OSCILLATORY AND ASYMPTOTIC CHARACTERIZATION OF THE SOLUTIONS OF HIGHER ORDER FORCED DIFFERENTIAL EQUATIONS GENERATED BY DEVIATING ARGUMENTS (*)

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SOMMARIO. - *In questo lavoro si classificano tutte le soluzioni dell'equazione differenziale non lineare forzata con argomenti devianti:*

$$x^{(n)}(t) + \sum_{i=1}^{m} f_i(t, x[g_{i1}(t)], x[g_{i2}(t)], \dots, x[g_{ir}(t)]) = \Phi(t)$$

con riguardo al loro comportamento per $t \to \infty$ *e al loro carattere oscillatorio.*

SUMMARY. - *In this paper we classify all solutions of the nonlinear forced differential equation with deviating arguments:*

$$x^{(n)}(t) + \sum_{i=1}^{\infty} f_i(t, x[g_{i1}(t)], x[g_{i2}(t)], \dots, x[g_{ir}(t)]) = \Phi(t)$$

with respect to their behavior as $t \to \infty$ *and to their oscillatory character.*

1. Introduction.

Recently, Ladas-Ladde-Papadakis [3] and Ladas-Lakshmikantham-Papadakis [4] classified all solutions of the following linear retarded

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differential equations of the particular forms

$$x''(t)-\sum_{i=1}^{m} p_i(t)\, x[g_i(t)]=0$$

and

$$x^{(n)}(t)+(-1)^{n+1} p(t)\, x[g(t)]=0$$

with respect to their behavior as $t\to\infty$ and to their oscillatory character. Ladde [5] also generalized the results in [3] to the following nonlinear differential equations with retarded arguments

$$x''(t)-\sum_{i=1}^{m} f_i(t, x(t), x(g_i(t)))=0.$$

More recently, Staikos-Sficas [6] extended and improve the above results to the following nonlinear differential equation with deviating arguments

$$x^{(n)}(t)+f(t, x[g_1(t)], x[g_2(t)], \dots, x[g_m(t)])=0$$

where

$$\lim_{t\to\infty} g_j(t)=\infty,\quad j=1,2,\dots,m.$$

In their discussions they only treated with the unforced differential equations. In the present paper, we extend Staikos-Sficas's results to the following more general nonlinear forced differential equation with deviating arguments:

$$(*)\qquad x^{(n)}(t)+\sum_{i=1}^{m} f_i(t, x[g_{i1}(t)], x[g_{i2}(t)], \dots, x[g_{ir}(t)])=\Phi(t)$$

where the following conditions are always assumed to hold:

(i) $f_i\in C[[t_0,\infty)\times R^r, R],\quad i=1,2,\dots,m,$

(ii) $\Phi\in C[[t_0,\infty), R],$

(iii) $g_{ij}\in C[[t_0,\infty), R],\quad \lim_{t\to\infty} g_{ij}(t)=\infty,$

$$i=1,2,\dots,m;\quad j=1,2,\dots,r.$$

The oscillatory character is considered in the usual sence, i. e. a continuous real-valued function which is defined for all large t is called *oscillatory* if it has no last zero, and otherwise it is called *nonoscillatory*. Let S denote the set of all solutions of equation (*) and $S^{\sim}, S^0, S_1^{+\infty}, S_2^{+\infty}, S_1^{-\infty}, S_2^{-\infty}, S^{+\infty}, S^{-\infty}$ subsets of S defined as follows.

$$S^{\sim}=\{x(t)\in S:\ x(t) \text{ is oscillatory}\}.$$

$$S^0=\{x(t)\in S:\ x(t) \text{ is nonoscillatory and } x^{(j)}(t)\to 0 \text{ as } t\to\infty,\ j=0,1,\dots,n-1\}.$$

$S_1^{+\infty}=\{x(t)\in S:$ there exists an integer k, $0\le k\le n-1$, with $n+k$ odd and such that

(C_1) $\lim_{t\to\infty} x^{(j)}(t)=\infty$ for $j=0,1,\dots,k$,

(C_2) if $k\le n-2$, then $\lim_{t\to\infty} x^{(k+1)}(t)$ exists in R,

(C_3) if $k\le n-3$, then for $j=k+2,\dots,n-1$

$$\lim_{t\to\infty} x^{(j)}(t)=0,\ x^{(j)}(t)\neq 0,\ x^{(j)}(t)\,x^{(j+1)}(t)\le 0 \text{ for all large } t\}.$$

$S_2^{+\infty}=\{x(t)\in S:\ x(t)$ posses properties (C_1) - (C_3) for some integer k, $0\le k\le n-1$, with $n+k$ even$\}$

$S_1^{-\infty}=\{x(t)\in S:\ -x(t)$ possess properties (C_1) - (C_3) for some integer k, $0\le k\le n-1$, with $n+k$ odd$\}$.

