A RESULT CONCERNING A CLASS OF LADDER OPERATORS (*)

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SOMMARIO. - *Nella nota che segue vengono discusse esistenza e unicità di certi operatori chiusi definiti su di una somma diretta di potenze tensoriali simmetriche di uno spazio complesso di Hilbert. Il risultato ha interesse nella teoria assiomatica dei campi quantistici.*

SUMMARY. - *In the following note existence and uniqueness of certain closed operators defined on a direct sum of symmetric tensor powers of a complex Hilbert space are briefly discussed. The results are of interest in axiomatic quantum field theory.*

1. Introduction.

The purpose of the present note is to show that suitably defined ladder operators on the direct sum of symmetric tensor powers of a Hilbert space H are densely defined adjoint endowed operators, which satisfy to a well-known commutation rule with their adjoints. This result is of some stress as regards axiomatic quantum field theory (see [1], [2], [3], [4], [5]), especially together with the converse, also proved, which assures that a direct sum of symmetric tensor powers of H can be obtained, up to isometry, by the repeated applications of

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the above operator on a specific vector within its domain. We point out that, although analogous results are well-known [1], our approach holds under weaker assumptions than the ones in the current literature.

2. Existence.

Let H be any separable Hilbert space over the complex field C. Let $H^{\otimes 0}$ denote the *n-th* tensor product of H over C (we pose, as usual, $H^{\otimes 0}=C$ and $H^{\otimes 1}=H$). Let H_n be the symmetric *n-th* tensor pover of H over C [2]. Let $F=\sum_n H_n$ be the direct sum of these spaces (Fock space). Let us denote with $\varphi, \psi, \ldots$ elements in H, with (φ, ψ) the scalar product on H between them, and with $\|\varphi\|=(\varphi, \varphi)$ the norm of φ in H: further, let us denote with $\{\psi_i\}_{i \in I}$ a countably infinite orthonormal basis in H, I being a suitable set of indices. Thus, for any n, let us denote with $x_n, y_n, \ldots$ elements in H_n, with $(x_n, y_n)_n$ the canonically induced scalar product between them, and with $\|x_n\|_n$ the norm of x_n in H_n. As a countable orthonormal basis in H_n we take the one obtained by collecting together all the normalized vectors of the form $y_n^{i_1, i_2, \ldots, i_n}=\psi_{i_1} \otimes \psi_{i_2} \otimes \ldots \otimes \psi_{i_n}$ where i_j runs the same index set I (and the tensor product is intended to be symmetric, as said above). We could also write y_n^k, with $k \in I^n$; but as a more familiar form we write the following: $y_n^k=|n_1, n_2, \ldots ; n>$, where n_i gives the multiplicity of ψ_i within $\psi_{i_1} \otimes \psi_{i_2} \otimes \ldots \otimes \psi_{i_n}$ and $\sum_{i \in I} n_i=n$. Similarly, let us denote with $x, y, \ldots$ elements within F, with $\{x, y\}$ the canonically induced scalar product on F, and with $\|x\|_F$ the norm of x on F. A countably infinite basis in F can be obtained by collecting together all the orthonormal vectors y_n^k, when n assumes all the values between 0 and ∞: the basis is an orthonormal one, as F is a direct sum. Finally, we denote with A a linear operator, with A^* the adjoint of A, and with D_A, D_{A^*} their domains.

Let now us consider the operator-valued map $\mathscr{A}: \psi \rightarrow A(\psi)$ which joins to any $\psi \in H$ the linear operators $A(\psi)$ [3] on F, defined as follows:

(1) See [2], p. 85 and following.

(2) The extension to the antisymmetric case is not difficult, apart from a few minor modifications: however, it will not be considered here, for simplicity's sake.

