REGULARITY AND REPRESENTATION THEOREMS FOR A CLASS OF TRANSLATION EQUATION'S SOLUTIONS (*)

di Gian Luigi Forti (a Milano) (**)

Sommario. - *In questa Nota vengono dimostrati alcuni teoremi che garantiscono la continuità di un sistema di omeomorfismi e del suo sistema inverso. Tali risultati sono poi utilizzati per dimostrare la regolarità di certe soluzioni dell'equazione funzionale di traslazione* $f(f(x, u, v), v, w) = f(x, u, w)$ *e per fornire una rappresentazione di tali soluzioni.*

Summary. - *In this Note some theorems are given which ensure the continuity of a system of homeomorphisms and of its inverse system. Those results are used to prove the regularity of certain solutions of the translation functional equation* $f(f(x, u, v), v, w) = f(x, u, w)$ *and to give a representation of such solutions.*

1. Consider the translation equation $f(f(x, u, v), v, w) = f(x, u, w)$, where $f: X \times Y \times Y \to X$. This functional equation has been recently treated by C. T. Ng ([3], [4]), who, assuming the local compactness and the local connectedness of the space X, gives a representation of a class of continuous solutions of such an equation, using the following theorem:

Let X be a locally compact Hausdorff and locally connected topological space, and let Y be a topological space. Let $f: X \times Y \to X$ be a continuous mapping such that for each $y \in Y$, the mapping $f(\cdot, y): X \to X$ is a homeomorphism on X. Then the inverse system $f^{-1}: X \times Y \to X$

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(**) Indirizzo dell'Autore: Gian Luigi Forti, Istituto di Matematica « F. Enriques », via C. Saldini 50, 20133 Milano.

defined by $f^{-1}(x, y)=x'$ *if and only if* $f(x', y)=x$ *for all* $x \in X$ *and* $y \in Y$, *is also continuous on* $X \times Y$.

In section 2 of this Note some theorems are given which ensure, in hypotheses different from the above mentioned theorem, the continuity of a system of homeomorphisms and of its inverse system. In section 3 those results are used to give some theorems of regularity and representation for a class of solutions of the translation equation.

2. Let X, Y be sets and let $f: X \times Y \rightarrow X$ be a mapping such that for each $y \in Y$, the mapping $f(\cdot, y): X \rightarrow X$ is bijective; f^{-1} will denote the mapping of $X \times Y$ into X defined by: $f^{-1}(x, y)=x'$ if and only if $f(x', y)=x$.

Afterwords the following notations will be used:

if X is a uniform space, the set of the entourages of the uniformity will be noted by $\mathcal{U}$.

if $V \in \mathcal{U}$, then for each $x \in X$ it is $V(x)=\{z \in X: (z, x) \in V\}$ and $\overset{2}{V}$ is the set of pairs $(x, z) \in X \times X$, such that $(x, w) \in V$ and $(w, z) \in V$ for some $w \in X$;

if Z is a topological space, $C(Z; X)$ denotes the set of all continuous mappings of Z into X.

Let $\mathfrak{S}$ be a set of subsets of Z; for each $A \in \mathfrak{S}$ and each entourage V of X, let $W(A, V)$ be the set of all pairs of continuous mappings (g, h) of Z into X such that $(g(z), h(z)) \in V$ for each $z \in A$; as A runs through $\mathfrak{S}$ and V runs through $\mathcal{U}$, the finite intersections of the sets $W(A, V)$ form a fundamental system of entourages of a uniformity on $C(Z; X)$; this uniformity is called the uniformity of $\mathfrak{S}$-convergence and the uniform space obtained by endowing $C(Z; X)$ with the uniformity of $\mathfrak{S}$-convergence is denoted by $C_{\mathfrak{S}}(Z; X)$. In particular $C_s(Z; X)$, $C_c(Z; X)$ and $C_u(Z; X)$ denote the set $C(Z; X)$ endowed respectively with the uniformity of pointwise convergence, compact convergence and uniform convergence (see [2]).

LEMMA 1. *Let* X *be a uniform space, let* Y *be a topological space and let* $f: X \times Y \rightarrow X$ *be a mapping such that for each* $y \in Y$, $f(\cdot, y) \in C(X; X)$. *Let* $\tilde{f}: Y \rightarrow C(X; X)$ *be the mapping* $\tilde{f}(y)=f(\cdot, y)$.

