ON PROBABILITY SOUSLIN MEASURES (*)

by CORRADO SCARAVELLI and GIANCARLO TEPPATI (a Parma) (**)

SOMMARIO. - *In questa breve nota viene introdotta una caratterizzazione di misure di probabilità di Souslin, e viene provata, mediante il teorema di Choquet, Bishop e de Leeuw, la esistenza di misure di probabilità di Souslin su spazi localmente convessi.*

SUMMARY. - *In this short paper we give a characterization of probability Souslin measures, and by means of the Choquet, Bishop and de Leeuw theorem, we show that probability Souslin measures exist on locally convex spaces.*

1. Introduction.

In probability theory one is frequently confronted with the problem of determining the entire class of events in a given universe. Usually the difficulty is bypassed by choosing a priori Borel subsets: whether the choice is exhaustive is basically a matter of proper interpretation of the rest of the theory. But there are examples in which one can be interested to an extension of this formalism. For instance, classical statistical mechanics could perhaps be more comprehensive if one would allow for more general subsets than the Borel subsets (with the obvious requirement that the new class includes the class of all the Borel subsets).

The present note arise out of an attempt to see whether this point of view can be formalized. The class we propose for the use

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(**)Indirizzo degli Autori: Istituto di Matematica dell'Università, Via Università 12, 43100 Parma.

above sketched is the Souslin class of subsets, which has recently drawn some interest.

We propose a certain amount of definitions, partly already known in the literature, partly, as far as we know, not even introduced, expecially in probability theory. Then we prove the existence of which we call a probability Souslin measure on a locally convex space: this result is given by means of a well-known theorem on probability measures on LCS [Choquet — Bishop — de Leeuw (briefly C. B. de L.) theorem] which can be easily applied to our scheme.

2. Basic definitions and properties.

In this Section we recall some well known definitions and notions, which will be useful in the rest of this Note.

First we point out that on any class $\mathcal{C}$ of sets the Souslin operation, defined as follows, gives Souslin $\mathcal{C}$-sets:

$$C = \bigcup_{\bar{i} \in \bar{I}} \bigcap_{n=1}^{\infty} C_{\bar{i}/n}$$

where $\bar{I}$ is the system of all infinite vectors $\bar{i}$ with components $i_1, i_2, \dots,$ which are positive integers, $\bar{i}/n$ is the restriction of the vector $\bar{i}$ to n, and $C_{\bar{i}/n} \in \mathcal{C}$ for any $n \geq 1$ (1).

Thus we recall the following definition in which the notion of Souslin measure is introduced.

DEFINITION 1. Let X be a non-void set, and let $\mathcal{C}$ be a class of subsets of X, containing ϕ. The *Souslin $\mathcal{C}$-sets* in X are the sets obtained by the application of the Souslin operation to sets of $\mathcal{C}$. We agree that ϕ is a Souslin $\mathcal{C}$-set. Whenever $\mathcal{C}$ is the class of the closed subsets of X, we speak simply of *Souslin subsets*. We call Σ_X the class of all the Souslin sets of X, and $\bar{\Sigma}_X$ the σ-ring generated by Σ_X. A *Souslin measure* will be a measure on $\bar{\Sigma}_X$.

We refer to the literature for the elementary properties of the Souslin operation and Souslin sets (see [2], [4]). We quote the following facts, which are interesting from the point of view of the

(1) The notation here used implies that $C_{\bar{i}/n}$, although it depends on the first n components, is independent of all the remaining components.

Introduction: i) Borel sets are also Souslin sets (and not conversely); ii) certain subsets of Souslin sets are compact; iii) in a separable metric space, any Souslin subset can be obtained from a system of suitably nested closed sets, with diameters tending to zero. Points ii) and iii) are especially important in classical mechanics.

Now we are faced with the following problem: is the class of the Souslin measures non void ? The answer will be yes, at least whenever we choose locally convex spaces as a framework. The reason for this will be apparent in the next Section.

3. Existence of Souslin measures in convex compact sets.

Let now X a finite-dimensional complex vector space.

DEFINITION 2. Let C be a convex set in X. An element $e \in C$ is called an *extreme point* of C if the set $C - \{e\}$ is convex. Let we call $ex\, C$ the set of all extreme points of C.

The simplest result we will quote is the classical theorem of Minkowski, which is as follows. Let C be a convex compact set in a finite dimensional vector space and let $x \in C$.

Then extreme points $e_1, \dots, e_k$ and non-negative numbers $t_1, \dots, t_k$ exist such that $x = \sum_i t_i e_i$ and $\sum t_i = 1$. This result can be rewrited in a more suitable manner for our purpose. Let in fact C as above, and, for any point $c \in C$, let ε_c be the Souslin point mass at c, namely the Souslin measure defined as follows: $\varepsilon_c(D) = 1$ for any Souslin subset D of C which contains c, $\varepsilon_c(D) = 0$ otherwise. Thus, in the above formulation, let we abbreviate ε_{e_i} by ε_i and let we pose $\mu_s = \sum t_i \varepsilon_i$: μ_s turns out to be a non-negative regular probability Souslin measure on C. Let now f be any continuous linear functional on X; we can write for any $c \in C$ $f(c) = \sum t_i f(e_i) = \int_C f \, d\mu_s$. This is the motivation for the following Definition.

