MODULES WITH IRREDUNDANT SETS OF COGENERATORS (*)

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SOMMARIO. - *In questa nota si provano tre teoremi. Il primo caratterizza gli R-moduli che possiedono un insieme minimale di cogeneratori come estensioni di moduli semisemplici. Il secondo fornisce un procedimento per ottenere un insieme minimale di cogeneratori per i moduli suddetti. Il terzo, qualora l'anello R sia commutativo e noetheriano, dà una decomposizione diretta di R-moduli che siano estensioni essenziali del loro «socle».*

SUMMARY. - *In this paper we prove three theorems. The first theorem characterizes R-modules that possess an irredundant set of cogenerators as the essential extensions of semi-simple modules. The second theorem provides a process for exhibiting an irredundant set of cogenerators for such modules. If the ring R is commutative Noetherian, then theorem 3 provides a direct sum decomposition of R-modules that are essential extensions of their socle.*

1. Introduction.

All modules that are considered are assumed to be unitary left modules over a ring R with $1 \neq 0$. A system of generators of an R-module M can be characterized as a non-empty subset G of M such that any homomorphism $f\colon N \to M$ with $G \subseteq \operatorname{Im} f$ is an epimorphism. Dualizing this concept, we call a non-empty subset C of an R-module M *a set of cogenerators of* M if any homomorphism $f\colon M \to N$ with $C \cap \operatorname{Ker} f = \varnothing$ is a monomorphism. Since every non-zero R-module M possesses a set of cogenerators, namely $M - \{0\}$, it is of interest to

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determine all R-modules M that possess an *irredundant* set of cogenerators, that is a set of cogenerators C such that every proper subset of C fails to be a set of cogenerators of M.

An R-module is *finitely cogenerated* if it possesses a finite set of cogenerators. Finitely cogenerated is not dual to finitely generated. The correct dual is the Vamos notion of finitely embedded [3]. A finitely cogenerated R-module is finitely embedded, but not conversely. However, the two notions are equivalent in the important category of abelian groups. They are not equivalent in the category of vector spaces over an infinite field. But here the notion of finitely embedded coincides with that of finitely generated. In case the basic field is finite all three notions are equivalent.

2. Notations and Definitions.

The injective envelope of an R-module M is denoted by $E(M)$ and the socle by $S(M)$. The set of simple submodules of $S(M)$ is partitioned as follows: the simple submodules S_1 and S_2 belong to the same set of the partition if $S_1 \cong S_2$. A *homogeneous component of* $S(M)$ is the sum of the simple submodules in a set of the partition. This decomposes $S(M)$ into a direct sum $\bigoplus_{i \in I} T_i$ of its homogeneous components T_i.

Let A be a submodule of an R-module M, and let B be a nonempty subset of M. The *carrier* A: B of B into A is the left ideal of R consisting of all elements r such that $rB \subseteq A$. If $x \in M$, we write $0(x)$ for 0: x. If S is a right ideal of R, we let $A:_M S$ denote the submodule of M consisting of all elements x such that $Sx \subseteq A$.

A commutative Noetherian ring R is called a *local ring* if the set of non-units of R forms an ideal. It follows that a local ring has only one maximal ideal.

3. Modules with Essential Socle.

THEOREM 1. *An R-module M possesses an irredundant set of cogenerators if and only if it is an essential extension of its socle $S(M)$.*

PROOF. Suppose first that $C=(c_i)_{i \in I}$ is an irredundant set of cogenerators of M. Then every non-zero submodule of M must intersect C. In particular M is an essential extension of the submodule $\sum_{i \in I} Rc_i$, generated by C and $S(M) \subseteqq \sum_{i \in I} Rc_i$. Further $c_i \notin Rc_j$ for $i \neq j$, for otherwise every homomorphism f with domain M such that $f(c_i) \neq 0$ implies $f(c_j) \neq 0$, and consequently $C - \{c_j\}$ is a set of cogenerators. It follows that every submodule Rc_i of M is simple, and hence $S(M) = \sum_{i \in I} Rc_i$.