If there exists an open cover $\mathfrak{S}$ *of* X *such that* $\tilde{f}: Y \rightarrow C_{\mathfrak{S}}(X; X)$ *is continuous, then* f *is continuous.*

PROOF. Let $x_0 \in X$, $y_0 \in Y$ and let $V \in \mathcal{U}$. As $f(\cdot, y_0) \in C(X; X)$, there exists a neighbourhood U of x_0, with $U \subset A \in \mathfrak{S}$, such that for each

$x \in U$ it is $(f(x, y_0), f(x_0, y_0)) \in V$. By the continuity of $\tilde{f}$, the set W of all $y \in Y$ such that $(f(x, y), f(x, y_0)) \in V$ for all $x \in A$, is a neighbourhood of y_0, then for each $(x, y) \in U \times W$ it is $(f(x, y), f(x_0, y_0)) \in \overset{2}{V}$, i. e. f is continuous.

REMARK 1. If $\mathfrak{S}$ is an open cover of X, the topology of $C_{\mathfrak{S}}(X; X)$ is finer than the topology of $C_c(X; X)$, and the continuity of f implies the continuity of $\tilde{f}$ from Y to $C_c(X; X)$; the converse is not true. If X is locally compact, then f is continuous if and only if $\tilde{f}$ from Y to $C_c(X; X)$ is continuous ([2]).

THEOREM 1. *Let X be a uniform space, let Y be a topological space and let f: $X \times Y \to X$ be a mapping such that for each $y \in Y$, $f(\cdot, y)$ is a homeomorphism on X; if*

1) *there exists an open cover $\mathfrak{S}$ of X such that $\tilde{f}$: $Y \to C_{\mathfrak{S}}(X; X)$ is continuous;*

2) *noted $\tilde{Y} = \tilde{f}(Y)$, the set $\tilde{Y}^{-1}$ of the inverse homeomorphisms is equicontinuous on X;*

then f and f^{-1}: $X \times Y \to X$ are continuous.

PROOF. f is continuous by Lemma 1.

Let $\tilde{Y}^{-1}$ be endowed with the topology of $C_s(X; X)$, then the mapping $(x, \tilde{y}^{-1}) \to \tilde{y}^{-1}(x)$ of $X \times \tilde{Y}^{-1}$ into X is continuous. Indeed let $\tilde{y}_0^{-1} \in \tilde{Y}^{-1}$, $x_0 \in X$ and $V \in \mathcal{U}$; by 2) there exists a neighbourhood U of x_0 such that for each $\tilde{y} \in \tilde{Y}$ and for each $x \in U$ $(\tilde{y}^{-1}(x), \tilde{y}^{-1}(x_0)) \in V$; the set T of all $\tilde{y}^{-1} \in \tilde{Y}^{-1}$ such that $(\tilde{y}^{-1}(x_0), \tilde{y}_0^{-1}(x_0)) \in V$ is a neighbourhood of $\tilde{y}_0^{-1}$ and, for every $(x, \tilde{y}^{-1}) \in U \times T$, it is $(\tilde{y}^{-1}(x), \tilde{y}_0^{-1}(x_0)) \in \overset{2}{V}$.

If $\tilde{Y}$ and $\tilde{Y}^{-1}$ are endowed with the topology of $C_s(X; X)$, then the mapping ϕ: $\tilde{Y} \to \tilde{Y}^{-1}$, $\phi(\tilde{y}) = \tilde{y}^{-1}$, is continuous. It is enaugh to show that, for each $x_0 \in X$, the mapping $\tilde{y} \to \tilde{y}^{-1}(x_0)$ of $\tilde{Y}$ into X is continuous at every point $\tilde{y}_0 \in \tilde{Y}$. Let $V \in \mathcal{U}$ and let $u_0 = \tilde{y}_0^{-1}(x_0)$; by 2) there exists a neighbourhood U of x_0 such that, for each $\tilde{y} \in \tilde{Y}$ and for each $x \in U$, it is $(\tilde{y}^{-1}(x), \tilde{y}^{-1}(x_0)) \in V$; if in particular $\tilde{y} \in \tilde{Y}$ is such that $\tilde{y}(u_0) \in U$, then $(u_0, \tilde{y}^{-1}(x_0)) \in V$, i. e. $(\tilde{y}_0^{-1}(x_0), \tilde{y}^{-1}(x_0)) \in V$.