DEFINITION 3. Let X be a complex vector space, and let C be a non-void compact subset of X. Let f be any continuous linear functional on X, and let μ_s be a non-negative regular probability Souslin measure on C. We say that a point $x \in X$ is the barycenter of μ_s if $f(x) = \int_C f \, d\mu_s$ for any continuous linear functional on X. We say that x *represents*

μ whenever there it is at most one point which is the barycenter of μ_s. A regular non-negative Souslin measure μ_s on C is *supported* by a non-void subset D of C if $\mu_s(C-D)=0$.

The notions above quoted are well-known and can be easily found in the literature (see [5]). Anyway, they are necessary for illustrate the result we use in the sequel. The setting is the following: let C denote a convex compact subset of the locally convex complex space X, and let we denote by c an arbitrary element of C. Choquet (see [3]) has shown that a probability measure exists on C which represents x and is supported by $ex\, C$, whenever C is metrizable and μ is Borel; Bishop an de Leeuw (see [1]) have proved that if μ is more general than Borel, without any further assumption on C, the result is still valid.

The preceding result allows us to prove the existence of probability Souslin measures. At this purpose, we need of a definition in which we introduce a new σ-ring.

DEFINITION 4. Let A be a non-void set and B a non-void subset of A. We call $\tilde{\mathcal{C}}(A, B)$ the class of all the closed subsets of A which contain B, and the corresponding Souslin subsets will called $\tilde{\mathcal{C}}(A, B)$--Souslin sets. Analogously, we call $\tilde{\Sigma}(A, B)$ the σ-ring generated by the $\tilde{\mathcal{C}}(A, B)$-Souslin sets.

Thus we can state the following Lemma.

LEMMA. *Under the assumptions of Definitions 3, 4, let D be a non-void subset of C, and let μ_s be a non-negative regular Souslin measure on C, supported by any closed subset which contains D. Then μ_s is supported by any member of the σ-ring $\tilde{\Sigma}(C, D)$.*

PROOF. All that we need to prove is that for any closed D_1, D_2 which contains D, $\mu_s(C-(D_1 \cap D_2))=\mu_s(C-(D_1 \cup D_2))=0$. This follows from

$$\mu_s(C-(D_1 \cap D_2))=\mu_s(C-D_1)+\mu_s(C-D_2)=0$$

$$\mu_s(C-(D_1 \cup D_2))=\mu((C-D_1) \cap (C-D_2)) \leq \mu(C-D_1)=0.$$

In this way we are prepared to the following definition, in which we introduce the concept of extreme Souslin measure.

DEFINITION 5. Let X be a locally convex space, and let C be a convex compact non-void subset of X. Let μ_s be a non-negative regular measure, supported by any closed set which contains $ex\,C$, hence by $\tilde{\Sigma}(C, ex\,C)$. We call μ_s an *extreme Souslin measure* on C.

With this concept, we can give the main result of this note.

PROPOSITION. *Let X be a locally convex space, and let C be a compact convex non-void subset of X. Let $c \in C$: then a probability estreme Souslin measure on C exists which represents c.*

PROOF. The proof proceeds along several steps, which follows literally the ones of the standard proofs of C. B. de L. theorem that can be found in the literature (see [5]). We quote here only the principal features. First the non negative measures on C are partially ordered in the following way: $\mu_1 > \mu_2$ if $\mu_1(f) \geq \mu_2(f)$ for any f in C. Then it is proved that given μ_1, a maximal measure μ_2 in C exists such that $\mu_2 > \mu_1$. Subsequently, let μ a maximal measure such that $\mu > \varepsilon_c$, where $c \in C$: than $\mu(f) = \mu(\bar{f})$, where $\bar{f} = \{\inf h(x), h$ in the set of the continuous affine functions on X, $h \geq f\}$; then it is shown that any maximal measure μ vanishes on the Baire sets which are disjoint from $ex\,C$: in particular $\mu(D) = 0$ if D is a compact G_δ set disjoint from $ex\,C$. The last step is to show that a maximal measure is supported by any closed set which contains $ex\,C$: at this point the preceding Lemma applies, and the conclusion follows.

On the same spirit, the following corollary can be shown.

COROLLARY. *Under the assumption of the proposition, and with the notation of Definitions 4 and 5, for each point $c \in C$ there exists a non negative extreme probability Souslin measures ν_s on $\tilde{\Sigma}(C, ex\,C)$ such that ν_s represent c and $\nu_s(ex\,C) = 1$.*

PROOF. Let we denote by ν_s the measure in the Proposition. We extend μ_s to a non negative ν_s in the following way: for any $S \in \tilde{\Sigma}$, we pose $\nu_s(ex\,S) = \mu_s(S)$: then $\nu_s(ex\,C) = \mu_s(C) = 1$.

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