$S_2^{-\infty}=\{x(t)\in S:\ -x(t)$ possess properties (C_1) - (C_3) for some integer k, $0\le k\le n-1$, with $n+k$ even$\}$.

$$S^{+\infty}=S_1^{+\infty}\cup S_2^{+\infty}.$$

$$S^{-\infty}=S_1^{-\infty}\cup S_2^{-\infty}.$$

We now, introduce, the main conditions which will be used in the classification of the solutions of (*).

(α) *There exists an oscillatory function* $p(t)$ *such that*

$$p^{(n)}(t)=\Phi(t),\quad \lim_{t\to\infty} p^{(j)}(t)=0,\ j=0,1,\dots,n-1.$$

(β) *For every* $t \geq t_0$,

$$f_i(t, 0, \dots, 0) = 0, \quad i = 1, 2, \dots, m.$$

(γ) *For every constant* $c \neq 0$,

$$\sum_{i=1}^{\infty} \int^{\infty} t^{n-1} |f_i(t, c, \dots, c)| \, dt = \infty .$$

(δ) *For every constant* $c \neq 0$,

$$\sum_{i=1}^{\infty} \int^{\infty} |f_i(t, c g_{i1}(t), \dots, c g_{ir}(t))| \, dt = \infty .$$

Using condition (α), (*) may be written as

$$(**) \quad y^{(n)}(t) + \sum_{i=1}^{m} f_i(t, y[g_{i1}(t)] + p[g_{i1}(t)], \dots, y[g_{ir}(t)] + p[g_{ir}(t)]) = 0$$

where $y(t) = x(t) - p(t)$.

In order to obtain our results we need the following three lemmas.

LEMMA 1. *If* $x(t)$ *is a positive (negative) solution of* (*) *for* $t \geq t_0$, *then there is a* $t_1 \geq t_0$ *for which* $y(t) = x(t) - p(t)$ *is a solution of* (**) *for* $t \geq t_1$, *also there is an integer* l *with* $0 \leq l \leq n-1$, $n+l$ *odd if* $y^{(n)}(t) \leq 0$, $n+l$ *even if* $y^{(n)}(t) \geq 0$ *and such that for every* $t \geq t_1$

$$\text{(A)} \quad \begin{cases} y^{(\nu)}(t) > 0 \ (y^{(\nu)}(t) < 0) \ \textit{for} \ \nu = 0, 1, \dots, l, \\ (-1)^{\nu+1} y^{(\nu)}(t) > 0 \ ((-1)^{\nu+1} y^{(\nu)}(t) < 0) \ \textit{for} \ \nu = l+1, l+2, \dots, n, \end{cases}$$

$$\text{(B)} \quad x^{(\nu)}(t) \, y^{(\nu)}(t) > 0 \ \textit{for} \ \nu = 0, 1, \dots, n.$$

PROOF. Since (A) is Lemma 1 of [1], we only prove (B). If $x^{(\nu)}(t) < 0$ then $y^{(\nu)}(t) < -p^{(\nu)}(t)$ for $\nu = 0, 1, \dots, n$. Since $y^{(\nu)}(t)$ is positive or negative, $p^{(\nu)}(t)$ is negative or positive respectively, a contradiction to the oscillatory character of $p^{(\nu)}(t)$ for $\nu = 0, 1, \dots, n$.