Since the topology of $C_{\mathfrak{S}}(X; X)$ is finer than the topology of $C_s(X; X)$, the mapping $f^{-1}(x, y) = [\phi(\tilde{f}(y))](x)$ is continuous.

REMARK 2. The hypothesis 1) of Theorem 1 is equivalent to the following

1') there exists an open cover $\mathfrak{S}$ of X such that for every $A \in \mathfrak{S}$ the set $H_A = \{y \to f(x, y), x \in A\}$ is equicontinuous on Y ([2]).

The hypothesis 2) of Theorem 1 is equivalent to the following

2') for each $x \in X$ and for each $V \in \mathcal{U}$, there exists a neighbourhood U of x such that $\bigcap_{y \in Y} f(V(\tilde{y}^{-1}(x)), y) \supset U$.

COROLLARY 1. *Let X be a uniform space, let Y be a topological space and let f: $X \times Y \to X$ be a continuous mapping such that for each $y \in Y$ $f(\cdot, y)$ is a homeomorphism on X.*

If the hypothesis 2) of Theorem 1 holds then f^{-1} is continuous.

PROOF. $\tilde{f}$: $Y \to C_s(X; X)$ is continuous; indeed let $B \subset X$ a finite set, then fixed $y_0 \in Y$ and $x \in B$, the continuity of f implies that for every $V \in \mathcal{U}$ there exists a neighbourhood U_x of y_0 such that $y \in U_x$ implies $(f(x, y), f(x, y_0)) \in V$; set $U = \bigcap_{x \in B} U_x$, then for each $y \in U$ and for each $x \in B$ it is $(f(x, y), f(x, y_0)) \in V$.

Now, acting as in the proof of Theorem 1, the assertion is proved.

The following example will illustrate Theorem 1; at the same time the impossibility of making use in this case of the theorem in [4], shows that the two theorems are suitable in different situations.

EXAMPLE. Let $X = Q^+ - \{0\}$ and $Y = \{y \in Q: 0 < y < 1\}$ (Q is the field of rational numbers; Q^+ is the set of non negative rational numbers) be topological subspaces of Q.

Let f: $X \times Y \to X$ be defined by

$$f(x, y) = \begin{cases} \dfrac{1}{x} + y - 1 & \text{if } 0 < x < 1 \\[2ex] \dfrac{y}{x} & \text{if } x \geq 1 \end{cases}$$

It is easy to verify that $\tilde{f}$ is continuous from Y to $C_u(X; X)$ and therefore to $C_{\mathfrak{S}}(X; X)$, for every open cover $\mathfrak{S}$ of X.

$\widetilde{Y}^{-1}$ is composed by the following mappings

$$\widetilde{y}^{-1}(x)=\begin{cases} \dfrac{y}{x} & \text{if } 0<x\leq y \\ \dfrac{1}{x-y+1} & \text{if } x>y \end{cases}$$

For fixed $x_0 \in X$, for every $\widetilde{y} \in \widetilde{Y}$ it is $|\widetilde{y}^{-1}(x)-\widetilde{y}^{-1}(x_0)|<K|x-x_0|$, where $K=\mathrm{Max}\left(1;\dfrac{2}{x_0};\dfrac{2}{x_0^2}\right)$, therefore $\widetilde{Y}^{-1}$ is equicontinuous in x_0 and then on X; hence the hypotheses of Theorem 1 hold.

The following example (see [4]) on the contrary shows that if hypothesis 2) of Theorem 1 is dropped, the function f^{-1} does not result necessarily continuous.

EXAMPLE. Let $X=C^0[0,1]$ be the linear space of all continuous real-valued functions defined on the interval $[0,1]$ vanishing at the origin 0, endowed with the topology of uniform convergence. Let $Y=\left\{\dfrac{1}{n}: n\in N\right\}\cup\{0\}$ (N is the set of all positive integer) with the topology of the real line.