Conversely, suppose that M is an essential extension of its socle $S(M)$, and let $\{Rc_i\}_{i \in I}$ be the set of simple submodules of M. Then $C = (c_i)_{i \in I}$ is a set of cogenerators of M. To prove this, let f be a homomorphism with domain M such that $f(c_i) \neq 0$ for every $i \in I$, and suppose $\operatorname{Ker} f \neq 0$. Let $(c_j)_{i \in J}$ be a subset of C such that the sum $\sum_{j \in J} Rc_j$ is direct and $S(M) = \sum_{j \in J} Rc_j$. Consider a non-zero element $x = r_1 c_{j_1} + r_2 c_{j_2} + \ldots + r_n c_{j_n} \in S(M) \cap \operatorname{Ker} f$ with a minimal number n of non-zero components. Then $n \geqq 2$, and $f(x) = 0$ implies $f(r_1 c_{j_1}) = -f(r_2 c_{j_2} + \ldots + r_n c_{j_n})$. If $t \in 0(r_1 c_{j_1})$, then $0 = f(t r_1 c_{j_1}) = -f(tr_2 c_{j_2} + \ldots + tr_n c_{j_n})$. The choice of n implies that $tr_2 c_{j_2} + \ldots + tr\, c_{j_n} = 0$. It follows that $tr_2 c_{j_2} = \ldots = tr_n c_{j_n} = 0$, and therefore $0(r_1 c_j) = \bigcap_{k=2}^{n} 0(r_k c_{j_k})$. Since $0(r_1 c_{j_1})$ is a maximal left ideal of R, we must have $0(r_1 c_{j_1}) = 0(r_1 c_{j_1} + r_2 c_{j_2} + \ldots + r_n c_{j_n}) = 0(x)$. Consequently, Rx is a simple submodule of M and therefore $f(x) \neq 0$. This contradiction shows that C is a set of cogenerators of M. Since every non-zero submodule of M must have a non-empty intersection with C and since $C \cap Rc = \{c\}$ for every $c \in C$, the set of cogenerators C is irredundant.

Corollary 1. *Suppose that the R-module M is an essential extension of its socle. If C and C′ are irredundant sets of cogenerators of M, then there is a one to one correspondence between C and C′ such that if c and c′ are corresponding elements then* $Rc = Rc'$.

This corollary follows immediately from the proof of the previous theorem since both $\{Rc\}_{c \in C}$ and $\{Rc'\}_{c' \in C'}$ represent the se of simple submodules of M.

An R-module M is said to be *cocyclic* if it has a set of cogenerators consisting of a single element. The following corollary generalizes Theorem 3.1 [1, p. 16] from abelian groups to R-modules.

Corollary 2. *An R-module* $M \neq 0$ *is cocyclic if and only if it is a submodule of an injective envelope of a simple R-module.*

Let $\bigoplus_{i \in I} S_i$ be a direct sum decomposition of the socle $S(M)$ of an R-module M into simple submodules, and let x be a non-zero element of $S(M)$. x has only a finite number of non-zero components in $\bigoplus_{i\in I} S_i$; let these be $x_{i_1}, \dots, x_{i_n}$. Rx is a simple submodule of M if and only if $0(x) = \bigcap_{k=1}^{n} 0(x_{i_k})$ is a maximal left ideal of R. Since the left ideals $0(x_{i_k})$ are all maximal, $0(x)$ is maximal if and only if $0(x_{i_1}) = \dots = 0(x_{i_n})$. Thus every simple submodule of M is a submodule of exactly one homogeneous component of $S(M)$. Consequently, in order to exhibit an irredundant set of cogenerators of an R-module M, we can assume that all its simple submodules are isomorphic.