LEMMA 2 (Staikos-Sficas). *If* $y(t)$ *is as in Lemma* 1 *and for some* $j = 0, 1, \dots, n-2$

$$\lim_{t \to \infty} y^{(j)}(t) = c, \ c \in R$$

then

$$\lim_{t \to \infty} y^{(j+1)}(t) = 0.$$

LEMMA 3 (Staikos-Sficas). *Consider the linear differential equation*

(1) $$z'-\frac{a}{t}z+\frac{h(t)}{t}=0,$$

where a is a positive integer and $h(t)$ is continuous on $[T, \infty)$ where $T>0$. Let $u(t)$ be the solution of (1) on $[T, \infty)$ satisfying $u(T)=0$. If $\lim_{t\to\infty}|h(t)|=h^*$ exists in the extended real line R^* then $\lim_{t\to\infty}|u(t)|=u^*$ exists in R^*. In particular $h^*=\infty$ implies $u^*=\infty$.

2. Theorems.

The monotonicity of f_i, $i=1,2,\ldots,m$ are considered with respect to the order in R^r defined as follows:

$$X_i=(x_{i1},\ldots,x_{ir})\le Y_i=(y_{i1},\ldots,y_{ir})\Leftrightarrow$$

$$\Leftrightarrow x_{ij}\le y_{ij} \text{ for } i=1,2,\ldots,m,\ j=1,2,\ldots,r.$$

THEOREM 1. *Let the conditions (α), (β) and (γ) hold. If, for each $t\ge t_0$, $f_i(t, Y_i)$, $i=1,2,\ldots,m$, are nonincreasing (respectively, nondecreasing) with respect to y_i, $i=1,2,\ldots,m$, then for n even (respectively, odd)*

$$S=S^{\sim}\cup S^0\cup S^{+\infty}\cup S^{-\infty}.$$

while for n odd (respectively, even)

$$S=S^{\sim}\cup S^{+\infty}\cup S^{-\infty}.$$

In particular, for n odd (respectively, even) all bounded solutions of equation (*) are oscillatory, while for n even (respectively, odd) all bounded solutions of equation (*) are either oscillatory or tending monotonically to zero as $t\to\infty$ together with their first $n-1$ derivatives.

PROOF. Let $x(t)\in S-S^{\sim}$. Let

(2) $$y(t)=x(t)-p(t).$$

From (*), (α) and (2) we have

(3) $$y^{(n)}(t)+\sum_{i=1}^{m} f_i(t, x[g_{i1}(t)], x[g_{i2}(t)],\ldots, x[g_{ir}(r)])=0.$$

By the monotonicity of f_i, $i=1, 2, \dots, m$, we see that $y^{(n)}(t)$ is of constant sign for all large t. This and Lemma 1 implys that all derivatives $y^{(j)}(t)$, $j=0, 1, \dots, n-1$, are also of constant sign for all large t. Therefore, $\lim_{t\to\infty} y^{(j)}(t)$ exists in the extended real line R^* for every $j=0, 1, \dots, n-1$.

Suppose that $\lim_{t\to\infty} x(t) \neq 0$, then there exist $T \geq t_0$ and $M>0$ such that for every $t \geq T$ and for $i=1, 2, \dots, m$, $j=1, 2, \dots, r$

$$|x[g_{ij}(t)]| \geq M. \tag{4}$$

Let

$$q_i(t) = \int_T^t s^i y^{(i+1)}(s)\, ds$$

then we obtain

$$q_i(t) = tq'_{i-1}(\mathrm{t}) - T^i y^{(i)}(T) - iq_{i-1}(t).$$

Therefore, $q_{i-1}(t)$ is a solution of the differential equation

$$z' - \frac{i}{t} z + \frac{h_i(t)}{t} = 0 \tag{5}$$

where $h_i(t) = -T^i y^{(i)}(t) - q_i(t)$, $i=0, 1, \dots, n-1$. We see easily that this solution satisfies the initial condition $q_{i-1}(T)=0$.

Since

$$q_{n-1}(t) = \int_T^t s^{n-1} y^{(n)}(s)\, ds =$$

$$- \sum_{i=1}^{m} \int_T^t s^{n-1} f_i(s, x[g_{i1}(s)], \dots, x[g_{ir}(s)])\, ds,$$

from (3), (4) and conditions (α), (β) and monotonicity of f_i, $i=1,2,\dots,m$, we obtain

$$|q_{n-1}(t)| \geq \begin{cases} \sum_{i=1}^{m} \int_T^t s^{n-1} |f_i(s, M, \dots, M)|\, ds, & \text{if } x \text{ is eventually positive,} \\ \sum_{i=1}^{m} \int_T^t s^{n-1} |f_i(s, -M, \dots, -M)|\, ds & \text{if } x \text{ is eventually negative.} \end{cases}$$