Let $\{\Phi_n\}$ be a sequence of continuous functions on $[0,1]$ defined by $\Phi_0(t)=1$,

$$\Phi_n(t)=\begin{cases} \dfrac{1}{n+1} & \text{if } 0\leq t<\dfrac{1}{n+1} \\ \dfrac{1}{n+1}+n^2\left(t-\dfrac{1}{n+1}\right) & \text{if } \dfrac{1}{n+1}\leq t<\dfrac{1}{n} \\ 1 & \text{if } \dfrac{1}{n}\leq t\leq 1. \end{cases}$$

Consider the mapping $f: X\times Y\rightarrow X$ defined by $f(x,0)=x$, $f\left(x,\dfrac{1}{n}\right)=\Phi_n x$, where $\Phi_n x$ is the pointwise product of Φ_n and x. Since $\Phi_n(t)\neq 0$ for each n, the mapping f, as a function of the first variable, is a homeomorphism on X for every element of Y. To prove the existence of an open cover $\mathfrak{S}$ of X such that $\widetilde{f}: Y\rightarrow C_{\mathfrak{S}}(X;X)$ is continuous, is equivalent to show that for each $x_0\in X$ a neighbourhood A of x_0 exists such that $\widetilde{f}: Y\rightarrow C_u(A;X)$ is continuous at 0 ([2]). Indeed, for fixed $\varepsilon>0$ there exists $m\in N$ such that $|x_0(t)|<\varepsilon$ for $t\in\left[0,\frac{1}{m}\right]$, then for each $x\in X$ with $||x_0-x||<\varepsilon$ and for each $n>m$ it is

$$||\Phi_n x-x|| \leq ||\Phi_n x-\Phi_n x_0|| +$$

$$+ ||\Phi_n x_0-x|| \leq ||x-x_0|| + \sup_{t\in[0,1]} |\Phi_n(t)\, x_0(t)-x(t)| \leq$$

$$\leq \varepsilon + \operatorname{Max}\{ \sup_{t\in\left[0,\frac{1}{n}\right]} |\Phi_n(t)\, x_0(t)-x(t)| \,;\, \sup_{t\in\left[\frac{1}{n},1\right]} |x_0(t)-x(t)|\} \leq$$

$$\leq \varepsilon + \operatorname{Max}\{ \sup_{t\in\left[0,\frac{1}{n}\right]} (|\Phi_n(t)|\, |x_0(t)| + |x(t)|);\, \varepsilon\} \leq$$

$$\leq \varepsilon + \operatorname{Max}\{ \sup_{t\in\left[0,\frac{1}{m}\right]} |x_0(t)| + \sup_{t\in\left[0,\frac{1}{m}\right]} |x(t)|;\, \varepsilon\} \leq 4\varepsilon.$$

The set of mappings $x \rightarrow \frac{x}{\Phi_n}$ is not equicontinuous at $x=0$. For fixed $\varepsilon>0$, suppose there exists $\delta>0$ such that $||x||<\delta$ implies $\left\|\frac{x}{\Phi_n}\right\|<\varepsilon$ for all Φ_n; for each $n\in N$ let $x_n\in X$ be a function such that $||x_n||<\delta$ and $x_n\left(\frac{1}{n+1}\right)=\frac{\delta}{2}$, then

$$\frac{x_n\left(\frac{1}{n+1}\right)}{\Phi_n\left(\frac{1}{n+1}\right)}=\frac{n+1}{2}\,\delta>\varepsilon \text{ for } n>\frac{2\varepsilon}{\delta}-1.$$

The function f^{-1} is not continuous at (0,0). Indeed let $\{x_n\}$ be a sequence in X converging to 0 and such that $x_n\left(\frac{1}{n+1}\right)=\frac{1}{n+1}$, then

$$\left\| f^{-1}\left(x_n, \frac{1}{n}\right) - f^{-1}(0,0)\right\| = \sup_{t\in[0,1]} \left|\frac{x_n(t)}{\Phi_n(t)}\right| \geq \frac{x_n\left(\frac{1}{n+1}\right)}{\Phi_n\left(\frac{1}{n+1}\right)}=1.$$

3. In this section, using the previous results, some regularity and representation theorems for a class of solutions of the translation equation, are proved.

If Y is a topological space, $R \subset Y \times Y$ is called a closed total relation if R is a closed set in $Y \times Y$ and if for each pair $(u, v) \in Y \times Y$ it is $(u, v) \in R$ or $(v, u) \in R$.

For fixed $u_0 \in Y$, it is $R_{u_0} = \{u \in Y\colon (u_0, u) \in R\}$ and $R^{u_0} = \{u \in Y\colon (u, u_0) \in R\}$.

