ON LOCALLY COMPACT PARACOMPACT SPACES (*)

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SOMMARIO. - *Viene data una dimostrazione elementare del fatto che il prodotto cartesiano di un'infinità numerabile di spazi localmente compatti e paracompatti è uno spazio paracompatto. Infine si caratterizzano gli spazi ereditariamente paracompatti e perfettamente normali.*

SUMMARY. - *An elementary proof is given that the cartesian product of countably many locally compact and paracompact spaces is a paracompact space. Finally hereditarily paracompact and perfectly normal spaces are characterized.*

The aim of this short note is to show how natural and useful can be to describe some classes of paracompact spaces in terms of families of compact subspaces.

Actually the class of locally compact paracompact spaces, a rather well known one, is examined and very elementary proofs concerning the products of such spaces are given.

1. Bourbaki [2] characterized locally compact and paracompact spaces as being the union of disjoint locally compact and σ-compact spaces. The following two propositions give another characterization.

PROPOSITION 1. *Let X be a Hausdorff space which is the union of a locally finite family of compact subspaces* $\{K_\alpha\}_{\alpha\in I}$. *Then X is locally compact and paracompact.*

(*) Pervenuto in Redazione il 24 agosto 1978.

Lavoro eseguito nell'ambito del Gruppo Nazionale per l'Analisi Funzionale e le sue Applicazioni.

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PROOF. Given $x \in X$ there is a neighborhood U of x that intersects only a finite number of compact subspaces of the family $\{K_\alpha\}_{\alpha \in I}$. We then have $x \in U \subset \bigcup_{i=1}^{n} K_{\alpha_i}$, so that U is compact. X being locally compact is also regular. Let now $\mathcal{U}$ be an open covering of X. For every $\alpha \in I$ let $\mathcal{U}_\alpha$ be a finite subfamily of $\mathcal{U}$ which covers K_α. Consider $\mathcal{B} = \{A \cap K_\alpha \colon \alpha \in I \text{ and } A \in \mathcal{U}_\alpha\}$. This family covers X since it covers K_α for every α. Since $\{K_\alpha\}$ is locally finite and every K_α contains a finite number of elements of $\mathcal{B}$, also $\mathcal{B}$ is locally finite. Finally for every element of $\mathcal{U}$ one has $A \cap K_\alpha \subset A \in \mathcal{U}_\alpha \subset \mathcal{U}$. So $\mathcal{B}$ is a locally finite refinement of $\mathcal{U}$ which covers X. Since X is regular then it is paracompact [4, 6].

PROPOSITION 2. *Let X be a Hausdorff locally compact and paracompact space. Then X can be represented as a union of a locally finite family of compact subspaces.*

PROOF. For every $x \in X$ choose a relatively compact neighborhood V_x, and let $\mathcal{V} = \{V_x\}_{x \in X}$. Then there exists [4, 6] a closed and locally finite covering $\mathcal{C}$ which refines $\mathcal{V}$. For every $K \in \mathcal{C}$ we have $K = \bar{K} \subset V_x \subset \bar{V}_x$, so that K is compact. $\mathcal{C}$ is then a locally finite family of compact subspaces of X whose union is X.

From Propositions 1 and 2 one can conclude that a Hausdorff space X is locally compact and paracompact if and only if $X = \bigcup_{\alpha \in I} K_\alpha$, where $\{K_\alpha\}_{\alpha \in I}$ is a locally finite family of compact spaces.

Starting with this characterization one can easily recover the one given by Bourbaki. Let us say that $x \sim y$ iff there exist finitely many compact sets $K_{\alpha_1}, K_{\alpha_2}, \ldots, K_{\alpha_n}$ such that $x \in K_{\alpha_1}$, $y \in K_{\alpha_n}$ and $K_{\alpha_i} \cap K_{\alpha_{i+1}} \neq \Phi$ for $i = 1, 2, \ldots, n-1$.

In such a way X is decomposed in a family of disjoint clopen equivalence classes and each one is a σ-compact and locally compact space. This is easily shown following the same line of proof as given in Bourbaki.

Next proposition is easily established.