(3) Which is said « ladder operator » because of i).

i) for any n and for all basis vectors within H_n, the restriction $A_n(\psi)$ to H_n is a correspondence $H_n \to H_{n+1}$ given by:

$$A_n(\psi) \mid n_1, n_2, \dots; n> = \sum_{i \in I} (\psi, \psi_i)\sqrt{n_i+1} \mid n_1, \dots, n_i+1, \dots; n+1>$$

[we agree that $A_0(\psi)\, x_0 = \psi$ (x_0 being a complex number)];

ii) the domain $D_{A(\psi)}$ of $A(\psi)$ is given by taking all the vectors $x \in F$, $x = \sum_n x_n$ such that $A(\psi)\, x = \sum_n A_n(\psi)\, x_n$ and

$$\sum_n \|A_n(\psi)\, x_n\|_n^2 < \infty .$$

Thus we state the following result.

THEOREM 1. Let H be a separable complex Hilbert space, and let F be the Fock space, as above. The following statements are true:

a) $\forall x \in F$, $x = \sum_n x_n$, $x_n \in H_n$, $\forall \psi \in H$, $x \in D_{A(\psi)} \Leftrightarrow \sum_n \|A(\psi) x_n\|_n^2 < \infty$ and $A(\psi)\, x = \sum_n A_n(\psi)\, x_n$; moreover, $A(\psi)$ has an adjoint $A^*(\psi)$, defined on all the basis vectors of H_n, for any n, and for any $\psi \in H$;

b) for all $\psi \in H$, $A(\psi)$ and $A^*(\psi)$ are closed;

c) the operators $A(\psi)$ and $A^*(\psi)$ have a non-void dense common invariant domain within F;

d) $\forall \varphi \in H$, $[A^*(\varphi), A(\psi)] \subseteq (\psi, \varphi)\, 1_F$ (1_F being the identity operator on F), and $D_{A^*A} = D_{AA^*}$;

e) the map $\mathcal{A}$ from H to the set of all closed operators on F is sequentially continuous, whenever this set is endowed with the weak topology given by the set of seminorms

$$A \to |(Ax, y)|, \quad \text{with} \quad x \in D_{A(\psi)}.$$

PROOF. The first part of *a)* follows from the definition of $D_{A(\psi)}$. As concerns the second part, we observe that the adjoint of A is defined through its restriction to A_n (which is equal to the adjoint of the restriction of A to H_n) in the following way:

$$A_n^*(\psi) \mid n_1, \dots, n_i, \dots; n> = \sum_{i \in I} \sqrt{n_i}\, (\psi_i, \psi) \mid n_1, \dots, n_i - 1, \dots \;; n-1>.$$

Let now us prove statement *b)*. We first prove that any $A_n(\psi)$ is continuous in H_n. We have, for all the basis vectors in H_n and for any

n, $||A_n(\psi)\,|\,n_1, \dots, n_i, \dots; n>||^2_{n+1} \leq (n+1)\cdot||\,|n_1, \dots, n_i, \dots; n+1>||^2_{n+1} =$ $=(n+1)\cdot||\,|n_1, \dots, n_i, \dots\;\; ; n>||^2_{n+1} = n+1$.

Then we take a set $\{x_a\}_{a\in N}$ (N being the set of natural numbers) of vectors $x_a \in D_{A(\psi)}$, such that $\lim\limits_{a\to\infty} x_a = x$. We can put $\lim\limits_{a\to\infty} A(\psi)\, x_a = y$, with $||y||^2_F < \infty$. Thus, by projecting onto the subspaces H_n and taking into account the continuity of the projectors, we get (the notations are self-explaining):

$$\lim_{a\to\infty} x_{a,n} = x_n \text{ and } y_n = \lim_{a\to\infty} (A(\psi)\, x_a)_n = \lim_{a\to\infty} (A(\psi)\, x_{a,n-1}) =$$

$$= \lim_{a\to\infty} (A_{n-1}(\psi)\, x_{a,n-1}) = A_{n-1}(\psi) \lim_{a\to\infty} x_{a,n-1} = A_{n-1}(\psi)\, x_{n-1}.$$

Then, by putting $y_0 = 0$, we can write:

$$\sum_{n=1}^{\infty} ||A(\psi)\, x_{n-1}||^2_n = \sum_{n=1}^{\infty} ||y_n||^2_n = ||y||^2_F,$$

from which it follows that $x \in D_{A(\psi)}$; moreover:

$$A(\psi)\, x = \sum_{n=0}^{\infty} A_n(\psi)\, x_n = \sum_{n=0}^{\infty} y_{n+1} = y.$$

This means that $A(\psi)$ is closed, for all $\psi \in H$. Same reasoning for $A^*(\psi)$. *c*) is a direct consequence of *a*): we shall denote with $D = D_A = D_{A^*}$ such a domain.

d) follows from a simple calculation of the commutator between $A_n^*(\varphi)$ and $A_n(\psi)$.

