COMMUTATIVE INVERSE PROPERTY LOOPS AS GROUPOID WITH ONE LAW (*)

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SOMMARIO. - *Lo scopo di questo lavoro è di dare una caratterizzazione dei loops commutativi dotati di inverso come sottovarietà di gruppoidi con una sola identità.*

SUMMARY. - *The object of this paper is to give characterization of commutative inverse property loops as a subvariety of groupoids with a single identity.*

1. Introduction: One of the problems of varieties is to define some class of varieties by a single identity. Higman and Neumann [1] have solved this problem for groups and Abelian groups. Padmanabhan [4] has solved this problem for inverse loops and generalized the corresponding result for groups due to Higman and Neumann [1]. The object of this paper is to show that every variety of commutative inverse property loops which can be defined by a finite system of laws as a subvariety of the variety of commutative inverse property lopps can be defined by a single law as a subvariety of groupoids. This generalizes the corresponding results for Abelian groups proved by Higman and Neumann [1]. An interesting corollary is that commutative Moufang loops can be defined by means of a single law. For algebraic properties of commutative inverse property loops see, Bruck [3].

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2. Let $\langle Q, \cdot \rangle$ be a loop with identity e. Then Q is said to be a loop with CIP (commutative inverse property) if to each element x in Q there corresponds a unique element x^{-1} of Q such that

$$x^{-1}(x \cdot y) = (y \cdot x)\, x^{-1} = y, \; xy = yx \text{ for all } x, y \in Q. \tag{1}$$

According to Bruck [3, p. 292] the CIP loop Q also satisfies the identities

$$x \cdot x^{-1} = x^{-1} \cdot x = e, \; (x^{-1})^{-1} = x, \; (x \cdot y)^{-1} = y^{-1} \cdot x^{-1} = x^{-1} \cdot y^{-1}, \tag{2}$$

for all $x, y \in Q$.

Let $\langle Q, * \rangle$ be a groupoid, we say that the groupoid $\langle Q, * \rangle$ is an iso - CIP loop if there is a CIP loop $\langle Q, \bigcirc \rangle$ which is a principal isotope of $\langle Q, * \rangle$ such that '$*$' and '$\bigcirc$' are connected by the relation $x * y = x \bigcirc y^{-1}$.

THEOREM 1. *A necessary and sufficient condition that a groupoid* $\langle Q, * \rangle$ *is an iso - CIP loop is that*

$$z = (u * u) * [((v * v) * (z * (y * x))) * ((t * t) * (x * y))] \tag{3}$$

for all $x, y, z, u, v, t \in Q$.

Proof: *Sufficiency*: Let the identity (3) hold in $\langle Q, * \rangle$. First we shall show that $(*)$ is right cancellative. Let $z * a = w * a$ for some $a \in Q$. Now, by the given identity, any element a in Q is of the form $b * c$ for some $b, c \in Q$, so

$$z = (u * u) * [((v * v) * (z * (b * c))) * ((t * t) * (c * b))]$$

$$= (u * u) * [((v * v) * (w * (b * c))) * ((t * t) * (c * b))] = w.$$

Thus $(*)$ is right cancellative. Using the right cancellative, from (3), keeping x, y, z, v, t the same and changing u to w, and using (3), we get

$$u * u = w * w = \text{constant} = e \text{ (say) for all } u, w \in Q. \tag{4}$$

with the help of (4), (3) becomes

(5) $z=e * [(e * (z * (y * x))) * (e * (x * y))]$ for all $x, y, z \in Q$.

Putting $z=y * x$ in (5),

(6) $$y * x=e * [e * (e * (x * y))]$$

Putting $x=y$ in (5),

(7) $$z=e * [(e * (z * e)) * e].$$

Let $a * z=a * w$. Then, by (6),

$$z * a=e * [e * (e * (a * z))] =$$

$$=e * [e * (e * (a * w))] =w * a.$$

On using the right cancellative of (*), we have $z=w$. Thus (*) is left cancellative. Next, by (7),

$$e * (e * z)= e* (e * (e * ((e * (z * e)) * e))) =e * (e * (z * e)),$$

on using the left cancellative of (*), we have

(8) $$z=z * e.$$

With the help of (8), (7) becomes

(9) $$z=e * (e * z).$$

With the help of (9), (6) becomes

(10) $$y * x=e * (x * y).$$

Let us define $\langle Q, \bigcirc \rangle$ as follows.

(11) $$x \bigcirc y=x * y^{-1} \text{ and}$$

(12) $$x^{-1}=e * x \text{ for all } x, y \in Q.$$

Now we shall show that $\langle Q, \bigcirc \rangle$ is a CIP loop.

(13) $x \bigcirc e=x * (e * e)=x$ by (8), for all $x \in Q$.

(14) $e \bigcirc x=e * (e * x)=x$ by (9), for all $x \in Q$.

(13) and (14) imply that e is the identity of $\langle Q, \bigcirc \rangle$.

