A REMARK ON SURJECTIVITY OF QUASIBOUNDED P-COMPACT MAPS (*)

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Sommario. - *Usando la teoria spettrale per operatori non lineari, recentemente introdotta da M. Furi ed A. Vignoli, si dà un teorema di suriettività per operatori P-compatti e quasi limitati. Come corollario si ottiene un teorema dovuto a W. V. Petryshyn.*

Summary. - *In the framework of a spectral theory for nonlinear maps, recently introduced by M. Furi and A. Vignoli, we give a surjectivity theorem for quasibounded P-compact operators. As a consequence we obtain a result due to W. V. Petryshyn.*

I. Introduction.

Let E be a Banach space and $A: E \rightarrow E$ be a P-compact quasibounded operator (see definition below). In [4] W. V. Petryshyn proved that for any $\mu > M$, where M is the quasinorm of A, the operator $A - \mu I: E \rightarrow E$ is onto.

The purpose of this note is to show that Petryshyn's result remains true if we replace the condition $\mu > M$ with the weaker assumption $\mu > r^+(A)$, where $r^+(A)$ is the positive spectral radius of A. This result has been obtained using the spectral theory for non linear operators, introduced recently by M. Furi - A. Vignoli [2], and the topological degree for A-proper maps [5].

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II. Notations, definitions.

Let E be a real Banach space with the property that there exist a sequence $\{E_n\}$ of finite dimensional subspaces of E, $E_n \subset E_{n+1}$, $\overline{U_n E_n} = E$ and a sequence of linear projections $\{P_n\}$ such that $P_n E = E_n$ and $P_n x \to x$ for every $x \in E$.

DEFINITION 2.1. ([5]). A map $T: E \to E$ is said to be A-proper if for any n the operator $T_n = P_n T$ is continuous as a map from E_n into itself and if for any bounded sequence $\{x_k\}$ such that $x_k \in E_k$ and $T_k(x_k) \to g$, $g \in E$, there exist a subsequence $\{x_{k_j}\}$ and $x \in E$ such that $x_{k_j} \to x$ and $Tx = g$.

DEFINITION 2.2. ([4]). A map $T: E \to E$ is said to be P-compact if $T - \lambda I$ (where I denotes the identity of E) is A-proper for every $\lambda > 0$.

F. Browder and W. V. Petryshyn ([5]) defined a topological degree for A-proper maps having the basic properties of the classical Leray-Schauder degree.

Let B_r denote the closed unit ball of radius r centered at the origin, ∂B_r its boundary, $\overset{\circ}{B}_r$ its interior.

DEFINITION 2.3. A map $T: E \to E$ is said to be quasibounded (see A. Granas [3]) if there are positive real numbers A, B such that $\|Tx\| \leq A + B\|x\|$ for any $x \in E$.

Let $T: E \to E$ be quasibounded. The number $|T| = \inf\{B > 0:$ there exists $A > 0$ such that $\|T(x)\| \leq A + B\|x\|, x \in E\}$ is called the quasinorm of T.

It is easy to see that $|T| = \lim_{\|x\| \to +\infty} \sup \dfrac{\|Tx\|}{\|x\|}$.

Let E be a Banach space over the complex or real field K and let $T: E \to E$ be quasibounded and continuous.

In [2] M. Furi - A. Vignoli defined a spectrum $\Sigma(T)$ of T in the following way:

$$\Sigma(T) = \{\lambda \in K: d(T - \lambda I) = 0\}$$

where

$$d(T - \lambda I) = \lim_{\|x\| \to +\infty} \inf \frac{\|Tx - \lambda x\|}{\|x\|}.$$

We observe that the properties of Σ and d listed in [2] hold true even without the continuity assumption on T. In particular one can prove that $\Sigma(T)$ is compact.

Let E be a real Banach space and $T: E \to E$ be quasibounded. Define $r^+(T) = \sup\{\lambda \geq 0: \lambda \in \Sigma(T)\}$. If $\Sigma(T) = \Phi$ we put $r^+(T) = 0$. Clearly $r^+(T) \leq |T|$.

III. Results.

Let E be a real Banach space and let $T: E \to E$ be a quasibounded, P-compact map. Assume that $\lambda \notin \Sigma(T)\, U\, (-\infty, 0]$. Then there exists $r_0 > 0$ such that $(T - \lambda I)\, x \neq 0$ for any $x \in E$ with $\|x\| \geq r_0$. This implies that for any $r \geq r_0$ the Browder-Petryshyn degree of the A-proper map $T - \lambda I$ restricted to B_r is defined. Moreover we have that $\mathrm{Deg}(I - \lambda I, B_r, 0)$ is indipendent of $r \geq r_0$.

In fact let $s > r \geq r_0$. Then $\mathrm{Deg}(T - \lambda I, B_s \backslash \overset{\circ}{B}_r, 0) = \{0\}$ (see Property B_2 of [5]) and so

$$\mathrm{Deg}(T - \lambda I, B_s, 0) = \mathrm{Deg}(T - \lambda I, B_s \backslash \overset{\circ}{B}_r, 0) + \mathrm{Deg}(T - \lambda I, B_r, 0) = \mathrm{Deg}(T - \lambda I, B_r, 0)$$

(see Property B_4 of [5]).

Therefore we can define $\deg(T - \lambda I) := \mathrm{Deg}(T - \lambda I, B_r, 0),\ r \geq r_0$. We call $\deg(T - \lambda I)$ « surjectivity degree » of the map $T - \lambda I$.