In fact:

$$[A^*_{n+1}(\varphi)\, A_n(\psi) - A_{n-1}(\psi)\, A_n^*(\varphi)]\,|\,n_1, \dots, n_i, \dots; n> =$$

$$= \sum_{i\in I} (\psi, \psi_i)(\psi_i, \varphi)(n_i+1)\,|\,n_1, \dots, n_i, \dots; n> -$$

$$- \sum_{i\in I} (\psi, \psi_i)(\psi_i, \varphi)\, n_i\,|\,n_1, n_2, \dots; n> = \sum_{i\in I} (\psi, \psi_i)(\psi_i, \varphi)\,|\,n_1, \dots, n_i, \dots; n>,$$

from which we get the result by using completeness of the scalar product.

Concerning the part *e*), let us take a sequence $\psi_1, \psi_2, \dots$ such that $\lim\limits_{n\to\infty} \psi_n = 0$: our purpose is to show that the sequence $|(A(\psi_1)\, x_k, y_k)_k|$, $|(A(\psi_2)\, x_k, y_k)_k|, \dots$ satisfies to $\lim\limits_{n\to\infty} |(A(\psi_n)\, x_k, y_k)_k| = 0$, x_k and y_k

being vectors of D. Suppose now that $x_k \in H_{k-1}$ and $y_k \in H_k$: we have no loss of generality because of the fact that the result can be immediately extended by the continuity of the scalar product to any n, while the choice of x_k is a suitable one as vectors belonging to different tensorial powers are orthogonal.

Thus we have $\lim_{n \to \infty} |(A(\psi_n) x_k, y_k)_k| \leq \lim_{n \to \infty} \sqrt{k+1} \, |(\psi_n \otimes x_k), y_k)_k| = = \sqrt{k+1} \, |(\lim_{n \to \infty} \psi_n \otimes x_k), y_k)_k| = 0$; on the other side, the sequence $|(A(\psi_1) x_k, y_k)_k|, |(A(\psi_2) x_k, y_k)_k|, \ldots$ is all made of positive terms, from which it follows that $\lim_{n \to \infty} (A(\psi_n) x_k, y_k)_k = 0$.

This concludes the proof of the Theorem.

3. Uniqueness.

We are now interested to a result concerning a sort of uniqueness of our operators. In fact, given a suitable operator-valued mapping from a Hilbert space H to the set of linear closed operators on the Fock space F, we will show that F is completely determined, provided a set of postulates concerning the said operators is given.

THEOREM 2. Let $\mathcal{A}: \psi \to A(\psi)$ be a map from a Hilbert space H to the set of the closed linear adjoint-endowed operators, which have an invariant domain D, defined on a Fock space F: suppose moreover that an operator $A(\varphi)$ exists satisfying to:

i) for all ψ, φ in H: $[A(\varphi), A(\psi)] = 0$ and $[A^*(\varphi), A(\psi)] \subseteq \subseteq (\varphi, \psi) \, 1_F$;

ii) a unique-up-to-a-factor vector $\Omega \in D$ exists such that for all $\varphi \in H$ $A^*(\varphi)\, \Omega = 0_F$ (the null vector of F) and $\{\Omega, \Omega\} = 1$.

Let the vector space G_k *be* defined as follows:

$$G_k = \{A(\psi_1) \ldots A(\psi_k)\, \Omega, \text{ for any } \psi_1, \ldots, \psi_k \in H\};$$

then the mapping τ from H_k to G_k defined by

$$\tau: (\psi_1 \otimes \ldots \otimes \psi_k) \in H_k \to A(\psi_1) \ldots A(\psi_k)\, \Omega \in G_k$$

is isometric.

