NORM DECREASING ALGEBRA ISOMORPHISM OF BANACH ALGEBRAS (*)

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SOMMARIO. - *In questa nota dimostriamo che se T è un'algebra-isomorfismo a norma decrescente di un'algebra di Banach A_1 su un'algebra di Banach A_2, (A_i, $i = 1, 2$ essendo un $B^*, B(X)$, $B^{\#}$ o approssimante $B^{\#}$), allora T è un'isometria.*

SUMMARY. - *In this paper, we show that if T is a norm decreasing algebra isomorphism of a Banach algebra A_i onto another A_2 (A_i, $i = 1, 2$ being a B^*, $B(X)$, $B^{\#}$ or approximate $B^{\#}$) then T is an isometry.*

1. Introduction.

Wendel in [7], and [8], Rigelhof in [6] and Wood in [9] and [10] have all shown that norm decreasing algebra isomorphism of some group algebra onto another of the same kind implies an isometry. This is clear from he definition of such isomorphism. There are cases of algebras where * isomorphism implies isometry - B^* algebra is an example (see [3]). Also in [4] we have that an algebra isomorphism of a function algebra onto another is an isometry.

In this paper, we shall investigate other Banach algebras on which norm decreasing isomorphism implies isometry. There are two ways this problem could be tackled. One is to find the conditions on the algebra which makes our assertion true. The other is to find the form such an isomorphism takes as we have in the group algebra cases. The answer to our problem by using the first method is implicit in

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Bousall's paper [2]. For, it is easy to see that a norm decreasing algebra isomorphism of a Banach algebra onto another implies an isometry if and only if the algebras involved have minimal norms: recall that the norm $\|\ \|$ of a Banch algebra A is minimal if there is a second norm $|\ |$ in A such that $|x| \leq \|x\|$ $(x \in A)$ implies $|\ | = \|\ \|$. Let T be a norm decreasing algebra isomorphism of a Banach algebra A_1 onto another A_2 (A_i, $i=1, 2$, has minimal norm). Define the second norm in A_1 by $|x| = \|Tx\|$ $\forall x \in A_1$. Since T is norm decreasing $|x| = \|Tx\| \leq \|x\|$. But $\|\ \|$ being minimal implies $\|x\| = |x| = \|Tx\|$. Hence T is an isometry. Examples of Banach algebras with minimal norms are

(i) B^* algebras.

(ii) real Banach algebras of bounded real valued functions defined on an arbitrary set X with the usual algebraic operations and with norm $\|f\| = \sup_{x \in X} |f(x)|$.

(iii) Banach algebras $B(X)$ of operators in a Banach space X which contains all finite valued operators in X and

(iv) $B^{\#}$ annihilator algebras.

(See Theorems 10, 9, 8 and 4 of [2]).

We shall now prove our theorem for B^* and $B(X)$ without assuming the existence of minimal norms and then add that our assertion is true also for the approximate $B^{\#}$ algebras.

2. Definitions and Notations.

2.1. Definition: A Banach algebra A is a $B^{\#}$ algebra if for each $x \in A$ and every positive integer n, there exists $x^{\#} \in A$ (not necessarily unique) such that $x \neq 0$ and $\|(x^{\#} x)^n\|^{1/n} = \|x^{\#}\| \, \|x\|$.

It is easy to show that this definition implies $\rho(x^{\#} x) = \|x^{\#}\| \, \|x\|$ where $\rho(x)$ denotes the spectral radius of $x \in A$. See [2].

2.2. Definition: A Banach algebra A is an approximate $B^{\#}$ algebra if for each $x \in A$ and $\varepsilon > 0$ there exists $x^{\#} \in A$ (not necessarily unique) such that $x^{\#} \neq 0$ and

$$\rho(x^{\#} x) \geq \|x_{\#}\| \, \|x\| (1-\varepsilon).$$

This is the same as the definition given by Alexander in [1] but we do not need $x^{\#}=1$ in our own case.

The standard definitions concerning Banach algebra theory that we shall use can be found in [5]. We shall always consider Banach algebra over the complex field which we shall denote by $\underset{\sim}{C}$. Given an element $x \in A$, we write $\sigma(x)$ for the spectrum of x. We write X^* for the space of bounded linear functionals on the space X. This work forms a small part of the author in Ph. D. thesis. I take this opportunity to thank Dr. G. V. Wood of the University College of Swansea, by research supervisor, for interesting me in this work and for his help and general advice during my three years stay in Swansea. I also thank Dr. N. J. Kalton for his useful suggestions.

3. We shall now prove our main theorems and we prove the B^* case first because of its well known properties.

3. 1. **Theorem:** A norm decreasing algebra isomorphism T of a B^* algebra A_1 onto another A_2 is an isometry.

Proof: Let $x \in A_1$, then $\|x\|^2 = \|x^* x\|$. Also $\rho(x) \leq \|x\|$ and $\|x^* x\| = \rho(x^* x)$. (See 4.8.1 of [5]).