THEOREM 2. *Suppose that the R-module M is an essential extension of its socle $S(M) = \bigoplus_{i\in I} Rc_i$ where the simple direct summands Rc_i of $S(M)$ are isomorphic. Suppose further that the index set I is well-ordered; then an irredundant set of cogenerators of M is given by*

$$C = \{c_{i_0} + r_1 c_{i_1} + \dots + r_n c_{i_n} \mid i_0 < i_1 < \dots < i_n,$$

$$r_i \in R,\ 0(c_{i_0}) = 0(r_1 c_{i_1}) = \dots = 0(r_n c_{i_n})\}.$$

PROOF. We have to prove the following:

(i) Rc is a simple submodule of M for every $c \in C$.

(ii) If Rx is a simple submodule of M, then $Rx = Rc$ for some $c \in C$.

(iii) $Rc \neq Rc'$ for any two distinct elements c and c' of C.

(i) is clear, so suppose Rx is a simple submodule of M. Write $x = r_0 c_{i_0} + r_1 c_{i_1} + \dots + r_n c_{i_n}$ with $i_0 < i_1 < \dots < i_n$. Then $0(x) = 0(r_0 c_{i_0}) = 0(r_1 c_{i_1}) = \dots = 0(r_n c_{i_n})$. Since $c_{i_0} \in Rr_0 c_{i_0}$, there exists $t \in R$ such that $c_{i_0} = tr_0 c_{i_0}$. Hence $tx = c_{i_0} + tr_1 c_{i_1} + \dots + tr_n c_{i_n} = c$. Since $c_{i_0} \neq 0$, $tx \neq 0$ and $Rx = Rtx$ is a simple submodule of M. Consequently, $0(c_{i_0}) = 0(tr_1 c_{i_1}) = \dots = 0(tr_n c_{i_n})$. Thus $c \in C$ and $Rx = Rc$. This proves (ii).

Suppose finally that $Rc = Rc'$ where $c = c_{i_0} + r_1 c_{i_1} + \dots + r_n c_{i_n}$, $c' = c_{j_0} + s_1 c_{j_1} + \dots + s_m c_{j_m}$ with $i_0 < i_1 < \dots < i_n$ and $j_0 < j_1 < \dots < j_m$. There exists $t \in R$ such that $tc' = c$. Since $c_{i_0} \neq 0$, $t \notin 0(c_{j_0}) = 0(s_1 c_{j_1}) = \dots = 0(s_m c_{j_m})$. This means that the components of c and tc' are the same. Thus $\{i_0, i_1, \dots, i_n\} = \{j_0, j_1, \dots, j_m\}$. In particular $n = m$. The ordering of i_k and j_k implies then that $i_0=j_0$, $i_1=j_1, \dots, i_n=j_n$. Consequently $c_{i_0}=c_{j_0}=tc_{j_0}, r_k c_{i_k}=ts_k c_{jk}$ for $k=1, \dots, n$. But $c_{j_0}=tc_{j_0}$ implies $s_k c_{j_k}=ts_k c_{j_k}$ and hence $r_k c_{ik}=s_k c_{j_k}$ for $k=1, \dots, n$. Thus $c=c'$.

We note that the above Theorem is formulated and proved under the assumption of the axiom of choice. If the index set I is at most countable, the axiom of choice is not needed.

Throughout the remainder of this paper, the basic ring R will be assumed to be commutative Noetherian.

THEOREM 3. *Suppose that the R-module M is an essential extension of its socle* $S(M)=\bigoplus_{i\in I} T_i$ *where the* T_i *are the homogeneous components of* $S(M)$. *Then*

(*a*) $M=\bigoplus_{i\in I}(M\cap E(T_i))$

(*b*) $M\cap E(T_i)=\bigcup_{k=1}^{\infty}(0:{}_M P_i^k)$ *where each simple direct summand of* T_i *is isomorphic to* R/P_i.

