SOME REMARKS ON ISOMORPHISMS OF FUNCTION ALGEBRAS (*)

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SOMMARIO. - *Sia $B(X)$ l'algebra reale di tutte le funzioni limitate a valori reali $f: X \to \mathbb{R}$ sull'insieme $X \neq \emptyset$. In questa nota mostriamo che ogni isomorfismo Φ fra le algebre $B(X), B(Y)$ è indotto da una biiezione $\varphi: X \to Y$*

$$\Phi(f)(t) := f[\varphi^{-1}(t)].$$

Infine proviamo un teorema riguardante la rappresentazione di isomorfismi di certi sottomonoidi moltiplicativi di $B(X)$ *e accenniamo ad un'applicazione alla topologia generale.*

SUMMARY. - *$B(X)$ be the real algebra of all bounded real-valued functions $f: X \to \mathbb{R}$ on the set $X \neq \emptyset$. We prove in this note, that every algebra isomorphism $\Phi: B(X) \to B(Y)$ is induced by a bijection $\varphi: X \to Y$*

$$\Phi(f)(t) := f[\varphi^{-1}(t)].$$

Finally we prove a theorem concerning the representation of isomorphisms of certain multiplicative submonoids of $B(X)$ and mention an application in general topology.

1. Notation. Let $X \neq \emptyset$ be a set and $B(X)$ the real algebra of all bounded real-valued functions $f: X \to \mathbb{R}$ with the usual pointwise operations; 0 resp. 1 are the zero and unit element in $B(X)$. We denote by $S(X)$ the multiplicative monoid of the idempotents in $B(X)$ (we have $S(X) = \{\chi_M \mid M \subseteq X\}$ where the symbol χ_M is throughout reserved for the characteristic function of the set $M \subseteq X$; we further use

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the abbreviation χ_p for $\chi_{\{p\}}$), $P(X) := \{\chi_p \in S(X) \mid p \in X\}$ and $P_0(X) := P(X) \cup \{0\}$. Finally, if X and Y are sets, we put bij (X, Y) resp. iso $[B(X), B(Y)]$ for the set of all bijections $\varphi: X \to Y$ resp. (algebra)-isomorphisms $\Phi: B(X) \to B(Y)$.

2. The main results. At first we notice, that every $\varphi \in \text{bij}(X, Y)$ induces a canonical map

(1) $$\varphi^{\#}: B(X) \to B(Y);\ \varphi^{\#}(f) := f \circ \varphi^{-1}.$$

It is easy to show, that

$$\forall \varphi \in \text{bij}(X, Y): \varphi^{\#} \in \text{iso}[B(X), B(Y)]$$

and therefore the map

(2) $$\#: \text{bij}(X, Y) \to \text{iso}[B(X), B(Y)];\ \#(\varphi) := \varphi^{\#}$$

is defined and is injective.

REMARKS 1. If we would have taken into account that $B(X)$ together with the uniform norm $\|f\| := \sup |f(x)|$ is a Banachalgebra, we could also show that $\varphi^{\#}$ in (1) is a norm-isometry.

2. The operator in (1) is well known; see for example [2], 4.2, p. 76, where the symbol $\mathcal{C}(\varphi)$ is used instead of $\varphi^{\#}$.

The main theorem is

THEOREM 1. *The map* $\#$ *in* (2) *is bijective.*

This generalizes in some sense a result of Z. SEMADENI, [2], 7.7.1, p. 127. By the NAGASAWA-theorem ([3], 149) not only every algebra isomorphism has a representation of the form (2) but also every norm isometry $\Phi: B(X) \to B(Y)$ with $\Phi(0) = 0$ and $\Phi(1) = 1$ must be of the form (2). An immediate consequence of theorem 1, which has an important application in topology, is

THEOREM 2. *Let* $M \subseteq S(X)$ *and* $N \subseteq S(Y)$ *be submonoids,* $\Phi: [M \cup P_0(X)] \to [N \cup P_0(Y)]$ *a monoid-isomorphism and denote by* $M^* \subseteq M$ resp. $N^* \subseteq N$ *the monoid which is generated by* $M - P_0(X)$ *resp.* $N - P_0(Y)$. *Then there exists a* $\varphi \in bij(X, Y)$ *such that*

$$\varphi^{\#} \mid [M \cup P_0(X)] = \Phi.$$

Further, we get

$$\Phi(M)=N \Leftrightarrow \Phi(M-M^*)=N-N^*.$$

3. Proof of theoremes 1 and 2.

PROOF OF THEOREM 1. Let

$$\Phi \in \text{iso}\,[B(X), B(Y)]$$

be any fixed isomorphism. It is easy to show

$$\Phi\,[S(X)]=S(Y)$$

and

$$\Phi\,[P(X)]=P(Y)$$

and therefore the mappings

(3) $$\Phi_S:=\Phi \mid S(X): S(X) \rightarrow S(Y)$$

and

(4) $$\Phi_P:=\Phi \mid P(X): P(X) \rightarrow P(Y)$$

are bijections. If we denote by $\alpha: X \rightarrow P(X)$; $\alpha(x):=\chi_x$ and $\beta: Y \rightarrow P(Y)$; $\beta(y):=\chi_y$ the canonical bijections and consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & P(X) \\ \varphi \downarrow & & \downarrow \Phi_P \\ Y & \xrightarrow{\beta} & P(Y) \end{array}$$