THEOREM 2. *Let X be a uniform space, let Y be a topological space on which a closed total transitive relation $R \subset Y \times Y$ is defined.*

1) *If $f\colon X \times R \to X$ is a solution of the restricted translation equation $f(f(x, u, v), v, w) = f(x, u, w)$ for all $x \in X$, $(u, v), (v, w) \in R$ such that:*

(i) *for each $(u, v) \in R$, $f(\cdot, u, v)$ is a homeomorphism on X;*

(ii) *there exists $u_0 \in Y$ and two open covers $\mathfrak{S}$ and $\mathfrak{S}'$ of X such that the mappings $\phi\colon R_{u_0} \to C_{\mathfrak{S}}(X; X)$, $\phi(u) = f(\cdot, u_0, u)$, and $\psi\colon R^{u_0} \to C_{\mathfrak{S}'}(X; X)$, $\psi(u) = f(\cdot, u, u_0)$, are continuous;*

(iii) *for each $x \in X$ and $V \in \mathcal{U}$ there exists a neighbourhood U of x such that*

$$\Big\{\bigcap_{u \in R_{u_0}} f(V([\phi(u)]^{-1}(x)), u_0, u)\Big\} \cap \Big\{\bigcap_{u \in R^{u_0}} f(V([\psi(u)]^{-1}(x)), u, u_0)\Big\} \supset U;$$

then f is continuous and a continuous function $g\colon X \times Y \to X$ exists such that for each $u \in Y$, $g(\cdot, u)$ is a homeomorphism on X and for each $(u, v) \in R$ it is $f(x, u, v) = g(g^{-1}(x, u), v)$. Furthermore there is a unique representative function g for f which fulfils the condition $g(x, u_0) = x$ for all $x \in X$.

2) *If $g\colon X \times Y \to X$ is a function which fulfils the hypotheses of Theorem 1, then the function $f(x, u, v) = g(g^{-1}(x, u), v)$ is a continuous solution of the restricted translation equation such that $f(\cdot, u, v)$ is a homeomorphism on X for each $(u, v) \in R$.*

REMARK 3. *If* $R = Y \times Y$, then the continuity of ψ is forced by that of ϕ. In effect, the translation equation implies $f(x, u, u) = x$ for all $x \in X$ and all $u \in Y$, consequently

$$[\phi(u) \circ \psi(u)](x) = \phi(u)(f(x, u, u_0)) = f(f(x, u, u_0), u_0, u) = f(x, u, u) = x,$$

i. e. $\psi(u)=[\phi(u)]^{-1}$ and Theorem 1 implies the continuity of ψ.

REMARK 4. The hypothesis (iii) of Theorem 2 may be substituted by the following: (iii′) for each $x\in X$ and $V\in\mathcal{U}$ there exists a neighbourhood U of x such that

$$\{\bigcap_{u\in R_{u_0}} f(V([\phi(u)]^{-1}(x)), u_0, u)\}\cap\{\bigcap_{u\in K} f(V([\psi(u)]^{-1}(x)), u, u_0)\}\supset U,$$

where K is a closed set containing $Y-R_{u_0}$; hence if $R=Y\times Y$ $K=\emptyset$.

PROOF OF THEOREM 2. 1) Let $g\colon X\times Y\to X$ be defined by $g(x,u)=$ $=f(x,u_0,u)$ if $u\in R_{u_0}$, and $g(x,u)=x'$ where x' is the unique point of X such that $f(x',u,u_0)=x$ if $u\in R^{u_0}$. Since R is total, g is defined on all $X\times Y$, furthermore if $u\in R_{u_0}\cap R^{u_0}$ it is $f(f(x,u_0,u),u,u_0)=$ $=f(x,u_0,u_0)=f(f(x,u_0,u_0),u_0,u_0)=x$, in this case it is $x'=f(x,u_0,u)$ and g is well defined. By (i) $g(\cdot,u)$ is a homeomorphism on X for each $u\in Y$. The mapping $(x,u)\to f(x,u_0,u)$ of $X\times R_{u_0}$ into X, by (ii) and (iii), fulfils the hypotheses of Theorem 1, hence it is continuous with its inverse system of homeomorphism on $X\times R_{u_0}$, that is g and g^{-1} are continuous on $X\times R_{u_0}$. The same reasoning on the mapping $(x,u)\to$ $\to f(x,u,u_0)$, gives the continuity of g and g^{-1} on $X\times R^{u_0}$. R being closed and total, R_{u_0} and R^{u_0} are closed and $R_{u_0}\cup R^{u_0}=Y$, then g and g^{-1} are continuous on $X\times Y$.