Part (*a*) of this theorem is proved in [2, p. 111] for finitely embedded modules over a left H-ring. A ring R is a left H-ring if for any two non-isomorphic simple left R-modules S_1 and S_2, $\mathrm{Hom}_R(E(S_1), E(S_2))=0$. Commutative Noetherian rings are H-rings, but not conversely. Part (*b*) is also proved in [2, p. 113] for finitely embedded modules over a commutative Noetherian ring. However, the proof given generalizes almost word by word to R-modules with an essential socle. We therefore omit the proof of (*b*).

The proof of Theorem 3 is preceded by the following three lemmas.

LEMMA 1. *Let P be a prime ideal of R and x a non-zero element of* $E(R/P)$, *then* $0(x)$ *is P-primary.*

PROOF. Multiplication by x is a homomorphism from R to $E(R/P)$ with kernel $0(x)$. Hence $R/0(x)$ is isomorphic to a non-zero submodule of $E(R/P)$. Since $E(R/P)$ is indecomposable, $E(R/0(x))\cong E(R/P)$, so that $0(x)$ is irreducible. In particular $0(x)$ is a primary ideal of R. We shall show that it is P-primary. There exists $s\in R$ such that $0\neq sx\in R/P$, whence $0(sx)=P$. Since $0(x)\subseteq 0(sx)$, we have $0(x)\subseteq P$. Now let $r\in P$, and for each positive integer n let I_n denote the carrier $0(x)\colon r^n$ of r^n into $0(x)$. We have an ascending sequence $I_1\subseteq I_2\subseteq \dots$ of ideals of R. Since R is Noetherian there exists a positive integer n such that $I_n=I_{n+1}=\dots$ We want to show that $r^n\in 0(x)$. Suppose the contrary, then $r^n x$ is a non-zero element of $E(R/P)$ and there exists $t \in R$ such that $0 \neq tr^n x \in R/P$. Now $t \notin I_n$, for otherwise $tr^n x = 0$. But since every element of R/P is annihilated by every element of P, we have $tr^{n+1} x = 0$, and therefore $tr^{n+1} \in 0(x)$, that is $t \in I_{n+1} = I_n$, which is not so. Thus $r^n \in 0(x)$, and therefore $0(x)$ is P-primary.

COROLLARY. *Let P be a prime ideal of R and x a non-zero element of a direct sum $\bigoplus_I E(R/P)$ of I copies of $E(R/P)$. Then $0(x)$ is P-primary.*

PROOF. x has a finite number of non-zero components in the direct sum. Let these be $x_{i_1}, \dots, x_{i_n}$. By Lemma 1, $0(x_{i_k})$ is P-primary, $k = 1, \dots, n$. Since finite intersections of P-primary ideals of R are P-primary, $0(x) = \bigcap_{k=1}^{n} 0(x_{i_k})$ is P-primary.

LEMMA 2. *Let P_1 and P_2 be two distinct maximal ideals of R. If Q_1 and Q_2 are respectively P_1-primary and P_2-primary, then $Q_1 + Q_2 = R$.*

PROOF. Since the ring R is Noetherian, there exist positive integers m and n such that $P_1^m \subseteqq Q_1$ and $P_2^n \subseteqq Q_2$. If $P_1^m + P_2^n \neq R$, there would exist a maximal ideal P such that $P_1^m + P_2^n \subseteqq P$. But then $P_1^m \subseteqq P$ and $P_2^n \subseteqq P$, whence $P_1 = P = P_2$. Thus $P_1^m + P_2^n = R$ and hence $Q_1 + Q_2 = R$.