we recognise by (4) that Φ_P defines in a unique way a map

(5) $$\varphi:=\beta^{-1} \circ \Phi_P \circ \alpha \in \text{bij}\,(X, Y).$$

Now we will show

(6) $$\forall M \subseteq X: \Phi_S(\chi_M)=\chi_{\varphi(M)}.$$

To do this, let us denote by 2^X resp. 2^Y the powerset of X resp. Y, $\gamma: 2^X \rightarrow S(X)$; $\gamma(M):=\chi_M$ and $\delta: 2^Y \rightarrow S(Y)$; $\delta(Q):=\chi_Q$. Then by (3) Φ_S induces a unique natural map

$$\pi:=\delta^{-1} \circ \Phi_S \circ \gamma \in \text{bij}\,(2^X, 2^Y)$$

and we obtain immediately

(7) $$\forall M \subseteq X:\ \Phi_S(\chi_M) = \chi_{\pi(M)},$$

so we only need to show

$$\forall M \subseteq X:\ \pi(M) = \varphi(M) = \{\varphi(x) \mid x \in M\}.$$

But this is trivial for $M = \varnothing$ and a simple consequence of (4), (5) and (7) in the case $M = \{x\}$. In the general case we proceede directly.

$$x \in M \Rightarrow \chi_{\varphi(x)} = \Phi_S(\chi_x) = \Phi_S(\chi_x \cdot \chi_M) =$$

$$= \Phi_S(\chi_x) \cdot \Phi_S(\chi_M) = \chi_{\varphi(x)} \cdot \chi_{\pi(M)} \Rightarrow \varphi(x) \in \pi(M) \Rightarrow \varphi(M) \subseteq \pi(M).$$

$$z \in \pi(M) \Rightarrow \chi_{\varphi^{-1}(z)} = \Phi_S^{-1}(\chi_z) = \Phi_S^{-1}(\chi_z \cdot \chi_{\pi(M)}) =$$

$$= \Phi_S^{-1}(\chi_z) \cdot \Phi_S^{-1}(\chi_{\pi(M)}) = \chi_{\varphi^{-1}(z)} \cdot \chi_M \Rightarrow \varphi^{-1}(z) \in M \Rightarrow \pi(M) \subseteq \varphi(M).$$

This proves (6), which means

(8) $$\Phi_S = \varphi^{\#} \mid S(X).$$

Finally we extend (8) to $B(X)$:

(9) $$\Phi = \varphi^{\#}.$$

We show (9) indirect. Assume that $\Phi \neq \varphi^{\#}$. Then there exists a $g \in B(X)$ such that $\varphi^{\#}(g) \neq \Phi(g)$ and therefore a $z \in X$ with

$$g(z) = \varphi^{\#}(g)[\varphi(z)] \neq \Phi(g)[\varphi(z)].$$

Now put $f := g \cdot \chi_z \in B(X)$. Trivially, we have $f = \omega \chi_z$ with $\omega := g(z) \in \mathbb{R}$; this implies with (8)

$$\omega \chi_{\varphi(z)} = \omega \varphi^{\#}(\chi_z) = \omega \Phi(\chi_z) = \Phi(\omega \chi_z) =$$

$$= \Phi(f) = \Phi(g \cdot \chi_z) = \Phi(g) \cdot \Phi(\chi_z) = \Phi(g) \cdot \chi_{\varphi(z)}.$$

Thus we obtain

$$g(z) = \omega = \omega \chi_{\varphi(z)}[\varphi(z)] = \Phi(g) \cdot \chi_{\varphi(z)}[\varphi(z)] = \Phi(g)[\varphi(z)].$$

This contradiction proves (9), and so the proof of theorem 1 is complete.

PROOF OF THEOREM 2. As in the proof of theorem 1 we can show, that $\Phi\,[P(X)] = P(Y)$, which implies the existence of a $\varphi \in \mathrm{bij}(X, Y)$ such that

$$\varphi^{\#} \mid [M \cup P_0(X)] = \Phi.$$

Because $\Phi\,[P_0(X)] = P_0(Y)$ and $\Phi\,[M - P_0(X)] = N - P_0(Y)$ we obtain $\Phi(M^*) = N^*$; and therefore

$$\Phi(M) = N \Leftrightarrow \Phi(M - M^*) = N - N^*$$

(with $M - M^* \subseteq M \cap P_0(X)$ and $N - N^* \subseteq N \cap P_0(Y)$) because $M = (M - M^*) \cup M^*$ and $N = (N - N^*) \cup N^*$ are partitions.

4. A topological application.

4. A topological application. Let $L^+(X)$ denote the semiring of all nonnegative realvalued lower semicontinuous functions on the topological space X, $I(X)$ the multiplicative monoid of all idempotents in $L^+(X)$, $I^*(X)$ the submonoid of $I(X)$ which is generated by the set $I(X) - P(X)$ and $D(X) := I(X) - I^*(X)$. It is evident that $D(X) \subseteq P(X)$ and contains at most two elements.

THEOREM ([1]). *Two topological spaces X and Y are homeomorphic iff there exists a monoid-isomorphism*

$$\Phi\colon [I(X) \cup P(X)] \to [I(Y) \cup P(Y)]$$

such that

$$\Phi\,[D(X)] = D(Y).$$

REFERENCES

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[2] Z. SEMADENI, *Banach spaces of continuous functions I*, Warszawa, 1971.

[3] W. ZELAZKO, *Banach Algebras*, Warszawa, 1973.