Now the identity $f(x,u,v)=g(g^{-1}(x,u),v)$, $(u,v)\in R$, will be proved and then the continuity of f on $X\times R$ will be proved.

Let $x\in X$ and $(u,v)\in R$ be given arbitrarily.

If $(v,u_0)\in R$, then $f(f(x,u,v),v,u_0)=f(x,u,u_0)$ and so $f(x,u,v)=$ $=g(f(x,u,u_0),v)=g(g^{-1}(x,u),v)$.

If $(u_0,v)\in R$ and $(u,u_0)\in R$, then $f(x,u,v)=f(f(x,u,u_0),u_0,v)=$ $=g(f(x,u,u_0),v)=g(g^{-1}(x,u),v)$.

If $(u_0,u)\in R$, let y be the point such that $f(y,u_0,u)=x$, then $f(x,u,v)=f(f(y,u_0,u),u,v)=f(y,u_0,v)=g(y,v)=g(g^{-1}(x,u),v)$.

Now suppose that $g(x,u_0)=x$ for all $x\in X$. If $\hat{g}\colon X\times Y\to X$ is a continuous function representing f with the required properties, for each $(u,v)\in R$ it is $g(g^{-1}(x,u),v)=\hat{g}(\hat{g}^{-1}(x,u),v)$, and putting $u=u_0$ it is $g(x,v)=\hat{g}(x,v)$ for each $v\in R_{u_0}$, then $g=\hat{g}$ on $X\times R_{u_0}$. Analogously it is $g=\hat{g}$ on $X\times R^{u_0}$, then $g=\hat{g}$ on $X\times Y$.

2) By Theorem 1 g and g^{-1} are continuous on $X\times Y$, then f is continuous on $X\times R$, $f(\cdot,u,v)$ is a homeomorphism on X for each

$(u, v) \in R$ and $f(f(x, u, v), v, w) = g(g^{-1}(f(x, u, v), v), w) = g(g^{-1} \cdot$ $\cdot (g(g^{-1}(x, u), v), v), w) = g(g^{-1}(x, u), w) = f(x, u, w)$.

COROLLARY 2. *Let X be a uniform space, let Y be a topological space on which a closed total transitive relation $R \subset Y \times Y$ is defined.*

The following statements concerning f are equivalent:

1) *f: $X \times R \to X$ is a solution of the restricted translation equation, such that:*

(i) *the hypothesis* (i) *of Theorem* 2 *holds;*

(ii) *there exists $u_0 \in Y$ such that the functions $(x, u) \to f(x, u_0, u)$ and $(x, u) \to f(x, u, u_0)$ are continuous respectively on $X \times R_{u_0}$ and on $X \times R^{u_0}$;*

(iii) *the hypothesis* (iii) *of Theorem* 2 *holds with the previous u_0;*

2) *there exists a unique continuous function g: $X \times Y \to X$ which fulfils the hypotheses of Corollary* 1, *such that $g(x, u_0) = x$ for all $x \in X$ and $f(x, u, v) = g(g^{-1}(x, u), v)$, for each $(u, v) \in R$.*

COROLLARY 3. *Let X be a uniform space and let Y be a topological space. Let f: $X \times Y \times Y \to X$ be a solution of the unrestricted translation equation such that:*

1) *for each $(u, v) \in Y \times Y$, $f(\cdot, u, v) \in C(X; X)$;*

2) *there exist $u_0 \in Y$ and two open covers $\mathfrak{S}$ and $\mathfrak{S}'$ of X such that ϕ: $Y \to C_{\mathfrak{S}}(X; X)$, $\phi(u) = f(\cdot, u_0, u)$, and ψ: $Y \to C_{\mathfrak{S}'}(X; X)$, $\psi(u) = f(\cdot, u, u_0)$, are continuous.*

Then f is continuous.

PROOF. Set $g(x, u) = f(x, u_0, u)$ and $h(x, u) = f(x, u, u_0)$, by Lemma 1 g and h are continuous functions. For each $(u, v) \in Y \times Y$ it is $f(x, u, v) = f(f(x, u, u_0), u_0, v) = g(f(x, u, u_0), v) = g(h(x, u), v)$, then f is continuous.

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