by theorem 1 and corollary 1, the iteration

$$\alpha''_{n+1}(x) - 2\alpha_{n+1}(x) = -\alpha^3_n(x-1) + \sin\frac{4}{3}\pi x \tag{16}$$

with α_{n+1} T-periodic, provides a sequence $\{\alpha_n\}_n$ which is increasing if $\alpha_0 = \alpha$ and decreasing if $\alpha_0 = \beta$. In the last case, we denote the sequence by $\{\beta_n\}_n$. In table 1 the values of α_n and β_n are printed in the points 0; 0.5; 1; 1.5; for $n = 0, \dots, 5$.

TABLE 1.

	$x = 0$	$x = 0.5$	$x = 1$	$x = 1.5$
β_0	1.0000000	1.0000000	1.0000000	1.0000000
β_1	0.4999999	0.4556928	0.5443071	0.4999999
β_2	0.0617905	0.0208925	0.1077614	0.0617905
β_3	0.0002189	—0.0440256	0.0445505	0.0002189
β_4	—0.0000039	—0.0443021	0.0443075	—0.0000039
β_5	—0.0000044	—0.0443026	0.0443071	—0.0000044
α_5	—0.0000044	—0.0443026	0.0443071	—0.0000044
α_4	—0.0000049	—0.0443031	0.0443066	—0.0000049
α_3	—0.0002798	—0.0445251	0.0440613	—0.0002798
α_2	—0.0651996	—0.1060976	—0.0191471	—0.0651996
α_1	—0.4999999	—0.5443071	—0.4556928	—0.4999999
α_0	—1.0000000	—1.0000000	—1.0000000	—1.0000000

The test jj) has been carried out at each step and is fulfilled by α_i, $i \leq 3$ and β_i, $i \leq 3$. Therefore, although the computed sequences $\{\alpha_n\}_n$ and $\{\beta_n\}_n$ are monotone, a more rigorous analysis according to the previous ocnsiderations, states that $\alpha_3 < u < \beta_3$. At each step, the linear equation (16) is solved by a fourth-order Runge-Kutta method with step-size 0.001. The computations are performed on a CDC 170/720 and the CPU time required for all the computations in table 1, including the test jj), amounts to ~ 10 sec.

EXAMPLE 2. Let consider the solutions of

$$u''(x) = -\frac{1}{16} \sin u(x) - (x+1)\, u(x-1) + x \qquad 0 \leq x \leq 2$$

satisfying the boundary conditions

$$u(x) = x - \frac{1}{2} \qquad -1 \leq x \leq 0$$

$$u(2) = -\frac{1}{2}.$$

The problem is already been investigated in [5], [7], [15], and the existence and uniqueness of the solution has been proved.

For this problem, our theoretical approach is not satisfactory since, although it is easy to see that the function $\beta = -0.5$ is an upper solution, it is difficult to find a lower solution α such that $\alpha \leq \beta$.

However, by remark 3, the iteration $\{\beta_n\}_n$, defined by

$$\beta_{n+1}(x) - \frac{1}{16}\beta_{n+1}(x) = -\frac{1}{16} \sin \beta_n(x) - (x+1)\, \beta_n(x-1) + x - \frac{1}{16}\beta_n(x)$$
$$0 \leq x \leq 2$$

$$\beta_{n+1}(x) = x - \frac{1}{2} \qquad -1 \leq x \leq 0$$

$$\beta_{n+1}(2) = -\frac{1}{2}$$

where $\beta_0 = \beta$, is decreasing and converges to the solution u provided $u \leq -0.5$.

In table 2 the values of β_n are printed in 0.5; 1; 1.5 for $n=0, \ldots, 10$.

TABLE 2.

	$x = 0.5$	$x = 1$	$x = 1.5$
β_0	— 0.5000000	— 0.5000000	— 0.5000000
β_1	— 1.2980322	— 1.6108198	— 1.3963463
β_2	— 1.4869041	— 1.9726863	— 1.8269092
β_3	— 1.5307820	— 2.0578733	— 1.9322411
β_4	— 1.5407254	— 2.0772475	— 1.9564022
β_5	— 1.5429657	— 2.0816164	— 1.9618607
β_6	— 1.5434698	— 2.0825997	— 1.9630898
β_7	— 1.5435832	— 2.0828209	— 1.9633663
β_8	— 1.5436087	— 2.0828707	— 1.9634285
β_9	— 1.5436145	— 2.0828819	— 1.9634425
β_{10}	— 1.5436158	— 2.0828844	— 1.9634456

Here also the computed sequence $\{\beta_n\}_n$ is monotone and the test jj) is fulfilled by β_i, $i \leq 5$ and therefore $u \leq \beta_5$. Each linear equation is solved by a fourth order Runge-Kutta method with step-size 0.001 and the CPU time required for the computations is $\sim$ 10 sec.

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