Thus, by condition (γ),

$$\lim_{t\to\infty} |q_{n-1}(t)| = \infty$$

and consequently

$$\lim_{t\to\infty} |h_{n-1}(t)| = \infty.$$

Applying Lemma 3 for the differential equation

$$z' - \frac{n-1}{t} z + \frac{h_{n-1}(t)}{t} = 0$$

we obtain

$$\lim_{t\to\infty} q_{n-2}(t) = \pm\infty$$

and consequently

$$\lim_{t\to\infty} |h_{n-1}(t)| = \infty.$$

Therefore, we can apply again Lemma 3 for the differential equation

$$z' - \frac{n-2}{t} z + \frac{h_{n-2}(t)}{t} = 0,$$

to obtain that

$$\lim_{t\to\infty} q_{n-3}(t) = \pm\infty.$$

Following the same procedure, we obtain finally

$$\lim_{t\to\infty} q_0(t) = \pm\infty,$$

which gives that

$$\lim_{t\to\infty} y(t) = \pm\infty,$$

i. e.

$$\lim_{t\to\infty} x(t) = \pm\infty,$$

since for every $t \geq T$, $y(t) = y(T) + q_0(t)$.

Hence the only possible cases for a nonoscillatory solution $x(t)$ of equation (*) are the following ones:

CASE 1^0. $\lim_{t\to\infty} y(t) = 0$, i. e. $\lim_{t\to\infty} x(t) = 0$.

From Lemma 2, we have that for every $j = 1, 2, \dots, n-1$

$$\lim_{t\to\infty} y^{(j)}(t) = 0$$

and since $y^{(j)}(t)$, $j = 0, 1, \dots, n-1$ are eventually monotone, and $p^{(j)}(t) \to 0$ as $t \to \infty$ for $j=0, 1, \dots, n-1$, we have $x^{(j)}(t)$, $j=0, 1, \dots$ $\dots, n-1$ are eventually monotone. Hence $x(t) \in S^0$.

CASE 2^0. $\lim_{t \to \infty} y(t) = \infty$, i. e. $\lim_{t \to \infty} x(t) = \infty$.

Let k be the greatest integer with $0 \leq k \leq n-1$ and for every $j=0, 1, \dots, k$,

$$\lim_{t \to \infty} y^{(j)}(t) = \infty, \text{ i. e. } \lim_{t \to \infty} x^{(j)}(t) = \infty.$$

Obviously, if $k \leq n-2$, then

$$\lim_{t \to \infty} y^{(k+1)}(t), \text{ i. e. } \lim_{t \to \infty} x^{(k+1)}(t)$$

exists in R and they are nonnegative. If $k \leq n-3$, then from Lemma 2, for every $j=k+2, \dots, n-1$,

$$\lim_{t \to \infty} y^{(j)}(t) = 0, \text{ i. e. } \lim_{t \to \infty} x^{(j)}(t) = 0$$

and consequently it is easy to see that for all large t

$$x^{(j)}(t)\, x^{(j+1)}(t) \leq 0.$$

Finally in order to derive that for every $j=k+2, \dots, n-1$, $x^{(j)}(t) \neq 0$ for all large t, it is enough to verify that $x^{(n)}(t)$ is not identically zero for all large t. To do this, we see that, by (3), the monotonicity of f_i, $i=1, 2, \dots, m$ and conditions (α), (β), for $t \geq T$

$$|y^{(n)}(t)| \geq \Big| \sum_{i=1}^{m} f_i(t, M, \dots, M) \Big| \geq 0.$$

Therefore, for all large t

$$y^{(n)}(t) \not\equiv 0.$$

Thus, $x(t)$ possess properties (C_1), (C_2) and (C_3), which means that $x(t) \in S^{+\infty}$.

CASE 3^0. $\lim_{t \to \infty} y(t) = -\infty$, i. e. $\lim_{t \to \infty} x(t) = -\infty$.