Since T is an algebra isomorphism $\rho(x) = \rho(Tx)$ for $x \in A_1$. Therefore

$$\|x\|^2 = \|x^* x\| = \rho(x^* x) = \rho(Tx^* x)$$

$$\leq \|T(x^* x)\| \leq \|Tx^*\| \, \|Tx\|$$

$$\leq \|x^*\| \, \|Tx\| = \|x\| \, \|Tx\|.$$

Hence

$$\|x\| \leq \|Tx\|,$$

since $\|Tx\| \leq \|x\|$ by hypothesis, we have

$$\|Tx\| = \|x\| \quad x \in A_1.$$

3.2. **Theorem:** Let T be a norm decreasing algebra isomorphism of a Banach algebra of bounded linear operators $B(X_1)$ on a Banach space X_1 onto another $B(X_2)$. Then T is an isometry.

Proof: An algebra isomorphism, T, of a Banach algebra $B(X_1)$ onto another $B(X_2)$ is given by $T\varphi = U \varphi U^{-1}$ (see Theorem 2.5.19 of [5]) where U is a linear isomorphism of X_1 onto X_2 and $\varphi \in B(X_1)$.

Let φ_0 be a one-dimensional operator in $B(X_1)$ defined by

$$\varphi_0(x)=f(x)\,y \quad \text{fof} \quad f\in X_1^*; \quad x, y\in X_1$$

such that

(1) $$\|f\| = \|y\| = 1 \quad \dots \quad \dots \quad \dots$$

Then

(2) $$\|U\varphi_0 U^{-1}\| = \sup_{\|x'\|\leq 1} \|U\varphi_0(U^{-1}x')\| \quad x'\in X_2$$

$$= \sup_{\|x'\|\leq 1} |f(U^{-1}x')|\,\|Uy\| \quad \text{by (1)}$$

$$= \sup_{\|x'\|\leq 1} |(U^{-1^*}f)\,x'|\,\|Uy\|$$

$$= \|U^{-1^*}f\|\,\|Uy\| \quad \dots \quad \dots$$

Also

(3) $$\|\varphi_0\| = \sup_{\|x\|\leq 1} \|\varphi_0 x\|$$

$$= \sup_{\|x\|\leq 1} |f(x)|\,\|y\| \quad \text{by (1)}$$

$$= \|f\|\,\|y\|$$

$$= 1 \quad \dots \quad \dots \quad \dots$$

Since T is norm decreasing, we have, from (2) and (3) that

$$\|U^{-1^*}f\|\,\|Uy\| \leq 1.$$

Since (2) holds for all one-dimensional operators in $B(X_1)$ and $\|U^{-1^*}\| = \|U^{-1}\|$, we have

$$\|U^{-1}\|\,\|U\| = \|U^{-1^*}\|\,\|U\| = \sup_{\|f\|\leq 1} \|U^{-1^*}f\| \sup_{\|y\|\leq 1} \|Uy\| \leq 1.$$

But $1 \leq \|U^{-1}\|\,\|U\|$.

Hence

$$\|U\|\,\|U^{-1}\| = 1.$$

Define $V=\dfrac{U}{\|U\|}$, then $V^{-1}=\dfrac{U^{-1}}{\|U^{-1}\|}$ and V is an isometry.

Then, for any $\varphi \in B(X_1)$, we have

$$\|\varphi\| = \sup_{\|V^{-1}y\| = \|y\| \leq 1} \|\varphi(V^{-1}y)\|$$

$$= \sup_{\|y\| \leq 1} \|V\varphi V^{-1}y\|$$

$$= \sup_{\|y\| \leq 1} \|U\varphi U^{-1}y\|$$

$$= \|U\varphi U^{-1}\|.$$

Therefore T is an isometry.

3.3. THEOREM: A norm decreasing algebra isomorphism T of an approximate $B^{\#}$ algebra A_1 onto another A_2 is an isometry.

PROOF: We recall that for $x \in A_1$, $\rho(x) \leq \|x\|$ and $\rho(x) = \rho(Tx)$ for the same reason as in 3.1. Hence

$$\|x^{\#}\|\,\|x\|\,(1-\varepsilon) \leq \rho(x^{\#}x) = \rho(T(x^{\#}x))$$

$$\leq \|T(x^{\#}x)\| \leq \|Tx^{\#}\|\,\|Tx\|$$

$$\leq \|x^{\#}\|\,\|Tx\|.$$

Since ε is arbitrary and $x^{\#} \neq 0$, we have

$$\|x\| \leq \|Tx\| \leq \|x\| \qquad \text{by hypothesis.}$$

Hence

$$\|Tx\| = \|x\| \qquad x \in A_1.$$

3.4. REMARK: It is clear that Theorem 3.3 still holds if the word « approximate » is dropped.

We shall now indicate by an example that not every norm decreasing isomorphism of a Banach algebra A_1 onto another A_2 (or A_1) is an isometry.

3.5. EXAMPLE: Let A be a Banach algebra of the 2×2 matrix $A = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in C\right\}$ with norm defined by

$$\left\| \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\| = \max\{|a|, |b| + |c|\}.$$

Define T by

$$T\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & \frac{1}{2}b \\ 0 & c \end{pmatrix}$$

T is easily shown to be an algebra automorphism of A. But T is norm decreasing and not an isometry.

A similar definition of T can be given on $n \times n$ matrix algebra where $n > 2$. In case $n = 3$, for instance, we can define T thus:

(i) $$T\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} a & \frac{1}{2}b & \frac{1}{2}c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

or

(ii) $$T\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} a & b & \frac{1}{2}c \\ 0 & d & \frac{1}{2}e \\ 0 & 0 & f \end{pmatrix}.$$

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