LEMMA 3. *Let R be a commutative Artinian ring and suppose that the socle of R is a direct sum of n isomorphic minimal ideals $S_1, \dots, S_n$, $n \geqq 1$. Then R is a local ring.*

PROOF. Let P be a maximal ideal such that $S_i \cong R/P$, $i = 1, \dots, n$. Since R is an essential extension of $S_1 \oplus \dots \oplus S_n$, we have an inclusion map $R \to E(R/P) \oplus \dots \oplus E(R/P)$ (n times). Let P' be a maximal ideal of R, then we have a non-zero homomorphism $R \to E(R/P')$. By injectivity of $E(R/P')$ this homomorphism can be extended to a homomorphism $E(R/P) \oplus \dots \oplus E(R/P) \to E(R/P')$. Thus there exists a non-zero homomorphism $f: E(R/P) \to E(R/P')$. Let $x \in E(R/P)$ be such that $f(x) \neq 0$. Then $0(x) \subseteqq 0(f(x)) \subseteqq P'$. By Lemma 1, $0(x)$ is P-primary. Since R is Noetherian, there exists a positive integer n such that $P^n \subseteqq 0(x) \subseteqq P'$, and hence $P \subseteqq P'$. Since both P and P' are maximal, $P = P'$.

PROOF OF THEOREM 3. (a) Let x be a non-zero element of M.

Then x has only a finite number of non-zero components in $\bigoplus_{i \in I} E(T_i)$. Let these be $x_{i_1}, \dots, x_{i_n}$, so that $x = x_{i_1} + x_{i_2} + \dots + x_{i_n}$. All that we need to show is that $x_{i_k} \in M$ for $k = 1, \dots, n$. This is true if $n = 1$. So suppose that this is true for every non-zero element of M whose non-zero components in $\bigoplus_{i \in I} E(T_i)$ are less than n. By Lemma 1 Corollary, $0(x_{i_1})$ and $0(x_{i_2})$ are respectively P_{i_1}-primary and P_{i_2}-primary. Since P_{i_1} and P_{i_2} are distinct maximal ideals, Lemma 2 implies that $0(x_{i_1}) + 0(x_{i_2}) = R$. Write $1 = u_1 + u_2$ with $u_1 \in 0(x_{i_1}$ and $u_2 \in 0(x_{i_2})$. Then $x_{i_1} = u_2 x_{i_1}$, and $u_2 x = x_{i_1} + u_2 x_{i_3} + \dots + u_2 x_{i_n}$. By the induction hypothesis, $x_{i_1} \in M$. Hence $x - x_{i_1} = x_{i_2} + \dots + x_{i_n} \in M$, and the inductioin hypothesis implies that $x_{i_2}, \dots, x_{i_k} \in M$. This completes the proof of (*a*).

An immediate consequence of Theorem 3 (*a*) is

COROLLARY 1. *If the socle of an R-module M is a direct sum of mutually non-isomorphic simple submodules and if M is an essential extension of its socle, then M is a direct sum of cocyclic submodules.*

The following result is a well-known theorem in ring theory.

COROLLARY 2. *A commutative Artinian ring R is a finite direct sum of local Artinian rings.*

PROOF. Since R is an essential extension of its socle, Theorem 3 (*a*) implies $R = \bigoplus_{i \in I} (R \cap E(T_i))$, where $\bigoplus_{i \in I} T_i$ is a direct sum decomposition of $S(R)$ into its homogeneous components. I is finite and every T_i is a finite direct sum of isomorphic minimal ideals of R. By Lemma 3, the ideal $R \cap E(T_i)$ of R is a local Artinian ring. Hence R is a direct finite sum of local Artinian rings.

COROLLARY 3. *In a commutative Artinian ring R there is a one to one correspondence between the maximal ideals of R and the classes of isomorphic minimal ideals. In particular every simple R-module is isomorphic to a minimal ideal of R.*

PROOF. If $R = R_1 \oplus \dots \oplus R_n$ is a decomposition of R as a direct sum of local Artinian rings, then the maximal ideals of R are all of the form $R_1 \oplus \dots \oplus R_{i-1} \oplus P_i \oplus R_{i+1} \oplus \dots \oplus R_n$ where P_i is the maximal ideal of R_i, $i = 1, \dots, n$.

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