Let k, be the greatest integer with $0 \leq k \leq n-1$ and for every $j=0, 1, \dots, k$

$$\lim_{t\to\infty} y^{(j)}(t) = \lim_{t\to\infty} x^{(j)}(t) = -\infty .$$

Similar to the Case 2^0 we can prove that $-x(t)$ posses properties (C_1), (C_2) and (C_3), which means that $x(t) \in S^{-\infty}$. Therefore, we have derived the

$$S = S^{\sim} \cup S^0 \cup S^{+\infty} \cup S^{-\infty}.$$

In order to complete the proof of our theorem, we must verify that

$$S^0 \neq \Phi \text{ implies } n \text{ even (respectively, odd).}$$

In fact, if $x(t) \in S^0$, then by Lemma 1, $y(t) = x(t) - p(t)$ is a solution of (**). Let $x(t) > 0$, then $y(t) > 0$. Since $y(t)$ is bounded, Lemma 1 implies $(-1)^{j+1} y^{(j)}(t) > 0$ for $j = 0, 1, \dots, n$.
i. e.

$$-y^{(j)}(t)\, y^{(j+1)}(t) \geq 0, \text{ for } j = 0, 1, \dots, n-1.$$

Thus

$$(-1)^n y(t)\, (y'(t) \dots y^{(n-1)}(t))^2\, y^{(n)}(t) \geq 0$$

implies $(-1)^n y(t)\, y^{(n)}(t) \geq 0$. Since $y^{(n)}(t) \neq 0$ for all large t and $y(t)\, y^{(n)}(t) \geq 0$ (respectively, ≤ 0), we must have $(-1)^n = 1$ (respectively, $(-1)^n = -1$), which means that n is even (respectively, odd).

THEOREM 2. *Let the conditions* (α), (β), (γ) *and* (δ) *hold. If, for each* $t \geq t_0$, $f_i(t, Y_i)$, $i = 1, 2, \dots, m$, *are nonincreasing with respect to* Y_i, $i = 1, 2, \dots, m$, *then for n even*

$$S = S^{\sim} \cup S^0 \cup S_1^{+\infty} \cup S_1^{-\infty}$$

while for n odd, $n > 1$,

$$S = S^{\sim} \cup S_1^{+\infty} \cup S_1^{-\infty} .$$

PROOF. We assume that $S_2^{+\infty} \neq \Phi$ and we consider a solution $x(t) \in S_2^{+\infty}$ as well as the associated integer k. Since $n+k$ is even, we must always have $k \leq n-2$. Using the present conditions and arguing as in the proof of Theorem 1, one again obtains that (4) and all derivatives $y^{(j)}(t)$, hence $x^{(j)}(t)$, $j = 0, 1, \dots, n-1$, are of consant sign for all large t.

If for some integer d, $1 \leq d \leq n-1$, and for all large t

(6) $$x^{(d)}(t) > 0 \text{ and } x^{(d+1)}(t) \geq 0$$

then, by choosing $T_0 \geq t_0$ so that $x^{(d)}(T_0) > 0$ and using Taylor's theorem, we have for all large t

$$x[g_{ij}(t)] \geq \sum_{s=0}^{r} \frac{x^{(s)}(T_0)}{s!}[g_{ij}(t) - T_0]^s, \quad i=1,2,\dots,m; \quad j=1,2,\dots,r.$$

Hence there exists $T \geq T_0$ and $M > 0$ such that for $i=1,2,\dots,m$; $j=1,2,\dots,r$ and for every $t \geq T$

$$x[g_{ij}(t)] \geq M g_{ij}(t). \tag{7}$$

From (4), (7), conditions (α), (β), and the monotonicity of f_i, $i=1,2,\dots,m$, we have

$$y^{(n-1)}(t) = y^{(n-1)}(T) - \sum_{i=1}^{m} \int_T^t f_i(s, x[g_{i1}(s)], \dots, x[g_{ir}(s)])\, ds \geq$$

$$\geq y^{(n-1)}(T) - \sum_{i=1}^{m} \int_T^t f_i(s, Mg_{i1}(s), \dots, Mg_{ir}(s))\, ds =$$

$$= y^{(n-1)}(T) + \sum_{i=1}^{m} \int_T^t |f_i(s, Mg_{i1}(s), \dots, Mg_{ir}(s))|\, ds$$

and consequently, by condition (δ),

$$\lim_{t \to \infty} y^{(n-1)}(t) = \infty, \quad \text{i. e.} \quad \lim_{t \to \infty} x^{(n-1)}(t) = \infty,$$

which contradicts that $k \leq n-2$. Thus, (6) is impossible for any integer d with $1 \leq d \leq n-1$.