PROPOSITION 3. *The product of two locally compact and paracompact spaces is also locally compact and paracompact.*

In fact if $X = \bigcup_{\alpha \in I} K_\alpha$ and $Y = \bigcup_{\beta \in J} C_\beta$, then $X \times Y = \bigcup_{(\alpha, \beta) \in I \times J} K_\alpha \times C_\beta$ and the family $\{K_\alpha \times C_\beta\}_{(\alpha, \beta) \in I \times J}$ is a locally finite family of compact spaces.

Obviously the same is true for any finite number of locally compact and paracompact spaces.

We are now ready to prove the following.

THEOREM 1. *The product of a countable family of locally compact and paracompact spaces is a paracompact space.*

PROOF. Suppose that a countable family of locally compact and paracompact spaces $\{X_n\}^{\infty}_{n=1}$ be given. Since all this spaces are locally compact they are open in any their compactification; denote by cX_n any compactification of X_n (for example the one-point compactification). Denote also by $\mathcal{P}^*$ (**N**) the set of all finite subsets of the natural numbers **N**. Observe that the compact space $Z=\prod_{n=1}^{\infty} cX_n$ is a compactification of $X=\prod_{n=1}^{\infty} X_n$. If $M \in \mathcal{P}^*(\mathbf{N})$ let $A_M=\prod_{n=1}^{\infty} X_n'$, where $X_n'=X_n$ if $n \in M$ and $X_n'=cX_n$ if $n \in \mathbf{N}-M$. Now there are countably many sets like A_M; they are open in Z and we have $X=\prod_{n=1}^{\infty} X_n= \bigcap_{M \in \mathcal{P}_*(\mathbf{N})} A_M$.

It is important to observe that the product space X not only is a G_δ in Z, but that the open sets A_M form a basis of X in Z and that each A_M is locally compact and paracompact.

In fact each A_M is a product of finitely many locally compact and paracompact spaces and of countably many compact spaces so that, as follows from Proposition 3, it is a paracompact and locally compact space. Then X is paracompact though, in general, it is not locally compact. (See [3], proposition 30 C. 11, p. 540).

It can be observed that this result is a particular case of a theorem of Z. Frolík [5], and also of a more general one of Arhangel'skii [1]. The proof given here is however a more direct one, only given in terms of coverings of the space, and is perhaps more elementarily established.

REMARK. It can be observed that if all spaces X_n are supposed complete in the sense of Čech and paracompact, and if one has a direct proof that the product of finitely many Čech-complete and paracompact spaces is a paracompact Čech-complete space, the same proof of Theorem 1 holds to give the stronger result that the product of countably many Čech-complete and paracompact spaces is a Čech-complete and paracompact space. Finally I observe that the result of Theorem 1 is in some sense the best possible one since A. H. Stone has shown that the product of uncountably many copies of **N** with its natural topology is not normal.

2. Finally I give a result concerning the related subject of hereditary paracompactness. Precisely what follows is a characterization of a paracompact and perfectly normal space, that is a paracompact space in which every open set is an F_σ: such a space is hereditarily paracompact.

Let us provisionally give the following.

DEFINITION. A paracompact space X is said to be *strongly hereditarily paracompact* (s. h. p.) if for every subset A of X and for every open covering of A an open refinement can be found which covers A and is *σ-locally finite in X.*

It is useful to remember the following theorems.

THEOREM A. The topological product of a strongly hereditarily paracompact space and a metrizable space is s. h. p.

The proof of this theorem follows classical lines [7].

THEOREM B. Let X be a topological space and Y a metrizable space. In order that the product space $X \times Y$ be paracompact and hereditarily normal it is necessary and sufficient that either

(*a*) X is paracompact and perfectly normal, or

(*b*) X is paracompact hereditarily normal and Y discrete [3].

One can then easily (but rather indirectly) prove the following

THEOREM 2. *A paracompact space is perfectly normal if and only if it is strongly hereditarily paracompact.*

PROOF. It is known that if X is paracompact and perfectly normal then it is a s. h. p. space [6]. Let now X be a s. h. p. space and Y a metrizable and not discrete space. By Theorem A, $X \times Y$ is s. h. p. and hence paracompact and hereditarily normal. From Theorem B it then follows that X is paracompact and perfectly normal.

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