(15) $$(x^{-1})^{-1}=(e * x)^{-1}=e * (e * x)=x \text{ for all } x \in Q.$$

Since the equations $a * x=b$ and $y * a=b$ have unique solutions in the groupoid $\langle Q, * \rangle$. Thus the equations $a \bigcirc x=b$ and $y \bigcirc a=b$ have unique solutions in the systems $\langle Q, \bigcirc \rangle$. Thus we have proved that $\langle Q, \bigcirc \rangle$ is a loop. Further

$$x \bigcirc (y^{-1} \bigcirc x^{-1})=x * (y^{-1} * x)^{-1} =x * (e * (y^{-1} * x))=x * (x * y^{-1})$$

by (10) $=x \bigcirc (x \bigcirc y)^{-1}$, on using the left cancellative of $(\bigcirc)$, we have

(16) $(x \bigcirc y)^{-1}=y^{-1} \bigcirc x^{-1}$ for all $x, y \in Q$. From (5)

$$x^{-1}=e * [(e * (x^{-1} * (y * e))) * (e * (e * y))]$$

$$=[(x^{-1} \bigcirc y^{-1})^{-1} \bigcirc y^{-1}]^{-1} \text{ by (11) and (12)}$$

$$=[(y \bigcirc x) \bigcirc y^{-1}]^{-1} \text{ by (16), it implies that}$$

(17) $$x=(y \bigcirc x) \bigcirc y^{-1} \text{ for } x, y \in Q.$$

With the help of (16), we write

$$x^{-1} \bigcirc y^{-1}= [((x \bigcirc y) \bigcirc x^{-1}) \bigcirc (x \bigcirc y)^{-1}] \bigcirc y^{-1}$$

$$=(y \bigcirc (x \bigcirc y)^{-1}) \bigcirc y^{-1} \text{ by (16)}$$

$$=(x \bigcirc y)^{-1} \text{ by (16),}$$

it gives

(18) $$(x \bigcirc y)^{-1}=x^{-1} \bigcirc y^{-1} \text{ for all } x, y \in Q.$$

(16) and (18) imply that

(19) $$x \bigcirc y=y \bigcirc x \text{ for all } x, y \in Q.$$

From (17) and (19) imply that left inverse property and right inverse property are true for the loop $\langle Q, \bigcirc \rangle$. Thus $\langle Q, \bigcirc \rangle$ is a CIP loop. It follows that $\langle Q, * \rangle$ is an iso - CIP loop.

Necessity: Let the groupoid $\langle Q, * \rangle$ be an iso - CIP loop. Let $\langle Q, \bigcirc \rangle$ be the corresponding CIP loop with identity e. The binary operations (*) and (○) are connected by

(20) $x * y = x \bigcirc y^{-1}$ for all $x, y \in Q$ and (15) and (16) are true.

Putting $y=x$ in (20), it gives

(4) $$x * x = e.$$

Putting $x=e$ in (20), it gives

(12) $$x^{-1} = e * x.$$

Further we consider

$$\begin{aligned} z &= [((x \bigcirc y^{-1}) \bigcirc z^{-1}) \bigcirc (x \bigcirc y^{-1})^{-1}]^{-1} \text{ by (17)} \\ &= [(z \bigcirc (x \bigcirc y^{-1})^{-1})^{-1} \bigcirc (y \bigcirc x^{-1})]^{-1} \text{ by (16)} \\ &= [(z * (x * y))^{-1} * (y * x)^{-1}]^{-1} \text{ by (20),} \end{aligned}$$

on using (4) and (12), we get

(3) $$z = (u * u) * [((v * v) * (z * (y * z))) * ((t * t) * (x * y))].$$

This completes the proof of the theorem.

3. Let $w = w(x_1, \dots, x_n)$ be some word in the variables $x_1, \dots, x_n$ in the groupoid $\langle Q, * \rangle$.

THEOREM 2. *A necessary and sufficient condition that a groupoid* $\langle Q, * \rangle$ *is an iso - CIP loop, in which the law* $w(x_1, \dots, x_n) = e$ *holds, is that*

(21) $$z = ((u * u) * w) * [((v * v) * (z * (y * x))) * ((t * t) * (x * y))]$$

for all $x, y, z, u, v, t \in Q$.

Proof: The necessary part is an easy consequence of theorem 1 and the hypothesis $w=e$. We need only to prove the sufficient part. As in theorem 1, here also we can show that $x * a = y * a \Rightarrow x = y$. Thus from (21) we have

$(u * u) * w = (s * s) * w$ for all $u, s \in Q$, from which it follows that

(4) $$u * u = s * s = \text{constant} = e.$$

Putting $x=y=z=e$ in (21) we get $(e * w) * e=e=e * e$ and hence, by the right cancellation law we have

$$e * w=e, \tag{22}$$

which by virtue of (4) gives $w=e$.

The condition (22) reduces the equation (21) to (3) in $\langle Q, * \rangle$.

Thus, by theorem 1, $\langle Q, * \rangle$ is an iso-CIP loop, in which the identity $w=e$ is satisfied. This completes the proof of the theorem.

COROLLARY 1. *Every variety of abelian groups which can be defined by a finite system of laws as a subvariety of the variety of abelian groups can be defined by a single law as a subvariety of the variety of groupoids.*

Proof: An abelian group is a commutative inverse property loop in which the associative law $(xy)\,z=x\,(yz)$ also holds.

Corollary 1 is due to Higman and Neumann [1].

COROLLARY 2. *Commutative Moufang loops can be defined by a single law as a sub-variety of groupoids.*

Proof: A commutative Moufang loop is a commutative inverse property loop in which the identity $(xy)\,(zx)=[x\,(yz)]\,x$ also holds. For Moufang loops, see Bruck [2, p. 115].

REFERENCES

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