The following Lemma holds:

LEMMA 3.1. *Let $T: E \to E$ be P-compact and quasibounded. Assume that $\lambda \notin \Sigma(T)\, U\, (-\infty, 0]$. If $\deg(T - \lambda I) \neq \{0\}$ then $T - \lambda I$ is onto.*

PROOF: Let $p \in E$. Consider the homotopy $H: E\, x\, [0, 1] \to E$ defined by $H(x, t) \doteq Tx - \lambda x - tp$. Let us prove that there exists $r_0 > 0$ such that $Tx - \lambda x - tp \neq 0$ for any $x \in E$ with $\|x\| \geq r_0$ and for any $t \in [0, 1]$. In fact suppose that for any $n \in N$ there exists $x_n \in E$, $\|x_n\| \geq n$, and $t_n \in [0, 1]$ such that $Tx_n - \lambda x_n - t_n p = 0$. Since $t_n \in [0, 1]$ we may assume $t_n \to t_0 \in [0, 1]$.

Since $\dfrac{\| Tx_n - \lambda x_n - t_0 p \|}{\| x_n \|} \leq |t_n - t_0| \dfrac{\| p \|}{\| x_n \|}$, it follows that $d(T - \lambda I - t_0 p) = 0$.

Thus $d(T-\lambda I)=d(T-\lambda I-t_0 p)=0$ contradicting the assumption $\lambda \notin \Sigma(T) \cup (-\infty, 0]$.

The uniform continuity of $H(x,t)$ with respect to $x \in \partial B_{r_0}$, insures that $\operatorname{Deg}(T-\lambda I, B_{r_0}, 0) = \operatorname{Deg}(T-\lambda I-p, B_{r_0}, 0)$ (see property B_3 of [5]). Thus there exists $x \in E$ such that $Tx-\lambda x=p$ (see Property B_2 of [5]).

We are in a position of proving our result:

THEOREM 3.1. *Let $T: E \to E$ be a quasibounded, P-compact map. If $\lambda > r^+(T)$ then $T-\lambda I$ is onto.*

PROOF. Consider the homotopy $H: E x [0,1] \to E$ defined by $H(x,t) = tT(x) - \lambda x$ which is A-proper for every $t \in [0,1]$.

We want to show that there exists $r_0 > 0$ such that $H(x,t) \neq 0$ for every $x \in E$, $\|x\| \geq r_0$ and $t \in [0,1]$.

Assume the contrary. Then there exist $t_n \in [0,1]$ and $x_n \in E$, $\|x_n\| \geq n$ such that $t_n T(x_n) - \lambda x_n = 0$. We may assume, without loss of generality, that $t_n \to t_0$. Hence

$$\frac{\| t_0 T(x_n) - \lambda x_n \|}{\| x_n \|} \leq | t_n - t_0 | \frac{\| T(x_n) \|}{\| x_n \|} + \frac{\| t_n T(x_n) - \lambda x_n \|}{\| x_n \|} \leq$$

$$\leq |t_n - t_0| \, |T|.$$

It follows that $d(t_0 T - \lambda I) = 0$.

This is clearly impossible if $t_0 = 0$ since $d(0T - \lambda I) = \lambda$. Thus $t_0 > 0$.

But $0 = d(t_0 T - \lambda I) = t_0 \, d\left(T - \dfrac{\lambda}{t_0} I\right) > 0$ since $\dfrac{\lambda}{t_0} > r^+(T)$.

Since for any real number $r > 0$ the restriction of H to the subset ∂B_r is continuous in t uniformly with respect to $x \in \partial B_r$, it follows that $\operatorname{Deg}(T-\lambda I, B_{r_0}, 0) = \operatorname{Deg}(-\lambda I, B_{r_0}, 0)$, for any $r > r_0$. The right-hand side of the last equality is different from $\{0\}$ (see B_5 [5]). Hence $\deg(T-\lambda I) \neq \{0\}$ and the statement follows from Lemma 3.1.

COROLLARY 3.1. (Petryshyn [3]).
Let $T: E \to E$ be a quasibounded P-compact map. If $\lambda > |T|$ then $T - \lambda I$ is onto.

PROOF: Follows immediately from the fact that $r^+(T) \leq |T|$.

REMARK 3.1. We obesrve that if T is a monotone decreasing quasibounded operator defined in a Hilbert space H, then $\Sigma(T) \subset$

$\subset(-\infty, 0]$. In fact for any $\lambda>0$ we have

$$\liminf_{\|x\|\to\infty} \frac{\|\lambda x - T(x)\|}{\|x\|} = \liminf_{\|x\|\to\infty} \frac{\|\lambda x - T(x) + T(0)\|}{\|x\|} \geq \lambda$$

(see Proposition 2.1, pag. 21 of [1]).

Hence if, in addition, T is P-compact (for this it sufficies that T is either continuous, demicontinuous, or weakly continuous) then $T-\lambda I$ is onto for any $\lambda>0$.

REMARK 3.2. Obviously in Definition 2.2 we may require that $T-\lambda I$ is A-proper for any $\lambda<0$. A result analogous to Theorem 3.1 can be obtained for the class of quasibounded operators $T: E\to E$, which are P-compact in the above sense. More precisely we have that $T-\lambda I$ is onto provided that $\lambda<r^-(T)$, where $r^-(T)=\inf\{\mu\leq 0: : \mu\in\Sigma(T)\}$. As a consequence we obtain in Corollary 3 of [4].

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