Since (6) is satisfied for $d=k$, we must have $k=0$ and, in addition, n even. Thus, since by the monotonicity of f_i, $i=1,2,\dots,m$, and condition (β), for all large t

$$y^{(n)}(t) \geq 0, \quad \text{i. e.} \quad x^{(n)}(t) \geq 0,$$

from condition (C_3), we obtain for all large t,

$$y''(t) \geq 0, \quad \text{i. e.} \quad x''(t) \geq 0.$$

Hence, (6) is satisfied for $d=1$, a contradiction. It is proved, now, that $S_2^{+\infty}=\Phi$, which means that $S^{+\infty}=S_1^{+\infty}$. Similarly, we can prove that $S^{-\infty}=S_1^{-\infty}$.

THEOREM 3. *Let the conditions* (α), (β), (γ) *and* (δ) *hold. If, for each* $t\geq t_0$, $f_i(t, Y_i)$, $i=1,2,\dots,m$ *are nondecreasing with respect to* Y_i, $i=1,2,\dots,m$, *then for* n *even*

$$S=S^{\sim}\cup S_2^{+\infty}\cup S_2^{-\infty}$$

while for n *odd*, $n>1$

$$S=S^{\sim}\cup S^0.$$

PROOF. Using the present conditions and arguing as in the proof of Theorem 1, one again obtains (4) and all derivatives $y^{(j)}(t)$, hence $x^{(j)}(t)$, $j=0,1,\dots,n-1$ are of constant sign for all large t. Let $x(t)\in S^{+\infty}$ and k be the associated integer. If $k\geq 1$, then, by the mean-value theorem, for each $i=1,2,\dots,m$; $j=1,2,\dots,r$, and for all large t,

$$x[g_{ij}(t)]\geq x(T_0)+x'(T_0)[g_{ij}(t)-T_0] \tag{8}$$

where T_0 is chosen so that $x'(T_0)>0$. So, there exist $T>T_0$ and $M>0$ such that for $i=1,2,\dots,m$; $j=1,2,\dots,r$ and for every $t\geq T$,

$$x[g_{ij}(t)]\geq Mg_{ij}(t). \tag{9}$$

From this, by virtue of the monotonicity of f_i, $i=1,2,\dots,m$, and conditions (α), (β), we obtain

$$y^{(n-1)}(t)=y^{(n-1)}(T)-\sum_{i=1}^{m}\int_T^t f_i(s, x[g_{i1}(s)],\dots,x[g_{ir}(s)])\,ds\leq$$

$$\leq y^{(n-1)}(T)-\sum_{i=1}^{m}\int_T^t f_i(s, Mg_{i1}(s),\dots,Mg_{ir}(s))\,ds=$$

$$=y^{(n-1)}(T)-\sum_{i=1}^{m}\int_T^t |f_i(s, Mg_{i1}(s),\dots,Mg_{ir}(s))|\,ds$$

and consequently, by condition (δ), the contradiction

$$\lim_{t \to \infty} y^{(n-1)}(t) = -\infty. \tag{10}$$

Thus, k must be zero, which implies

$$y(t) \in S_2^{+\infty}, \text{ if } n \text{ is even,}$$

$$y(t) \in S_1^{+\infty}, \text{ if } n \text{ is odd.}$$

Since, by the monotonicity of f_i, $i=1,2,\dots,m$, and conditions (α), (β), $x^{(n)}(t) \le 0$ for all large t, in the case of odd n,

$$x''(t) > 0 \text{ for all large } t.$$

Moreover, $x(t)$ and $x'(t)$ are evenually positive and hence (8), (9) and the contradiction (10) can again be derived in the considered case of odd n.

Therefore, for n even $S^{+\infty} = S_2^{+\infty}$, while for n odd $S^{+\infty} = \Phi$. Similarly, we can prove that for n even $S^{\ \infty} = S_2^{-\infty}$, while for n odd $S^{-\infty} = \Phi$. This proves the theorem, since, by Theorem 1, the solutions of equation (*) admit the decomposition

$$S = S^{\sim} \cup S^0 \cup S^{+\infty} \cup S^{-\infty}, \text{ if } n \text{ is odd,}$$

and

$$S = S^{\sim} \cup S^{+\infty} \cup S^{-\infty}, \text{ if } n \text{ is even.}$$

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