PROOF. We divide the proof in three steps. First of all we note that, by denoting with G_1 the set $G_1 = \{A(\psi)\, \Omega \mid \psi \in H\}$, we have in G_1

$\{A(\varphi)\Omega, A(\psi)\Omega\} = \{\Omega, A^*(\varphi)A(\psi)\Omega\} = \{\Omega, A(\psi)A^*(\varphi)\Omega\} + \{\Omega, (\varphi,\psi)\Omega\} = (\varphi, \psi)$, by using i) and ii). In the second place we observe that in H_k for all simmetrized $\Psi = (\psi_1 \otimes \ldots \otimes \psi_i \otimes \ldots \otimes \psi_k)$ and $\Phi = (\varphi_1 \otimes \ldots \otimes \varphi_i \otimes \ldots \otimes \varphi_k)$ we have:

$$(\Psi, \Phi)_k = \sum_i (1/\sqrt{n_1 m_i})\,(\psi_1, \varphi_i)\,(\Psi^{(1)}, \Phi^{(i)})_k$$

where n_1 (n_i) is the number of the j indices for which $\psi_j = \psi_1$ (ψ_i) and m_i is the number of the j indices for which $\varphi_j = \varphi_i$, and where:

$$\Psi^{(1)} = (\psi_2 \otimes \ldots \otimes \psi_k)$$

$$\Phi^{(i)} = (\varphi_1 \otimes \ldots \otimes \varphi_{i-1} \otimes \varphi_{i+1} \otimes \ldots \otimes \varphi_k).$$

In fact, by using the continuity of the scalar product, we have:

$$(\Psi, \Phi)_k = (\sum_i (1/\sqrt{n_i k})\,\psi_i \otimes \Psi^{(i)},\ \sum_j (1/\sqrt{k m_j})\,\varphi_j \otimes \Phi^{(j)})_k =$$

$$= \sum_{ij} (1/k)\,(1/\sqrt{m_j n_i})\,(\psi_i, \varphi_j)\,(\Psi^{(i)}, \Phi^{(j)})_k =$$

$$= (1/k) \sum_j (1/\sqrt{n_1 m_j})\,(\psi_1, \varphi_j)\,(\Psi^{(1)}, \Phi^{(j)})_k + \ldots$$

$$\ldots + (1/k) \sum_j (1/\sqrt{n_k m_j})\,(\psi_k, \varphi_j)\,(\Psi^{(k)}, \Phi^{(j)})_k =$$

$$= \sum_i (1/\sqrt{n_1 m_i})\,(\psi_1, \varphi_i)\,(\Psi^{(1)}, \Psi^{(i)})$$

because of the fact that all the separately written terms are equal each other and that all the possible products of the components of the expansion of $\Psi \otimes \Phi$ are contained in each of these terms.

The last step consists in the use of the induction. As a matter of fact we know that the statement holds for G_1. Let us suppose that it is true for $n-1$: then we have, by using isometry:

$$\{A(\psi_1) \ldots A(\psi_k)\,\Omega,\ A(\varphi_1) \ldots A(\varphi_k)\,\Omega\} =$$

$$= \{A(\psi_2) \ldots A(\psi_k)\,\Omega,\ A^*(\psi_1) \ldots A(\psi_k)\,\Omega\} =$$

$$= \sum \frac{1}{\sqrt{n_1 m_i}}\,(\psi_1, \varphi_i)\,\{A(\psi_2) \ldots \mathrm{A}(\psi_k)\,\Omega,\ A(\varphi_1) \ldots A(\varphi_{i-1})$$

$$A(\varphi_{i+1}) \dots A(\varphi_k)\,\Omega\} = \sum_i \frac{1}{\sqrt{n_1 m_i}} (\psi_1, \varphi_i)(\Psi^{(1)}, \Phi^{(i)})_k.$$

At last, because of the precedently proved step, we have:

$$\{A(\psi_1) \dots A(\psi_k)\,\Omega,\ A(\varphi_1) \dots A(\varphi_k)\,\Omega\} = (\Phi, \Psi)_k.$$

Thus the theorem is proved.

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